HOW INTEGRABILITY WORKS (FOR ADS/CFT)*

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(Received October 14, 2016)

In this article, we give a pedagogical introduction into integrable models. First, we review how classical integrable models can be constructed by demanding the existence of a higher spin conserved charge. The primary examples are the sinh-Gordon and sine-Gordon theories whose classical integrability is exactly shown by exploiting the Bäcklund transformation. Lagrangian quantization together with the LSZ reduction formula provide insight into the functional properties and analytical structure of the scattering matrices. These are the inputs in the S-matrix bootstrap program, which determine the scattering matrices from global symmetries, crossing and unitarity properties. Then it is shown how the scattering matrices can be used to calculate the large volume spectrum of the theory. We end the paper by overviewing the literature where the various steps of the analogue integrable developments in the AdS/CFT correspondence were developed.

DOI:10.5506/APhysPolB.47.2451

1. Introduction

The world does not seem to be integrable, so why shall we care about integrable theories at all? They are very special/simple as there is no particle production in scattering processes, moreover, they are also very rare, compared to non-integrable theories and exist only in very specific points of the moduli space of all theories. They are mostly 2 dimensional (2D) and there is only a few systems among them which have physical relevance so far.

Despite these facts, we think that integrable models can be very useful in many respects. First, they can be solved exactly. Exact solutions can give precious insight into the structure of solutions, enable to test new ideas and methods, and provide alternative approaches to define quantum field

* Presented at the LVI Cracow School of Theoretical Physics "A Panorama of Holography", Zakopane, Poland, May 24–June 1, 2016.

theories. Recently, there has been intensive research and relevant progress in analyzing the AdS/CFT correspondence. String theory, viewed as a 2D quantum field theory, on the $\mathrm{AdS}_5\times \mathrm{S}^5$ background is integrable and is dual to the maximally supersymmetric 4D gauge theory. As a consequence, we can use integrable methods to solve a 4D interactive quantum field theory and to quantize string theory on a curved background. We think this is enough motivation to start to investigate integrable theories.

The article is organized as follows: we start by showing how classical integrable models can be constructed by demanding the existence of a higher spin conserved charge. Having analyzed the free boson theory, we derive integrable potentials leading to the sinh-Gordon and sine-Gordon models. Their classical integrability is exactly shown by exploiting the Bäcklund transformation. Lagrangian quantization and perturbation theory defines the model completely. The LSZ reduction formula relates the scattering matrix to correlation functions and provides a way to continue the S-matrix analytically to the whole complex plane of the (single) Mandelstam variable. This representation implies unitarity and crossing symmetry, which are the inputs in the S-matrix bootstrap program. In this program, scattering matrices are fixed from global symmetries, crossing and unitarity properties, and from the maximal analiticity requirement. Then it is shown how the scattering matrices can be used to calculate the large volume spectrum of the theory. We end the paper by overviewing the literature where the various steps of the analogue integrable developments in the AdS/CFT correspondence were developed.

2. Classical integrability

In this section, we construct integrable field theories by demanding the existence of a higher spin charge.

2.1. Notion of integrability

We start with 1+1 dimensional field theories, which are defined by actions of the form of

$$S=\int \mathrm{d}x\mathrm{d}t\,\mathcal{L}(\phi,\partial_\mu\phi)\,;\qquad \mathcal{L}=\tfrac{1}{2}(\partial_t\phi)^2-\tfrac{1}{2}(\partial_x\phi)^2-V(\phi) \tag{1}$$

with fields vanishing at spatial infinities.

Variation of the action

$$\begin{aligned}\delta S &= \int \mathrm{d}^2x\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\mu\delta\phi\right]\\ &= \int \mathrm{d}^2x\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right]\delta\phi+\int \mathrm{d}^2x\ \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right]\end{aligned} \tag{2}$$

can vanish in two interesting ways:

— for generic variations vanishing at time- and space-boundaries, we obtain the equation of motion (e.o.m.)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = -\partial_t^2 \phi + \partial_x^2 \phi - V'(\phi) = 0\,; \tag{3}$$

— for continuous symmetry variations of the fields ϕ satisfying the e.o.m., we obtain conservation laws (Noether theorem)

$$\partial_\mu J^\mu = \partial_t J_t - \partial_x J_x = 0\,; \qquad J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta\phi \tag{4}$$

such that the conserved charge generates the symmetry transformation

$$Q = \int J_t \mathrm{d}x = \int \pi \delta\phi \mathrm{d}x\,; \qquad \{Q, \phi\} = \delta\phi\,. \tag{5}$$

Conservation follows from

$$\dot{Q} = \int \partial_t J_t \mathrm{d}x = \int \partial_x J_x \mathrm{d}x = J(+\infty) - J(-\infty) = 0 \tag{6}$$

since fields vanish at spatial infinities.

Every relativistic theory has conserved energy and momentum, but *integrable theories* have infinitely many functionally-independent additional conserved charges. Let us construct such *relativistic* integrable theories.

2.2. Relativistic invariance

Lorentz covariance helps to organize physical quantities. A Lorentz boost in 1+1 dimensions is simply given by

$$(t, x) \to \left(\frac{t + vx}{\sqrt{1 - v^2}}, \frac{x + vt}{\sqrt{1 - v^2}} \right)\,, \tag{7}$$

where the speed of light was normalized to 1. By introducing rapidity, $v = \tanh \varLambda$, the Lorentz transformation becomes a hyperbolic rotation

$$\begin{pmatrix} t \\ x \end{pmatrix} \to \begin{pmatrix} \cosh \varLambda & \sinh \varLambda \\ \sinh \varLambda & \cosh \varLambda \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{8}$$

which, in the Euclidean version $t \to i\tau$, corresponds to rotation. Light-cone coordinates diagonalize this "rotation"

$$x_\pm = \tfrac{1}{2}(t \pm x) \to e^{\pm \varLambda} x_\pm\,. \tag{9}$$

Light-cone derivatives

$$\partial_\pm = \frac{\partial}{\partial x_\pm} = \partial_t \pm \partial_x \tag{10}$$

and light-cone components of the current can be used to write the conservation laws in the form of

$$\partial_t J_t - \partial_x J_x = \partial_+ J_- + \partial_- J_+ = 0\,; \qquad J_\pm = \tfrac{1}{2}(J_t \pm J_x)\,. \tag{11}$$

2.3. Integrable Lagrangians

In the following, we search for such potentials V which admit infinitely many independent conservation laws.

We start with the $V = 0$ theory such that the equation of motion takes the form of

$$\partial_+\partial_-\phi = 0\,. \tag{12}$$

Clearly, any ∂_- differential polynomial of $\partial_-\phi$ leads to a conserved current with

$$J_- = \left(\partial_-^{j_1}\phi\right)^{k_1} \dots \left(\partial_-^{j_n}\phi\right)^{k_n}\,; \qquad J_+ = 0\,. \tag{13}$$

Similarly, by changing $\partial_- \to \partial_+$, we also get another set of infinitely many conserved charges leading to the conclusion that the free massless boson is actually integrable.

Let us switch on the potential. This changes the equation of motion to

$$\partial_+\partial_-\phi = -V'(\phi) \tag{14}$$

and prevents $J_- = \partial_-\phi$ from being a conserved current. We cannot even modify $J_+ = 0$ such that it remains conserved. Neglecting those combinations which are derivatives by themselves (as their integrals should provide non-zero conserved charges), we analyze the only candidate with two derivatives

$$J_- = T_- = \tfrac{1}{2}(\partial_-\phi)^2 \tag{15}$$

and search for $J_+ = \Theta_+$ such that the current J is conserved (11). We find that

$$\partial_+ T_- = \partial_-\phi\partial_+\partial_-\phi = -\partial_-\phi V'(\phi) = -\partial_- V = -\partial_-\Theta_+ \tag{16}$$

such that $J_+ = \Theta_+ = V$ and the conserved charge is

$$Q_{-1} = \int J_t \mathrm{d}x = \int \left[\tfrac{1}{2}(\partial_-\phi)^2 + V(\phi)\right] \mathrm{d}x\,. \tag{17}$$

Similar calculations can be done by replacing ∂_- with ∂_+ leading to the conserved charged Q_1. These two charges are the light-cone components of the energy and momentum $Q_{\pm 1} = E \pm P$ with

$$E = \int \left[\tfrac{1}{2}(\partial_t\phi)^2 + \tfrac{1}{2}(\partial_x\phi)^2 + V(\phi)\right] \mathrm{d}x\,; \qquad P = \int \partial_x\phi\partial_t\phi\,\mathrm{d}x \tag{18}$$

which are conserved for any V.

Let us go on in searching for higher spin conserved charges. The term containing 3 derivatives cannot be conserved

$$\partial_+\tfrac{1}{3}(\partial_-\phi)^3 = -(\partial_-\phi)^2V'(\phi) = -\partial_-\phi\partial_-V \neq \partial_-(\text{something})\,. \tag{19}$$

With 4 derivatives, we have two candidates

$$\partial_+\tfrac{1}{4}(\partial_-\phi)^4 \;=\; -(\partial_-\phi)^3V'(\phi) = -(\partial_-\phi)^2\partial_-V\,, \tag{20}$$

$$\partial_+\left(\partial_-^2\phi\right)^2 \;=\; -2\left(\partial_-^2\phi\right)\partial_-V'(\phi) = -V''\partial_-(\partial_-\phi)^2 \tag{21}$$

but individually none is a total ∂_- derivative. However, if we demand that

$$V'' = \alpha V \tag{22}$$

and combine them as

$$T_-^{(4)} = \alpha\tfrac{1}{4}(\partial_-\phi)^4 + \left(\partial_-^2\phi\right)^2\,; \qquad \Theta_+^{(2)} = \alpha(\partial_-\phi)^2V\,, \tag{23}$$

then we have higher spin conserved charges

$$Q_{\pm 3} = \int\left[T_\pm^{(4)} + \Theta_\pm^{(2)}\right]\mathrm{d}x = \int\left[\frac{\alpha}{4}(\partial_-\phi)^4 + \left(\partial_-^2\phi\right)^2 + \alpha(\partial_-\phi)^2V\right]\mathrm{d}x\,. \tag{24}$$

Potentials which satisfy $V'' = \alpha V$ are — among others — the sine-Gordon and sinh-Gordon potentials

$$V(\phi) = -\frac{m^2}{\beta^2}\cos\beta\phi\,; \qquad V(\phi) = \frac{m^2}{b^2}\cosh b\phi\,. \tag{25}$$

They can be obtained from each other by the analytical continuation

$$\beta \to ib\,.$$

Exercises

In this part, we list some problems to deepen the understanding of the subject.

1. *Find static solutions of the equation of motions in the sine-Gordon theory! What about the sinh-Gordon theory?*
2. *Construct conserved charges of spin 5!*
3. *Analyze the free massive boson limit $b \to 0$. What happens for the conserved charges?*
4. *Instead of assuming a conserved charge at spin 3, search for the first conserved charge at spin 5. What are the possible potentials now?*
5. *Prove the existence of infinitely many conserved charges in the sine-Gordon theory!*

Observe that there is a Bäcklund transformation

$$\partial_+(\phi_l + \phi_r) = \frac{2m}{\beta\sigma} \sin \frac{\beta}{2}(\phi_l - \phi_r)\,, \tag{26}$$

$$\partial_-(\phi_l - \phi_r) = -\frac{2m}{\beta}\sigma \sin \frac{\beta}{2}(\phi_l + \phi_r) \tag{27}$$

for any σ which ensures that if ϕ_l solves the sine-Gordon equation of motion, and together with ϕ_r they satisfy the first order equations above, then ϕ_r solves the sine-Gordon equation, too. The Bäcklund transformation can be used to generate a new solution from an old one, additionally, it proves the existence of infinitely many conserved charges. Indeed, by combining the equations, we easily get

$$\sigma\partial_+ \cos \frac{\beta}{2}(\phi_l + \phi_r) + \sigma^{-1}\partial_- \cos \frac{\beta}{2}(\phi_l - \phi_r) = 0\,. \tag{28}$$

Expanding ϕ_r around ϕ_l in powers of σ

$$\phi_r = \sum_{n=1}^{\infty} \sigma^n \phi^{(n)} \tag{29}$$

and plugging back to Eq. (28), after expanding in σ, we obtain infinitely many conservation laws.

Literature

In this part, we list some literature and suggest further readings. The similar program to find integrable potentials with more scalar fields leads to the affine Toda field theories. Their conserved charges was originally constructed by different means in [1].

3. Quantum integrability in the Lagrangian framework

In this section, we quantize the previously introduced theories. We start with the Lagrangian quantization, which is based on quantizing the free massive boson first, and taking into account the interaction perturbatively. This is adequate for the sinh-Gordon theory, where the perturbation does not change the particle spectrum. We quantize the sine-Gordon theory later in the bootstrap scheme.

3.1. Lagrangian quantization

In quantizing the sinh-Gordon theory, we choose a free theory first $(b \to 0)$,

$$\mathcal{L}_0 = \frac{1}{2}(\partial_t\varphi)^2 - \frac{1}{2}(\partial_x\varphi)^2 - \frac{m^2}{2}\varphi^2 \,, \tag{30}$$

quantize this theory and then add the perturbation

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \,; \qquad \mathcal{L}_I = \frac{m^2}{b^2}\sum_{n=2}^{\infty}\frac{b^{2n}}{(2n)!}\varphi^{2n} \,. \tag{31}$$

The free massive boson can be quantized easily. The quantum field

$$\varphi_0(x,t) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi 2\omega(k)} \left(a(k)e^{-i\omega(k)t+ikx} + a^+(k)e^{i\omega(k)t-ikx}\right) \tag{32}$$

can be expressed in terms of the creation and annihilation operators

$$\left[a(k), a^+\left(k'\right)\right] = 2\pi 2\omega(k)\delta\left(k-k'\right) \,; \qquad \omega(k) = \sqrt{k^2+m^2} \,. \tag{33}$$

The Hilbert space of the model has multiparticle states

$$a^+(k_1)\dots a^+(k_n)|0\rangle = |k_1,\dots k_n\rangle \,; \qquad a(k)|0\rangle = 0 \tag{34}$$

with definite energies

$$H_0|k_1,\dots,k_n\rangle = \sum_i \omega(k_i)|k_1,\dots,k_n\rangle \tag{35}$$

and momenta

$$P|k_1,\dots,k_n\rangle = \sum_i k_i|k_1,\dots,k_n\rangle \,, \tag{36}$$

where the free energy and momentum operators are normal ordered as

$$H_0 = \int\limits_{-\infty}^{\infty} : \left[\frac{1}{2}(\partial_t\varphi_0)^2 + \frac{1}{2}(\partial_x\varphi_0)^2 + \frac{m^2}{2}\varphi_0^2\right] : \mathrm{d}x \tag{37}$$

and as

$$P = \int_{-\infty}^{\infty} : \partial_x \varphi_0 \partial_t \varphi_0 : \, . \tag{38}$$

Generic correlators can be written in terms of the two-point function

$$\langle 0 | T \left(\varphi_0(x,t) \varphi_0 \left(x', t' \right) \right) | 0 \rangle = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik_0(t-t') + ik_1(x-x')} \tag{39}$$

using Wick theorem.

The interacting theory is defined by its correlators, which can be calculated perturbatively

$$\begin{aligned} &\langle 0 | T \left(\varphi(x_1, t_1) \ldots \varphi(x_n, t_n) \right) | 0 \rangle \\ &= \frac{\langle 0 | T \left(\varphi_0(x_1, t_1) \ldots \varphi_0(x_n, t_n) \exp \left\{ -i \int \mathrm{d}^2 x \mathcal{L}_I(\varphi_0(x)) \right\} \right) | 0 \rangle}{\langle 0 | T \left(\exp \left\{ -i \int \mathrm{d}^2 x \mathcal{L}_I(\varphi_0(x)) \right\} \right) | 0 \rangle} \end{aligned} \tag{40}$$

by expanding the exponential terms. The Feynman rules summarize the way one can systematically perform the computations. In momentum space, they read as:

- — draw all topologically distinct diagrams;
- — associate a propagator $\frac{i}{k^2 - m^2 + i\epsilon}$ for each inner line;
- — introduce $im^2 b^{2n-2}$ for each vertex of $2n$ legs and demand momentum conservation;
- — integrate for inner momenta not fixed by momentum conservations $\int \frac{\mathrm{d}^2 k}{(2\pi)^2}$;
- — divide by the symmetry factor of the graph.

The simplest problem is the calculation of the two-point function. Immediately, however, at one loop, we face with a divergent integral, which can be regularized and compensated by a counterterm: $\frac{\delta m^2}{2} \varphi^2$. Calculating other correlators at one loop, we arrive at the same divergence. Interestingly, the induced counterterm Lagrangian for higher point functions has exactly the same form as the original one. Thus, the divergences can be absorbed into the renormalization of the mass term

$$V(\varphi) - V_{\mathrm{CT}}(\varphi) = \frac{m^2 - \delta m^2}{b^2} (\cosh b\varphi - 1) \tag{41}$$

which at one loop reads as

$$\delta m^2 = -m^2 b^2 \int_0^{\Lambda} \frac{\mathrm{d}p}{\sqrt{p^2+m^2}} \,. \tag{42}$$

The fact that the form of the Lagrangian is not changed at the quantum level, merely the coefficients are renormalized, implies that the quantum equation of motions has the same structure as the classical one and indicates that the sinh-Gordon theory is integrable at the quantum level, too.

The pole of the propagator tells us the mass of the particle, while the scattering matrix is related to the higher point correlation functions via the reduction formula. In deriving the reduction formula, one uses the fact that particles are localized excitations, which for asymptotically large times are well-separated and non-interacting. Thus, we can switch off the interaction adiabatically for large times and suppose that the quantum field is proportional to a free field

$$\lim_{t\to\mp\infty} \varphi(x,t) \approx \lim_{t\to\mp\infty} Z^{\frac{1}{2}} \varphi_0^{\mathrm{in/out}}(x,t) \,, \tag{43}$$

where Z takes care of the canonical normalization of the fields and the limit is understood in the weak sense, *i.e.* for the matrix elements of the operators. Asymptotic annihilation/creation operators can be defined in terms of the asymptotic fields as

$$\begin{aligned} a^{\mathrm{as}}(k) &= i \int \mathrm{d}x\, e^{i\omega(k)t-ikx} \overleftrightarrow{\partial_t} \varphi_0^{\mathrm{as}}(x,t) \,, \\ a^{\mathrm{as}}(k)^{+} &= -i \int \mathrm{d}x\, e^{-i\omega(k)t+ikx} \overleftrightarrow{\partial_t} \varphi_0^{\mathrm{as}}(x,t) \end{aligned} \tag{44}$$

and they create the free asymptotic states

$$|k_1,\ldots,k_n\rangle^{\mathrm{as}} = a^{\mathrm{as}}(k_1)^{+} \ldots a^{\mathrm{as}}(k_1)^{+} |0\rangle \,. \tag{45}$$

Asymptotic completeness means that both the initial and final states form a complete set. Thus, they can be expressed in terms of each other by the so-called scattering matrix

$$S_{\mathrm{fi}} = \langle \mathrm{final} | \mathrm{initial} \rangle \,. \tag{46}$$

As an operator, the scattering matrix is the time-evolution operator in the interaction picture which implies that it is unitary and commutes with the symmetries.

The simplest non-trivial S-matrix element is

$$_{\mathrm{out}}\langle k_3,k_4|k_1,k_2\rangle_{\mathrm{in}}=S(k_1,k_2|k_3,k_4)(2\pi)^2 2\omega(k_1)2\omega(k_2)\delta(k_1-k_3)\delta(k_2-k_4)\,.$$

In a Lorentz invariant theory, the S-matrix depends only on the relativistically invariant Mandelstam variables $s=(k_1+k_2)^2$, $t=(k_1-k_3)^2$ and $u=(k_1-k_4)^2$, where $s+t+u=4m^2$. In 2D, the kinematics is further restricted since $t=4m^2-s$.

The scattering matrix can be expressed in terms of the correlation functions via the reduction formulas, which can be obtained as follows: one first expresses the asymptotic creation and annihilation operators in terms of the free asymptotic fields (44). The asymptotic fields can be expressed at $t=-\infty$ with the interacting field, which, using the identity $f(-\infty)=f(\infty)-\int_{-\infty}^{\infty}\partial_t f(t)$, can be further decomposed into the disconnected, $f(\infty)$, and the remaining connected contributions. In the connected piece, we use the dispersion relation $\omega^2=k^2+m^2$ to replace the time derivatives with space derivatives, which we integrate by parts. Dropping the surface terms and repeating the same procedure for each asymptotic creation/annihilation operators, we obtain the following reduction formula:

$$_{\mathrm{out}}\langle k_3,k_4|k_1,k_2\rangle_{\mathrm{in}}=\text{disc.}+Z^{-2}\bar{\mathcal{D}}_4\bar{\mathcal{D}}_3\mathcal{D}_2\mathcal{D}_1\langle 0|T(\varphi(1)\varphi(2)\varphi(3)\varphi(4))|0\rangle\,, \tag{47}$$

where $\varphi(i)$ stands for $\varphi(x_i,t_i)$ and

$$\mathcal{D}_i=-\int \mathrm{d}^2x_i e^{-i\omega(k_i)t_i+ik_ix_i}\Box_i\,;\qquad -\Box_i=-\partial_{t_i}^2+\partial_{x_i}^2-m^2\,. \tag{48}$$

Disconnected terms appear whenever one of the outgoing momenta equals to any of the incoming ones. The physical meaning of the operator $\mathcal{D}_i$ is to amputate a leg of the correlator and to put it on-shell. Clearly, in momentum space, $\Box_i$ picks up the residue of the pole of the propagator, while the inverse Fourier transformation puts the particle on the mass shell: $\omega^2+k^2=m^2$. For initial states, we obtain the operator $\bar{\mathcal{D}}_i=-\int\mathrm{d}^2x_i e^{i\omega(k_i)t_i-ik_ix_i}\Box_i$. Comparing the two operators $\mathcal{D}_i$ and $\bar{\mathcal{D}}_i$, the only difference is in the sign of the two-momentum (ω,k). From this, we can read the crossing symmetry of the scattering matrix

$$S(k_1,k_2|k_3,k_4)=S\left(k_1,\bar{k}_3|\bar{k}_2,k_4\right)\,. \tag{49}$$

Let us go into the center-of-mass frame $k_1\equiv(\omega(k),k)$ and $k_2=(\omega(k),-k)$. Then we have $s=4m^2+4k^2$ and $t=-4k^2$. Crossing symmetry implies that

$$S(s+i\epsilon)=S(t-i\epsilon)=S\left(4m_1^2-s-i\epsilon\right)\,. \tag{50}$$

Switching from the $i\epsilon$ prescription to the $-i\epsilon$ prescription is equivalent to time reversal, which changes the scattering matrix to its inverse

$$S(s+i\epsilon) = S(s-i\epsilon)^{-1}\,. \tag{51}$$

We can learn about the non-analytical domains of the S-matrix from the unitarity relation $SS^\dagger = \mathbb{I}$. For this, we write the scattering matrix as $S = \mathbb{I} + iT$ giving

$$i\left(T - T^\dagger\right) = -TT^\dagger\,. \tag{52}$$

Taking matrix element between initial and final two-particle states and inserting a complete system, we can write

$$\langle p_3, p_4 | i\left(T - T^\dagger\right) | p_1, p_2\rangle = -\sum_{n\in\mathcal{H}} \langle p_3, p_4 | T | n\rangle\langle n | T^\dagger | p_1, p_2\rangle\,. \tag{53}$$

The vacuum does not contribute. One-particle terms give pole singularities at m^2 and at $3m^2$. There are cuts on the real line starting at the multiparticle thresholds $(nm)^2$. The physical value of the S-matrix is at $S(s+i\epsilon)$, where $s > 4m^2$. This analytical structure of the S-matrix is displayed in Fig. 1.

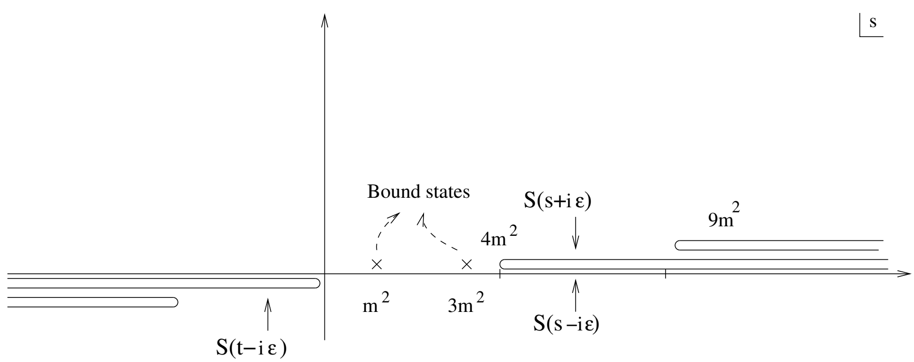


Fig. 1. The analitycal structure of the S-matrix.

It is useful to introduce rapidity parametrization which, in the center-of-mass frame, reads as $k = m\sinh\frac{\theta}{2}$, such that the rapidity difference is θ. The rapidity is related to the s variable as

$$s = 4m^2\left(1 + \sinh^2\frac{\theta}{2}\right) = 4m^2\cosh^2\frac{\theta}{2}\,. \tag{54}$$

This resolves the cut starting at $4m^2$ and maps the first sheet of the complex s-plane to the strip $0 < \mathrm{Im}(\theta) < \pi$. The bound-state poles are located on the imaginary axes at $\theta = i\zeta$ and at $\theta = i(\pi - \zeta)$. Crossing symmetry translates to the rapidity parameter as

$$S(i\pi - \theta) = S(\theta)\,, \tag{55}$$

while the S-matrices on the two sides of the cut are related as

$$S(\theta) = S(-\theta)^{-1}\,. \tag{56}$$

As we already developed the technique to calculate the correlation function, using the reduction formula, we can elaborate order-by-order the scattering matrix of the sinh-Gordon theory. The perturbative result in the sinh-Gordon theory reads as

$$S(\theta) = 1 - \frac{ib^2}{4\sinh\theta} - \frac{b^4\left(\frac{\pi}{\sinh\theta} - i\right)}{32\pi\sinh\theta} + \frac{ib^6\left(\frac{\pi}{\sinh\theta} - i\right)^2}{256\pi^2\sinh\theta} + O\left(b^8\right)\,. \tag{57}$$

In the next section, we obtain an all-order exact expression for this quantity.

Exercises

1. *Quantize the free massive boson and derive the Feynman rules!*
2. *Calculate the mass counterterm at first order in perturbation theory!*
3. *Calculate the scattering matrix at first order in perturbation theory! (Pay particular attention to the normalization of these quantities, especially the arguments of the delta functions.)*
4. *Construct quantum integrable potentials! Start with a ϕ^4 theory and calculate at 1-loop the $S_{2\to4}$ particle process. Introduce a ϕ^6 potential which cancels this contribution at tree level. Repeat the procedure for $S_{2\to n}$.*

Literature

A more detailed exposition of the section can be found in [2] and [3]. For the analytical structure of the scattering matrix, see also [4, 5].

4. Quantum integrability: the S-matrix bootstrap

The S-matrix bootstrap combines the analytic S-matrix theory with integrability in an axiomatic framework.

4.1. Bootstrap quantization

We first assume that the Hilbert space is spanned by free multiparticle states with relativistic dispersion relation

$$E(p) = \omega(p) = \sqrt{p^2 + m^2}\,; \qquad E(p)^2 - p^2 = m^2\,. \tag{58}$$

Using the rapidity parametrization, we can write

$$E(\theta) = \omega(\theta) = m\cosh\theta\,; \qquad p(\theta) = m\sinh(\theta)\,. \tag{59}$$

Light-cone components diagonalize the action of boosts and can be written as

$$(E \pm p)(\theta) = Q_{\pm 1}(\theta) = me^{\pm\theta}\,. \tag{60}$$

In an integrable theory, these are the first members of an infinite family of conserved charges which can be labeled by their spin s: $Q_s(\theta) = q_s e^{s\theta}$. A multiparticle initial state is denoted as

$$|\theta_1, \dots \theta_n\rangle_{\text{in}}\,; \qquad \theta_1 > \cdots > \theta_n\,, \tag{61}$$

while final states as

$$|\theta_1, \dots \theta_m\rangle_{\text{out}}\,; \qquad \theta_m > \cdots > \theta_1\,. \tag{62}$$

Both bases diagonalize the action of the (infinitely many) conserved charges

$$Q_s|\theta_1, \dots \theta_n\rangle_{\text{in}} = \sum_{i=1}^{n} q_s e^{s\theta_i}|\theta_1, \dots \theta_n\rangle_{\text{in}}\,. \tag{63}$$

The scattering matrix connects the two bases of the Hilbert space, thus they relate initial and final states

$$S_{n\to m} = {}_{\text{out}}\big\langle\theta'_1, \dots \theta'_m\big|\theta_1, \dots \theta_n\big\rangle_{\text{in}} \tag{64}$$

and its absolute square describes the probability with which the initial state evolves into the final state. Let us formulate the requirements for this S-matrix. As the scattering matrix is the time evolution operator in the interaction picture, it must commute with the symmetries, *i.e.* with their generators the conserved charges Q_s. Thus, if we evaluate them before and after the scattering, they have to coincide

$$\sum_{i=1}^{n} q_s e^{s\theta_i} = \sum_{i=1}^{m} q_s e^{s\theta'_i}\,. \tag{65}$$

These are functionally-independent polynomial equations ($x_i = e^{\theta_i}$) for infinite different values of the spin s. They can be satisfied for a finite number of $\{\theta_i\}$ and $\{\theta'_j\}$ only if the two sets completely agree: $\{\theta_i\} = \{\theta'_j\}$. In particular, this means that the number of particles in the initial and final states coincides $n = m$, *i.e.* there is no particle creation in quantum integrable theories.

Conserved charges generate symmetry transformations: H generates uniform shift in time, while P generates uniform shift in space. Higher spin charges, however, generate momentum-dependent shifts in space-time. As all rapidities are different, by acting with a higher spin charge, we can spatially separate the particle interactions and factorize the multiparticle scattering amplitudes into the product of two-particle scatterings

$$S_{n\to n}(\theta_1,\dots,\theta_n) = \prod_{i,j} S_{2\to 2}\left(\theta_i,\theta_j\right) . \tag{66}$$

The full information of the multiparticle scattering matrix is contained in the $S_{2\to 2}(\theta_1,\theta_2)$ two-particle elastic scatterings only, thus we focus on its determination from now on. Lorentz boost acts on the rapidity as $\theta \to \theta + \Lambda$ and, as it is a symmetry, we can write

$$S_{2\to 2}(\theta_1,\theta_2) = S(\theta_1-\theta_2)\,. \tag{67}$$

Unitarity translates into

$$S(\theta)S(-\theta) = 1\,, \tag{68}$$

while crossing symmetry to

$$S(\theta) = S(i\pi-\theta)\,. \tag{69}$$

Absence of bound state implies that we have no poles in the physical strip.

We now solve these functional relations. In order to have the right periodicity and good asymptotic property, let us consider the logarithmic derivative of the scattering matrix

$$\phi(\theta) = -i\frac{\mathrm{d}}{\mathrm{d}\theta}\log S(\theta)\,. \tag{70}$$

It has the properties

$$\phi(\theta) = \phi(-\theta)\,; \qquad \phi(\theta) = -\phi(i\pi-\theta) = -\phi(i\pi+\theta)\,. \tag{71}$$

Let us write $\phi(\theta)$ as

$$\phi(\theta) = \oint_{\theta} \frac{\mathrm{d}\theta'}{2\pi i}\frac{\phi\left(\theta'\right)}{\sinh\left(\theta'-\theta\right)}\,, \tag{72}$$

where the contour surrounds θ infinitesimally. Now, we blow up the contour and deform it to $\mathrm{Im}(\theta) = 0$ and to $\mathrm{Im}(\theta) = \pi$. In doing so, we pick up the residues of the possible bound-state poles or zeros. Assuming we have no pole but a single pair of zeros at $i\alpha$ and $i(\pi - \alpha)$ [1] leads to

$$\phi(\theta) = -\frac{1}{\sinh(i\alpha - \theta)} - \frac{1}{\sinh(i(\pi - \alpha) - \theta)} \,. \tag{73}$$

After integrating and exponentiating, we obtain

$$S(\theta) = \frac{\sinh\theta - i\sin\alpha}{\sinh\theta + i\sin\alpha}\,; \qquad \alpha > 0\,. \tag{74}$$

By comparing to the perturbative result, we can conjecture that

$$\alpha = \frac{\pi b^2}{8\pi + b^2}\,. \tag{75}$$

4.2. Non-diagonal scatterings

We can perform an analytical continuation $b \to i\beta$ leading to an S-matrix in the sine-Gordon theory

$$S(\theta) = \frac{\sinh\theta + i\sin\alpha}{\sinh\theta - i\sin\alpha}\,; \qquad \alpha = \frac{\pi\beta^2}{8\pi - \beta^2}\,. \tag{76}$$

In this case, we find a pole in the physical strip ($\mathrm{Im}(\theta) \in [0, \pi]$), and we have to introduce more particles in the spectrum. It turns out that the spectrum is not consistent with the bound states of this particle only and we have to introduce a mass degenerate doublet. These two particles correspond to the quantizations of the two static solutions of the equations of motion

$$\phi_{\text{static}}(x) = \pm\frac{4}{\beta}\arctan(e^{mx})\,. \tag{77}$$

They are called the soliton and the anti-soliton. In the quantum theory, they are degenerate in mass and during the scattering process, they mix with each other. They form a doublet representation of the non-local symmetry of the model $U_q(\widehat{\mathrm{sl}}_2)$. To derive this symmetry, one can start by quantizing the free massless compactified boson. This $c = 1$ conformal field theory is integrable and has infinitely many conserved charges at the quantum level. They are quantization of the charges which appeared at the classical level (13). After switching on the perturbation, which leads to the sine-Gordon

[1] Having neither a zero nor a pole implies that $\phi(\theta) = 0$, *i.e.* $S = \pm 1$. These solutions correspond to free boson and free fermion.

theory, a similar calculation to the classical one shows that there are conserved higher spin charges. Additionally to these charges, one can also show the existence of non-local charges. These charges generate an $U_q(\hat{\mathrm{sl}}_2)$ algebra. In the following, we use the invariance of the scattering matrix for an $U_q(\mathrm{sl}_2)$ subalgebra to find the matrix part of the soliton scattering matrix. This $U_q(\mathrm{sl}_2)$ subalgebra is generated by Q_+, Q_- and Q_0, which act on the two-particle soliton–anti-soliton doublet as

$$Q_0 = \sigma_0 \, ; \qquad Q_\pm = e^{\lambda\theta}\sigma_\pm q^{\pm\sigma_0} \, , \tag{78}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{79}$$

and q is a parameter of the representation with $q = e^{i\lambda}$. From conformal perturbation theory, it can be shown that $\lambda = \frac{8\pi}{\beta^2} - 1$. The non-local nature of the symmetry manifests in the action of the charges on two-particle states. These are defined by the coproduct formula

$$\Delta(Q_0) = Q_0 \otimes \mathbb{I} + \mathbb{I} \otimes Q_0 \, ; \qquad \Delta(Q_\pm) = Q_\pm \otimes \mathbb{I} + q^{Q_0} \otimes Q_\pm \, . \tag{80}$$

The scattering of the soliton doublet on itself is described by a 4-by-4 matrix, which maps a two-particle state with rapidities θ_1 and θ_2 into the state with rapidities θ_2 and θ_1. This commutes with the symmetries

$$\Delta_{21}(Q)S_{12}(\theta_1 - \theta_2) = S_{12}(\theta_1 - \theta_2)\Delta_{12}(Q) \tag{81}$$

and is the function of the rapidity difference only. The charge Q_0 is nothing but the topological charge and its conservation restricts the scattering into the form

$$S(\theta) \; = \begin{pmatrix} S_{++}^{++}(\theta) & 0 & 0 & 0 \\ 0 & S_{+-}^{+-}(\theta) & S_{+-}^{-+}(\theta) & 0 \\ 0 & S_{-+}^{+-}(\theta) & S_{-+}^{-+}(\theta) & 0 \\ 0 & 0 & 0 & S_{--}^{--}(\theta) \end{pmatrix} . \tag{82}$$

Further invariance with $Q_\pm$ gives

$$S(\theta) = \rho(\theta) \begin{pmatrix} a(\theta) & 0 & 0 & 0 \\ 0 & b(\theta) & c(\theta) & 0 \\ 0 & c(\theta) & b(\theta) & 0 \\ 0 & 0 & 0 & a(\theta) \end{pmatrix} \tag{83}$$

with

$$a(\theta) = 1\,; \qquad b(\theta) = \frac{\sin(i\lambda\theta)}{\sin\lambda(\pi + i\theta)}\,; \qquad c(\theta) = \frac{\sin(\lambda\pi)}{\sin\lambda(\pi + i\theta)}\,. \tag{84}$$

The scalar coefficient can be fixed from the unitary

$$S_{ij}^{kl}(\theta_1 - \theta_2) S_{lk}^{mn}(\theta_2 - \theta_1) = \delta_i^m \delta_j^l \tag{85}$$

and the crossing symmetry requirements

$$S_{ij}^{kl}(\theta_1 - \theta_2) = S_{j\bar{l}}^{\bar{i}k}(i\pi - \theta_1 + \theta_2)\,. \tag{86}$$

They translate to the prefactor as

$$\rho(\theta)\rho(-\theta) = 1\,; \qquad \rho(i\pi - \theta) = \rho(\theta)\frac{\sin(i\lambda\theta)}{\sin\lambda(\pi + i\theta)}\,. \tag{87}$$

These equations can be solved by an infinite product of the form of

$$\rho(\theta) = \prod_{l=1}^{\infty}\left[\frac{\mathit{\Gamma}\left(\lambda(2l-2) + \frac{i\lambda\theta}{\pi}\right)\mathit{\Gamma}\left(1 + \lambda 2l + \frac{i\lambda\theta}{\pi}\right)}{\mathit{\Gamma}\left(\lambda(2l-1) + \frac{i\lambda\theta}{\pi}\right)\mathit{\Gamma}\left(1 + \lambda(2l-1) + \frac{i\lambda\theta}{\pi}\right)}\Big/(\theta \to -\theta)\right],$$

where some CDD ambiguity still remains, *i.e.* we can multiply the solution with any function satisfying $f(\theta) = f(i\pi - \theta) = f(-\theta)^{-1}$. These functions, however, will introduce additional zeros or poles in the physical strip what we would like to avoid, as each pole/zero must have a physical origin. Let us analyze the analytical structure of the solution we found.

The soliton–anti-soliton scatterings ($S_{+-}^{+-}(\theta) = \rho(\theta)b(\theta)$ and $S_{+-}^{-+}(\theta) = \rho(\theta)c(\theta)$) have poles at $i\pi(1 - \frac{n}{\lambda})$, they are in the physical strip for $n = 1, \ldots, [\lambda]$. Thus, for $\lambda > 1$, the soliton and the anti-soliton can form bound states. The lightest one should correspond to the fundamental excitation of the field ϕ and should be the continuation of the sinh-Gordon particle. It is possible to extract the scattering matrix of this bound state from the scattering matrices of the soliton–anti-soliton doublet (see exercises) and compare its analytical continuation to the sinh-Gordon scattering matrix. The analysis gives the relation

$$\lambda = \frac{8\pi}{\beta^2} - 1 \tag{88}$$

which proves the previously conjectured relation for the sinh-Gordon model.

Exercises

1. *Assume that* $\alpha < 0$ *in (74)! Interpret the pole in the S-matrix at* $\theta = -i\alpha$ *as a boundstate and calculate its mass! Determine* α *for which the mass is* m*, i.e. the bound state is the particle itself!*
2. *By shifting the trajectories of the particles show that the scattering of the bound state on the original particle can be written as* $S(\theta-\frac{i\alpha}{2})S(\theta+\frac{i\alpha}{2})$*! Determine* α *for which it equals to* $S(\theta)$*!*
3. *Show that the sine-Gordon scattering matrix satisfies the Yang–Baxter equation*

$$\begin{aligned} &S_{ij}^{pr}(\theta_1-\theta_2)S_{pk}^{lq}(\theta_1-\theta_3)S_{rq}^{mn}(\theta_2-\theta_3) \\ &= S_{jk}^{pr}(\theta_2-\theta_3)S_{ir}^{qn}(\theta_1-\theta_3)S_{qp}^{lm}(\theta_1-\theta_2)\,. \end{aligned}$$

4. *Calculate the mass and the scattering matrix of the first soliton–anti-soliton bound state!*

Literature

The S-matrix bootstrap in the sine-Gordon theory was developed in [6]. The quantum group symmetry was derived in [7] together with our calculation of the matrix part of the scattering matrix. For further details, see also [5] and [2].

5. Finite volume energy spectrum

In the previous sections, we determined the particle content of 2D integrable quantum field theories together with their scattering matrices using the S-matrix bootstrap approach. In the present section, we will use these quantities to calculate the energy spectrum in finite volume.

The spectrum in infinite volume is very simple

$$E(\theta_1,\ldots,\theta_n)=\sum_i m\cosh\theta_i\,. \tag{89}$$

We see that above m, we have a continuum of energy levels. This continuous spectrum is replaced with discrete energy levels in finite volume. We analyze first how the energy levels are discretized for a free system, then we turn to the investigation of the effect of the interaction. The leading finite size correction originates from the momentum quantization which is described by the Bethe–Yang equations. This part contains all polynomial finite size corrections in the inverse of the volume and would give an exact answer in quantum mechanical systems. In a quantum field theory, additionally, there are vacuum polarization effects, which result in exponentially small corrections. There is a systematic way of taking them into account iteratively and there is an exact description, called the Thermodynamic Bethe Ansatz, which sums them all up.

Suppose now that a free quantum field theory is put into a finite volume L with periodic boundary condition. Periodicity means that fields satisfy $\phi(x+L)=\phi(x)$. As the conserved momentum P generates the space translation $e^{iPL}\phi(x,t)e^{-iPL}=\phi(x+L,t)$, its spectrum has to be quantized $e^{iPL}=1$. This means for a one-particle state with momentum p that

$$e^{ipL}=e^{imL\sinh\theta}=1\,;\qquad p_n=\frac{2\pi}{L}n\,. \tag{90}$$

The spectrum of a free theory is composed of non-interacting particles

$$E(n_1,\dots,n_k)=\sum_{i=1}^{k}\sqrt{m^2+\left(\frac{2\pi}{L}n_i\right)^2}\,. \tag{91}$$

Clearly, the spectrum is discrete. It becomes denser and denser for $L\to\infty$ and gives the continuous spectrum in the infinite volume limit. Let us see now how we can take into account the interaction among the particles.

5.1. Diagonal theories: the sinh-Gordon model

Let us consider the quantum mechanics of a particle in finite volume in the presence of a localized potential. The wave function, additionally to the free propagation phase e^{ipL}, picks up the transmission phase $T(p)$ when passing through the potential, thus the periodicity requirement is modified to $e^{ipL}T(p)=1$. We use this observation for the particle-like excitation of the quantum field.

We will treat the excitation of the quantum field as particles, but this is correct as far as their size (m^{-1}) is much smaller than the volume L. In this approximation, we can use the quantum mechanical argumentation above. This is further supported by the absence of particle production in integrable theories, where the only interaction is condensed into the phase that a particle acquires when it scatters on the others. As a consequence, the particle number is a conserved quantity and can be used to label the multiparticle finite volume states. In describing the finite size effects, we proceed in this particle number.

The one-particle quantization condition is not changed compared to the free theory since there is no other particle to scatter on

$$e^{im\sinh\theta L}=1\leftrightarrow m\sinh\theta_n=\frac{2\pi}{L}n\,. \tag{92}$$

The energy is given by

$$E(n) = m\cosh\theta_n \,. \tag{93}$$

In the case of two particles with momentum p_1, p_2, we have to take into account their scattering interactions. Thus, when we move the first particle around the circle, we do not only pick up the phase e^{ip_1L} but also scatter on the other particle

$$e^{imL\sinh\theta_1}S(\theta_1-\theta_2) = e^{imL\sinh\theta_1}e^{i\delta(\theta_1-\theta_2)} = 1\,. \tag{94}$$

Here, we introduced the phase of the scattering matrix

$$S(\theta_1-\theta_2) = e^{i\delta(\theta_1-\theta_2)}\,. \tag{95}$$

We can compare the momentum/rapidity to the free case by solving the equation

$$mL\sinh\theta_1 + \delta(\theta_1-\theta_2) = 2\pi n_1\,. \tag{96}$$

The correction compared to free particle quantization is

$$p_n = p_n^{\mathrm{free}} - \frac{\delta(p_1,p_2)}{L} \tag{97}$$

which is of the order of $O(L^{-1})$ and vanishes when $L\to\infty$. In an analogous way, we can write the equation for the second particle

$$mL\sinh\theta_2 + \delta(\theta_2-\theta_1) = 2\pi n_2\,. \tag{98}$$

Since the unitarity property $S(\theta)S(-\theta) = 1$ translates into the phase as $\delta(\theta)+\delta(-\theta) = 0$, we have

$$mL(p_1+p_2) = 2\pi(n_1+n_2)\,. \tag{99}$$

That is, the total momentum is quantized in the units of $2\pi/L$. This is expected as the momentum is conserved and generates space translation $e^{iPL} = 1$. The energy of the state is given by

$$E(n_1,n_2) = m\cosh\theta_1 + m\cosh\theta_2\,. \tag{100}$$

In the case of N particles, the quantization condition is

$$e^{imL\sinh\theta_1}S(\theta_1-\theta_2)S(\theta_1-\theta_3)\ldots S(\theta_1-\theta_n) = 1 \tag{101}$$

and similar equations hold for each θ_i. These are called the Bethe–Yang equations which determine the finite volume energy levels via the formula

$$E(\theta_1,\ldots,\theta_n) = \sum_i m\cosh\theta_i\,. \tag{102}$$

This approximation is valid whenever $m^{-1} \ll L$. In the following, we extend our findings to non-diagonal scattering theories.

5.2. Non-diagonal theories: the sine-Gordon model

We analyze the sine-Gordon model in the $\lambda < 1$ region where there are no bound states, only the soliton–anti-soliton doublet $|\pm\rangle$ is in the spectrum.

The one-particle states are not part of the spectrum with periodic boundary condition. If one takes the $L \to \infty$ limit of the periodic theory, then only the $Q = 0$ topological charge sector can be reached. If we would like to describe the charge Q states, we have to demand a boundary condition in which the field φ jumps $Q\frac{2\pi}{\beta}$ when we go around the circle $\varphi(x + L, t) = \varphi(x, t) + \frac{2\pi Q}{\beta}$.

The $Q = \pm 1$ one-particle states can be then described as free states

$$e^{iML\sinh\theta} = 1 \leftrightarrow ML\sinh\theta = \frac{2\pi}{L}n\,. \tag{103}$$

Here, we denote the mass of the doublet by M.

In the case of two-particle states with charge $Q = 2$, we have two solitons on the circle. Since the scattering is diagonal $S^{++}_{++}(\theta)$, we can use the diagonal equations as before

$$e^{iML\sinh\theta_1} S^{++}_{++}(\theta_1 - \theta_2) = 1\,. \tag{104}$$

By parity symmetry, we have the same equation for $Q = -2$. The $Q = 0$ sector is more complicated as the scattering mixes up solitons and anti-solitons and we have to find such a state which is invariant under the scatterings. Since parity is conserved, the even and the odd combinations of the states $s_+(\theta_1)s_-(\theta_2) \pm s_-(\theta_1)s_+(\theta_2)$ diagonalize the scattering matrix with eigenvalues $S^{\pm}(\theta)$. Here, we denoted the soliton with rapidity θ by $s_+(\theta)$ and similarly the anti-soliton by s_-. The quantization condition then reads as

$$e^{iML\sinh\theta_1} S^{\pm}(\theta_1 - \theta_2) = 1\,. \tag{105}$$

If we consider a theory in a finite volume L which contains all topological charges, then the two-particle BY equation can be formulated as

$$e^{iML\sinh\theta_1} S^{kl}_{ij}(\theta_1 - \theta_2)\Psi_{kl} = \Psi_{ij}\,, \tag{106}$$

where Ψ_{ij} is the coefficient matrix of the two-particle state which diagonalizes the scatterings $\Psi_{ij}s_i(\theta_1)s_j(\theta_2)$.

In the case of n particles, the matrix we have to diagonalize can be obtained by bringing the doublet with θ_1 around all the other particles

$$e^{iML\sinh\theta_1 L} S^{j_1 j_n}_{k_{n-1} i_n}(\theta_1 - \theta_n)\ldots S^{k_2 j_3}_{k_1 i_3}(\theta_1 - \theta_3) S^{k_1 j_2}_{i_1 i_2}(\theta_1 - \theta_2)\Psi_{j_1\ldots j_n} = \Psi_{i_1\ldots i_n}\,. \tag{107}$$

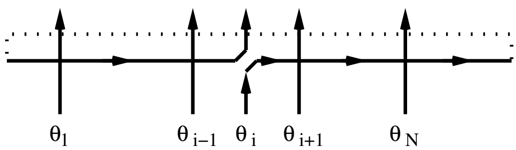
Basically, we have to diagonalize the following Bethe–Yang matrix:

$$T_1(\theta_1;\theta_2,\dots,\theta_n)_{i_1i_2\dots i_n}^{j_1j_2\dots j_n}=S_{k_{n-1}i_n}^{j_1j_n}(\theta_1-\theta_n)\dots S_{k_1i_3}^{k_2j_3}(\theta_1-\theta_3)S_{i_1i_2}^{k_1j_2}(\theta_1-\theta_2)\,. \tag{108}$$

If we bring the l^{th} particle around the circle, we need to diagonalize the following matrix:

$$\begin{aligned}&T_l(\theta_1,...;\theta_l;...,\theta_n)_{i_1i_2\dots i_n}^{j_1j_2\dots j_n}\\&=S_{k_{l-1}i_{l-1}}^{i_lj_{l-1}}(\theta_l,\theta_{l-1})...S_{k_li_{l+2}}^{k_{l+1}j_{l+2}}(\theta_l,\theta_{l+2})S_{i_li_{l+1}}^{k_lj_{l+1}}(\theta_l,\theta_{l+1})\end{aligned} \tag{109}$$

depicted graphically as



These matrices commute and can be diagonalized on the same basis. To show this, first we define the monodromy matrix

$$T(\theta|\theta_1,\dots,\theta_n)_{i,i_1,i_2,\dots i_n}^{j,j_1,j_2,\dots j_n}=S_{k_{n-1}i_n}^{j\,j_n}(\theta,\theta_n)\dots S_{k_1i_2}^{k_2j_2}(\theta,\theta_2)S_{i\,i_1}^{k_1j_1}(\theta,\theta_1) \tag{110}$$

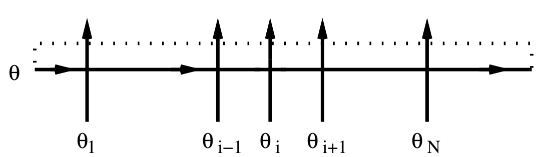
by bringing an auxiliary test particle around the circle. One can show that $\mathrm{Tr}(T(\theta))$ reduces to T_l for $\theta=\theta_l$. This is a consequence of the fact that the scattering matrix reduces to the permutation matrix for vanishing arguments: $S_{ij}^{kl}(0)=-\delta_i^l\delta_j^k$. One can derive the following commutation relation for monodromy matrices

$$S_{12}(\theta_1-\theta_2)T_1(\theta_1)T_2(\theta_2)=T_2(\theta_2)T_1(\theta_1)S_{12}(\theta_1-\theta_2)\,, \tag{111}$$

where only the auxiliary space of the monodromy matrix is indicated, together with the space where it acts. Taking the trace of the equation, it follows that transfer matrices commute for different arguments and the BY matrices can be diagonalized simultaneously.

In describing the finite size spectrum of the sine-Gordon theory, we need to diagonalize the transfer matrix

$$\tilde{T}(\theta)=\mathrm{Tr}\left(\tilde{S}(\theta-\theta_N)\dots\tilde{S}(\theta-\theta_1)\right)\,;\qquad S(\theta)=\rho(\theta)\tilde{S}(\theta)\,. \tag{112}$$



Interestingly, $\tilde{T}$ is the transfer matrix of the inhomogeneous XXZ Heisenberg spin chain. We can see that the calculation of the finite size energy spectrum of a non-diagonal quantum field theory is reduced to the solution of an inhomogeneous spin chain.

Once the eigenvalue $\lambda(\theta|\theta_1,\ldots,\theta_N)$ of $\tilde{T}(\theta)$ has been determined, the BY equations can be written as

$$e^{iML\sinh(\theta_j)}\prod_{i=1}^{N}\rho(\theta-\theta_i)\lambda(\theta_j|\theta_1,\ldots,\theta_N)=-1\,. \tag{113}$$

These equations determine the rapidities θ_i and the energy can be written as

$$E(\theta_1,\ldots,\theta_N)=M\sum_{i=1}^{N}\cosh(\theta_i)\,. \tag{114}$$

In the following, we diagonalize the transfer matrix of the XXZ spin chain.

5.3. Solution of the inhomogeneous XXZ spin chain

The Hilbert space of the Heisenberg spin chain of N sites is $\mathcal{H}=\otimes_{i=1}^{N}\mathbb{C}^2$. Each element of the monodromy matrix $\tilde{S}(\theta-\theta_1)\ldots\tilde{S}(\theta-\theta_N)$ is an operator on $\mathcal{H}$. Let us write the monodromy matrix in the form of

$$\tilde{S}(\theta-\theta_1)\ldots\tilde{S}(\theta-\theta_N)=\begin{pmatrix} A(\theta) & B(\theta)\\ C(\theta) & D(\theta)\end{pmatrix}, \tag{115}$$

where the matrix structure is written explicitly for the auxiliary space. We need to diagonalize the transfer matrix, which is the trace of the monodromy matrix

$$\tilde{T}(\theta)=A(\theta)+B(\theta)\,. \tag{116}$$

It is easy to find one eigenvector of this matrix $(+,\ldots,+)=|+,\ldots,+\rangle\equiv|0\rangle$. The corresponding eigenvalue follows from

$$A(\theta)|0\rangle \;=\; \prod_{i=1}^{N}S_{++}^{++}(\theta-\theta_i)|0\rangle =: \tilde{a}(\theta)|0\rangle\,, \tag{117}$$

$$D(\theta)|0\rangle \;=\; \prod_{i=1}^{N}S_{-+}^{-+}(\theta-\theta_i)|0\rangle =: \tilde{d}(\theta)|0\rangle\,. \tag{118}$$

This eigenstate is called the pseudo-vacuum state. One can check that $C(\theta)$ acts as an annihilation operator

$$C(\theta)|0\rangle=0\,, \tag{119}$$

while $B(\theta)$ acts as a creation operator. A general eigenvector can be written as

$$B(v_1)\dots B(v_m)|0\rangle\,, \tag{120}$$

where the parameters v_i satisfy non-trivial relations what we determine in the following. In finding the eigenvectors, the algebraic relations of the elements of the monodromy matrix are crucial

$$\begin{aligned} B(\theta_1)B(\theta_2) &= B(\theta_2)B(\theta_1)\,, &(121)\\ A(\theta_1)B(\theta_2) &= f(\theta_2-\theta_1)B(\theta_2)A(\theta_1)+g(\theta_2-\theta_1)B(\theta_1)A(\theta_2)\,, &(122)\\ D(\theta_1)B(\theta_2) &= f(\theta_1-\theta_2)B(\theta_2)D(\theta_1)+g(\theta_1-\theta_2)B(\theta_1)D(\theta_2)\,, &(123)\end{aligned}$$

where one can show that $f=\frac{a}{b}$ and $g=-\frac{c}{b}$.

We need to find the constraints on the parameters v_i such that the above state is an eigenstate of the Hamiltonian. If we have one single $B(v)$, then the calculation goes as follows:

$$\begin{aligned} A(\theta)B(v)|0\rangle &= f(v-\theta)B(v)A(\theta)|0\rangle+g(v-\theta)B(\theta)A(v)|0\rangle\\ &= f(v-\theta)\tilde{a}(\theta)B(v)|0\rangle+g(v-\theta)\tilde{a}(v)B(\theta)|0\rangle\,, \end{aligned} \tag{124}$$

and similar equations for D

$$\begin{aligned} D(\theta)B(v)|0\rangle &= f(\theta-v)B(v)D(\theta)|0\rangle+g(\theta-v)B(\theta)D(v)|0\rangle\\ &= f(\theta-v)\tilde{d}(\theta)B(v)|0\rangle+g(\theta-v)\tilde{d}(v)B(\theta)|0\rangle\,. \end{aligned} \tag{125}$$

The basic idea is to combine the two equations such that the “unwanted” $B(\theta)|0\rangle$ term disappears. The condition is

$$g(v-\theta)\tilde{a}(v)+g(\theta-v)\tilde{d}(v)=0 \tag{126}$$

or

$$1=\prod_{k=1}^{N}\frac{\sinh\lambda(v-\theta_k)}{\sinh\lambda(v-\theta_k-i\pi)}\,. \tag{127}$$

One can use the same strategy for higher m. The resulting equations can be encoded as follows. One first defines the functions

$$Q(\theta)=\prod_\beta \sinh\lambda(\theta-v_\beta)\,;\qquad T_0(\theta)=\prod_j \sinh\lambda\left(\theta-\frac{i\pi}{2}-\theta_j\right)\,. \tag{128}$$

The transfer matrix eigenvalue $\tilde{t}(\theta)$ satisfies the so-called TQ relation

$$\tilde{t}(\theta)Q(\theta)=Q(\theta+i\pi)T_0\left(\theta-\frac{i\pi}{2}\right)+Q\left(\theta-i\pi\right)T_0\left(\theta+\frac{i\pi}{2}\right)\,. \tag{129}$$

As the transfer matrix is regular at v_β, which is a zero of $Q(\theta)$, the r.h.s. of the equation has to vanish, too:

$$\frac{T_0\left(v_\alpha - \frac{i\pi}{2}\right) Q\left(v_\alpha + i\pi\right)}{T_0\left(v_\alpha + \frac{i\pi}{2}\right) Q\left(v_\alpha - i\pi\right)} = -1\,. \tag{130}$$

These are the so-called Bethe Ansatz equations. The eigenvalue of the Bethe–Yang matrix is simply

$$\tilde{t}(\theta_j) = \lambda(\theta_j|\theta_1,\ldots,\theta_n) = \frac{Q\left(\theta_j + i\pi\right)}{Q(\theta_j)} T_0\left(\theta_j - \frac{i\pi}{2}\right)\,. \tag{131}$$

We should keep in mind that the BA or BY equations describe the spectrum of the quantum field theory only asymptotically, when the size is larger then the size m^{-1} of the smallest particle. In smaller volume, vacuum polarization effects are no longer negligible.

Exercises

1. *Show that the two-particle scattering matrix can be reconstructed from the volume dependence of the energy of a two-particle state!*
2. *Derive the commutation relation for the matrix elements of the monodromy matrix!*
3. *Derive the Bethe Ansatz equations!*

Literature

The polynomial finite size effects are considered in [8], while the exponentially small ones in [9] and [10]. The Thermodynamic Bethe Ansatz [11] sums up these corrections. On the algebraic Bethe Ansatz, see the lecture notes [12].

6. The bootstrap program for AdS/CFT

The AdS/CFT correspondence conjectures an equivalence between string theories on anti-de Sitter spaces and conformal field theories on the boundaries of these spaces [13]. In particular, the IIB string theory on the $\mathrm{AdS}_5\times \mathrm{S}^5$ background is equivalent to the maximally supersymmetric 4D gauge theory [14]. This string theory was shown to be integrable classically [15]. The light-cone gauge fixing breaks the $\mathrm{psu}(2,2|4)$ symmetry of the problem to the centrally extended $\mathrm{su_c}(2|2)^2$ algebra and gives masses to the excitations with a non-relativistic dispersion relation

$$\epsilon(p) = \sqrt{1 + 16g^2 \sin^2\frac{p}{2}}\,, \tag{132}$$

where g is the dimensionless combination of the string tension and the AdS radius. The 8 bosonic and 8 fermionic massive excitations form the tensor product of two fundamental representations of $\mathrm{su_c}(2|2)$ implying the factorization of the scattering matrix [16]

$$S(p_1,p_2) = S_0(p_1,p_2)\,\mathbb{S}(p_1,p_2)\otimes\mathbb{S}(p_1,p_2)\,. \tag{133}$$

The factor $\mathbb{S}$ is a 16-by-16 matrix which can be fixed from invariance under the charges Q of the $\mathrm{su_c}(2|2)$ algebra

$$\varDelta(Q)\mathbb{S}(p_1,p_2) = \mathbb{S}(p_1,p_2)\varDelta(Q)\,. \tag{134}$$

There are analogous relations to unitarity and crossing symmetry relations [17] which determine $S_0(p_1,p_2)$ [18]. The resulting scattering matrix has poles related to bound states [19]. These bound states form short $4n$-dimensional atypical representations of the symmetry algebra with dispersion relation $\epsilon(p)=\sqrt{n^2+16g^2\sin^2\frac{p}{2}}$. Their scattering matrix can be fixed either from fusion [20] or from invariance under the symmetry of the problem [21]. The full spectrum of excitations together with their dispersion relations and scatterings characterize the theory in infinite volume. These data can be then used to determine the spectrum in finite volume, the problem we are eventually interested in. The diagonalization of the transfer matrix is equivalent to the determination of spectrum of the inhomogenous Hubbard model and the large volume spectrum is determined by the Bethe–Yang equations [22]. The leading exponential corrections are related to virtual particles [10] and they are summed up by the TBA equations [23–25] (for a review, see [26]). These equations recently got an elegant reformulation in terms of the quantum spectral curve [27].

The author thanks the organizers for the kind invitation and hospitality in Zakopane, and Tamas Gombor and Marton Lajer for reading the manuscript and providing useful feedback.

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