

LECTURES ON HIGHER SPIN BLACK HOLES IN AdS_3 GRAVITY*

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(Received October 13, 2016)

Three-dimensional gravity is known as a powerful arena to explore non-perturbative aspects of quantum gravity. A particularly useful corner has been the Chern–Simons formulation of higher spin gravity in AdS_3 : this setup allows us to explore and test the lore behind black-hole mechanics in a theory that lacks a metric description. In this lectures, we will review recent developments in this subject; in particular, we will emphasize on various definitions of black holes in AdS_3 and how to probe them using the observables that naturally arise in the Chern–Simons theory.

DOI:10.5506/APhysPolB.47.2479

1. Introduction

General relativity teaches us a powerful lesson: *Gravity is Geometry*. Higher spin theories stand as an excellent candidate to test our geometrical intuition and explore repercussions of violent modifications of general relativity. A higher spin theory is characterized by being somewhat crowded by symmetries: the theory introduces massless higher spin fields whose gauge symmetries and interactions spoil the standard notions of causality and curvature that we hold sacred otherwise. Very natural concepts in general relativity, such as black holes, become rather puzzling in higher spin gravity. The aim of these lectures is to explore potential definitions of black holes in gravitational theories that lack such a geometrical description.

An important appeal of higher spin gravity is that it allows us to introduce non-linear and non-geometrical features classically. These are features we expect to arise in quantum gravity, but are generically difficult to quantify. Within higher spin gravity, there is a rather powerful example: in three

* Presented at the LVI Cracow School of Theoretical Physics “A Panorama of Holography”, Zakopane, Poland, May 24–June 1, 2016.

dimensions, the massless higher spin sector can be consistently described using the Chern–Simons theory. Depending on the gauge group we assign to the theory, we will have a different spectrum of higher spin fields. For example, these include pure AdS_3 gravity [1, 2], gravity coupled to Abelian gauge fields, and a tower of massless spin- s fields coupled to a gravity, among many other examples. This can be viewed as truncations of the interacting Prokushkin–Vasiliev higher spin theory [3, 4] which includes in addition massive scalar fields.

The Chern–Simons sector captures the chiral algebra of the dual two-dimensional CFTs which have an extended symmetry algebras of $\mathcal{W}$ -type [5–8]. In this context, holography has provided a useful framework to organize our understanding of higher spin gravity. It is rather clear how to define, *e.g.*, correlation functions, currents and sources. What this description lacks is the addition of light primary fields that are generic in CFTs. Nevertheless, the Chern–Simons sector will suffice to probe how deviations from general relativity can affect our understanding of gravity and, in particular, the mechanics behind black holes. For recent developments on holographic examples of $\text{AdS}_3/\text{CFT}_2$ involving higher spin fields, we refer the reader to [9, 10] and references within.

As we mentioned before, our main goal here is to be capable of describing *black holes* in higher spin gravity. In the absence of a metric, which is crucial for giving a notion of causality, it is rather non-trivial to think of a definition of black hole at a non-linear and non-local level. Is it the horizon its defining property? Is it high mass density? Is it the thermodynamic nature the defining feature? Or the fact that it is the fastest scrambler? Or something else, such as the ringdown pattern? All of these are, in my opinion, valid starting points. The remarkable property of general relativity is that these facts are generically implied by each other. But as we modify violently the interactions, it is not clear if we should expect that this different properties are still so intimately tied: perhaps one could find important deviations from the usual lore in a two derivative theory of gravity.

In the following, I will present one general strategy our community has taken to define a black hole in higher spin gravity. This strategy was initially put forward in the original proposal of [11] and further refined in later work, as we will elaborate in the following sections. I leave it as a challenge to the reader to further explore and question this starting point: any deviations from the lore of GR would be very interesting! And as we will see, some of these deviations are already present given the modest starting point we take.

These lectures will cover only three topics in this field: Euclidean black holes (Section 3), extremal black holes (Section 4), and Wilson lines in higher spin gravity (Section 5). There are also many other topics in higher spin

gravity that I will not explore. I will provide references as appropriate, but unfortunately the field is rather large so many interesting corners will be left out of this lectures.

2. AdS₃ higher spin gravity

The simplest way to craft a higher spin theory exploits the Chern–Simons formulation of three-dimensional gravity: general relativity with a negative cosmological constant can be reformulated as a $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ Chern–Simons theory [1, 2, 12]. A high spin theory can be crafted by simply taking instead $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$, which will produce an interacting higher spin theory for symmetric tensors of spin $s = 2, 3, \dots, N$ [13]. There are, of course, other ways to build higher spin theories, but here we restrict the attention to these models. For a more complete discussion on properties of these theories, see for example [9, 14, 15].

The action of the $\mathrm{SL}(N, \mathbb{R}) \times \mathrm{SL}(N, \mathbb{R})$ Chern–Simons theory is given by

$$S = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] , \quad S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) . \quad (1)$$

Here, $\mathcal{M}$ is the 3-manifold that supports the $\mathfrak{sl}(N, \mathbb{R})$ algebra valued connections A and $\bar{A}$, and the trace ‘Tr’ denotes the invariant quadratic form of the Lie algebra. The equations of motion following from (1) are

$$dA + A \wedge A = 0, \quad d\bar{A} + \bar{A} \wedge \bar{A} = 0. \quad (2)$$

The conventions here follow those in [16].

The metric and higher spin fields are obtained from the Chern–Simons connection as symmetric, traceless tensors that transform in the spin- s representation of $\mathrm{SL}(2, \mathbb{R})$. For example, the metric and the spin three field can be expressed as follows:

$$g_{\mu\nu} \sim \mathrm{Tr}(e_\mu e_\nu) , \quad \phi_{\mu\nu\rho} \sim \mathrm{Tr}(e_{(\mu} e_\nu e_{\rho)}) , \quad (3)$$

where, in line with the pure gravity case, one defines

$$e = \frac{\ell}{2} (A - \bar{A}) , \quad \omega = \frac{1}{2} (A + \bar{A}) \quad (4)$$

and we introduced the AdS radius ℓ . The metric and higher spin fields can then be expressed in terms of trace invariants of the vielbein [6, 7], with the total number of inequivalent invariants being $N - 1$ for $\mathfrak{sl}(N, \mathbb{R})$. This definition for metric-like fields is appropriate for the principal embedding of $\mathfrak{sl}(2, \mathbb{R})$ in $\mathfrak{sl}(N, \mathbb{R})$.

The relation between the Chern–Simons level and the gravitational couplings is

$$k = \frac{\ell}{8G_3\epsilon_N}, \quad \epsilon_N \equiv \text{Tr}_f(L_0 L_0) = \frac{1}{12}N(N^2 - 1), \quad (5)$$

in accordance with the pure gravity limit. The notation Tr_f denotes a trace in the fundamental representation of $\mathfrak{sl}(N, \mathbb{R})$. The central charge of the asymptotic symmetry group is [5, 6]

$$c = 12k\epsilon_N = \frac{3\ell}{2G}. \quad (6)$$

The standard way to parametrize solutions to (2) is by gauging away the radial dependence, *i.e.*

$$A = b(r)^{-1} (a(x^+, x^-) + d) b(r), \quad \bar{A} = b(r) (\bar{a}(x^+, x^-) + d) b(r)^{-1}. \quad (7)$$

Here, r is the holographic radial direction, and $x^\pm = t \pm \phi$ are the boundary coordinates. In Lorentzian signature, we will consider solutions with $\mathbb{R} \times D_2$ topology; the compact direction on D_2 is described by $\phi \sim \phi + 2\pi$. In Euclidean signature, we will analytically continue $x^\pm$ to complex coordinates $(z, \bar{z})$, and the topology of the bulk is now a solid torus with $z \sim z + 2\pi \sim z + 2\pi i\tau$. Here, τ is the modular parameter of the boundary torus. $b(r)$ is a radial function that is normally taken to be e^{rL_0} ¹.

The connections $a(x^+, x^-)$ and $\bar{a}(x^+, x^-)$ contain the information that characterizes the state in the dual CFT. In the absence of sources, there is systematic procedure to label them: a suitable set of boundary conditions on the connections results in $\mathcal{W}$ -algebras as asymptotic symmetries [5–8, 19]. These are commonly known as Drinfeld–Sokolov boundary conditions. To be concrete, for $\mathfrak{sl}(N) \times \mathfrak{sl}(N)$ the connections take the form of

$$a_z = L_1 - \sum_{s=2}^N J_{(s)}(z) W_{-s+1}^{(s)}, \quad \bar{a}_{\bar{z}} = - \left(L_{-1} - \sum_{s=2}^N \bar{J}_{(s)}(\bar{z}) W_{s-1}^{(s)} \right), \quad (8)$$

while $a_{\bar{z}} = \bar{a}_z = 0$. Here, $\{L_0, L_{\pm 1}\}$ are the generators of the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra in $\mathfrak{sl}(N)$, and $W_j^{(s)}$ are the spin- s generators with $j = -(s-1), \dots, (s-1)$; note that $W_j^{(2)} = L_j$. And $J_{(s)}(z)$ are dimension- s currents whose algebra is $\mathcal{W}_N$, and the same for the barred sector.

¹ How to choose $b(r)$ is very important when considering Lorentzian properties of the solutions and it is usually overlooked. See [17, 18] for a recent discussion on this topic.

Our general arguments and results will not be very sensitive to the choice of gauge group, but for the sake of simplicity, our explicit computations will involve connections valued in either the Lie algebra $\mathfrak{sl}(2)$ (standard spin-2 gravity on AdS_3) or $\mathfrak{sl}(3)$ (a graviton coupled to a single spin-3 field).

3. Euclidean black holes

We will start the discussion with the most successful (and elegant) definition: Euclidean black holes in AdS_3 . We will review our current understanding of the solutions and its properties in the Chern–Simons formulation of higher spin gravity. This section is a collection of results in [11, 15, 20–23].

Any definition of black holes should include at least two inputs:

1. A quantitative definition of physical observables; in particular, a definition of conserved charges (such as mass and angular momentum) and its counterparts potentials (such as temperature and angular velocity).
2. A notion of regularity and smoothness. The aim here would be to find a notion of horizon. But even more broadly, we need to clearly argue if a solution, at least in Euclidean signature, lacks singularities.

Let us elaborate first on how to obtain conserved charges. The properties and values of these observables are intimately tied to the boundary conditions we use. For instance, in AdS spacetimes, we are mostly accustomed to Dirichlet boundary conditions and to implement a notion of asymptotically AdS spaces (AAdS). But let me emphasize: there is more than one choice! This occurs even in AdS_3 gravity, where some non-trivial examples are shown in [24, 25] and more recently, a broad analysis was presented in [26] which are important deviations from the standard Brown–Henneaux boundary conditions [27].

In higher spin gravity, we of course have similar choices, but in addition, there are further complexities as we turn on sources. More concretely, consider the Chern–Simons connections in (7) and (9), and that we impose AAdS boundary conditions. From the CFT perspective, it is natural to capture the currents in a_z and the sources in $a_{\bar{z}}$, and *vice versa* for $\bar{a}$ [11]. From the gravitational perspective, the canonical prescription is to encode in $(a_\phi, \bar{a}_\phi)$ the currents [28–31]. These two choices, a_z *versus* a_ϕ , amount for different partition functions as shown in [23]: the a_z prescription, denoted *holomorphic* black hole, corresponds to a Lagrangian deformation of the theory; the a_ϕ prescription, denoted *canonical* black hole, corresponds to a Hamiltonian deformation. It is important to make a distinction between these two, since the Legendre transformation that connects these two prescriptions is non-trivial.

In the remainder of these lectures, we will mostly use the canonical description. Moreover, we are interested in stationary black hole solutions, hence $(a, \bar{a})$ are constant flat connections that contain both charges and sources. This, in particular, implies that the ϕ -component will be always written as

$$a_\phi = L_1 - \sum_{s=2}^N Q_{(s)} W_{-s+1}^{(s)}, \quad \bar{a}_\phi = L_{-1} - \sum_{s=2}^N \bar{Q}_{(s)} W_{s-1}^{(s)}, \quad (9)$$

where $(Q_{(s)}, \bar{Q}_{(s)})$ are constants (not functions) and they represent the conserved charges associated to the zero modes of each higher spin current $(J_{(s)}, \bar{J}_{(s)})$. The a_t component will contain the information about the potentials, which we will usually denote as μ_s ². Hence, a solution that contains both charges and potentials will be interpreted in the CFT as being part of a canonical ensemble, where

$$Z_{\text{can}}[\tau, \alpha_s, \bar{\alpha}_s] = \text{Tr}_{\mathcal{H}} \exp 2\pi i \left[\sum_{s=2}^N \left(\alpha_s J_0^{(s)} - \bar{\alpha}_s \bar{J}_0^{(s)} \right) \right]. \quad (10)$$

Here, $J_0^{(s)}$ and $\bar{J}_0^{(s)}$ denote the zero modes of the corresponding currents; $Q_{(s)}$ and $\bar{Q}_{(s)}$ would be the eigenvalues of these operators. For $s > 3$, we have

$$\mu_s = \frac{i\alpha_s}{\text{Im}(\tau)}, \quad \bar{\mu}_s = -\frac{i\bar{\alpha}_s}{\text{Im}(\tau)}, \quad (11)$$

which are the chemical potential associated to each operator; recall that τ is the complex structure of the torus. For $s = 2$, we have

$$J_0^{(2)} = L_0 - \frac{c}{24}, \quad \bar{J}_0^{(2)} = \bar{L}_0 - \frac{c}{24}, \quad (12)$$

and the CFT Hamiltonian and angular momentum are $H = L_0 + \bar{L}_0 - \frac{c}{12}$ and $J = L_0 - \bar{L}_0$, respectively. For the potentials, the relation with the complex structure of the torus is

$$\alpha_2 = \tau = \frac{i\beta}{2\pi} (1 + \Omega), \quad \bar{\alpha}_2 = \bar{\tau} = \frac{i\beta}{2\pi} (-1 + \Omega), \quad (13)$$

with β the inverse temperature and Ω the angular velocity.

The feature that distinguishes black holes from other solutions is a smoothness condition, and this brings us to the second bullet point mentioned above. In a metric formulation of gravity, the Euclidean section of

² For a quantitative and general definition of μ_s in terms of (a_t, a_ϕ) see, for example, [23]. Here, we will just define them via examples.

a black hole has the property that the compact Euclidean time direction smoothly shrinks to zero size at the horizon of the black hole, resulting in a smooth cigar-like geometry as in figure 1. In the Chern–Simons formulation of gravity, this property is normally thought to generalize to the idea that a black hole is a flat gauge connection defined on a solid torus, where the holonomy along the thermal cycle of the torus belongs to the centre of the group, *i.e.*

$$\mathcal{P} \exp \left(\oint_{\mathcal{C}_E} a \right) \cong e^{2\pi(\tau a_z + \bar{\tau} a_{\bar{z}})} \cong e^{2\pi i L_0}, \quad (14)$$

and similarly in the barred sector; here L_0 denotes the Cartan element of $\mathfrak{sl}(2)$ ³, and $\mathcal{C}_E$ is the thermal cycle $z \sim z + 2\pi i\tau$ which is contractible in the bulk.

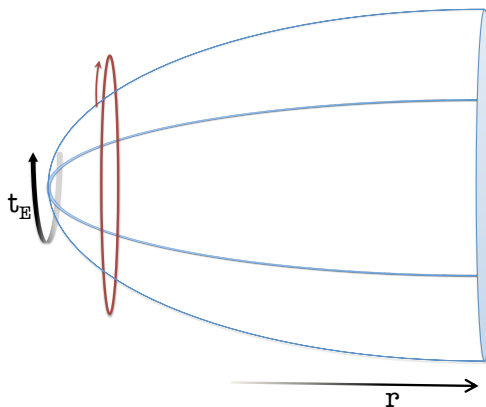


Fig. 1. (Colour on-line) Topology of the Euclidean higher spin black hole for a static solution, where the compact direction is Euclidean time $t = it_E$. The black (red) curve depicts the cycle along which the smoothness condition (14) is imposed, and it is independent of the radial position. In Euclidean signature, the geometry ends at a finite value of r : in a metric-like formulation of gravity, this end point would be the horizon.

Smoothness condition (14) is a robust and successful definition of Euclidean black holes. It reproduces in an elegant manner many properties that we expect from a thermal state in the dual CFT_2 . This definition has also unveiled novel properties of systems in the grand canonical ensemble of

³ Depending on the gauge group, the choice of centre in the r.h.s. of (14) is not unique [32]. The choice used here has the feature that it is smoothly connected to the BTZ solution. The interpretations of other choices are discussed in [33, 34].

$\mathcal{W}_N$, such as microscopic features of the entropy [30, 35, 36], ensemble properties [21, 23] and novel phase diagrams [37], and it inspires new observables related to entanglement entropy [33, 38, 39].

It is perhaps worth emphasizing that there exist several ways to compute the entropy of higher spin black holes, all giving the same result. In the original proposal of [11], the entropy was inferred by demanding integrability of the thermodynamic laws. For a Hamiltonian derivation of the entropy, see *e.g.* [22, 28, 40]. The entropy can also be understood as the on-shell value of the appropriate action functional in a microcanonical ensemble, where the charges at infinity are held fixed [21]. The punchline is that the entropy of a higher spin black hole reads

$$S = -2\pi i k \operatorname{Tr} [(a_z + a_{\bar{z}})(\tau a_z + \bar{\tau} a_{\bar{z}}) - (\bar{a}_z + \bar{a}_{\bar{z}})(\tau \bar{a}_z + \bar{\tau} \bar{a}_{\bar{z}})] . \quad (15)$$

More interestingly, one can exploit the holonomy conditions to cast the entropy directly as a function of the charges only. Using the smoothness conditions (14), one finds that (15) can be written equivalently as [21]

$$S = 2\pi k \operatorname{Tr} [(\lambda_\phi - \bar{\lambda}_\phi) L_0] , \quad (16)$$

where λ_ϕ and $\bar{\lambda}_\phi$ are diagonal matrices containing the eigenvalues of the angular component of the connection (9), which carries the values of the charges $(Q_{(s)}, \bar{Q}_{(s)})$.

3.1. Example

To illustrate the discussion in this section, we will consider black holes in the $\mathrm{SL}(3) \times \mathrm{SL}(3)$ Chern–Simons theory. In this case, we define⁴:

$$\begin{aligned} a_+ &= L_1 - Q_{(2)} L_{-1} - \frac{Q_{(3)}}{4} W_{-2} , \\ a_- &= \mu_3 \left(W_2 + 2Q_{(3)} L_{-1} + Q_{(2)}^2 W_{-2} - 2Q_{(2)} W_0 \right) , \\ \bar{a}_- &= - \left(L_{-1} - Q_{(2)} L_1 + \frac{Q_{(3)}}{4} W_2 \right) , \\ \bar{a}_+ &= \mu_3 \left(W_{-2} - 2Q_{(3)} L_1 + Q_{(2)}^2 W_2 - 2Q_{(2)} W_0 \right) . \end{aligned} \quad (17)$$

For simplicity, we have turned off rotation, *i.e.* $Q_{(2)} = \bar{Q}_{(2)}$ and $Q_{(3)} = -\bar{Q}_{(3)}$; this as well implies that τ is purely imaginary ($\tau = i\beta$) and $\bar{\mu} = -\mu$. The interpretation of these connections as thermal states depends on the

⁴ Note that the equations of motion, flatness condition, simply imposes that $[a_+, a_-] = 0 = [\bar{a}_-, \bar{a}_+]$ as can be checked explicit for (17).

boundary conditions used to define the classical phase space. The holomorphic black hole is given by the following connections:

$$a_h = a_+ dz + a_- d\bar{z}, \quad \bar{a}_h = \bar{a}_+ dz + \bar{a}_- d\bar{z}. \quad (18)$$

In this notation, the components $(a_z, \bar{a}_{\bar{z}})$ contain the information of the charges of the system: $(Q_{(2)}, Q_{(3)})$ are the zero modes of the stress tensor and dimension-3 current of the $\mathcal{W}_3$ asymptotic symmetry group that organizes the states in this theory. (β, μ) are their respective sources which are fixed by the smoothness condition (14). The second prescription, *i.e.* the canonical black hole, is given by

$$a_c = a_+ d\phi + (a_+ + a_-) dt, \quad \bar{a}_c = -\bar{a}_- d\phi + (\bar{a}_+ + \bar{a}_-) dt. \quad (19)$$

For this prescription, again, $(Q_{(2)}, Q_{(3)})$ are the zero modes of the currents in $\mathcal{W}_3$. The quantitative difference between the holomorphic and canonical definitions lies in the spatial components of the connection; both a_c and a_h have the same time component.

Smoothness condition (14) enforces relations between the parameters $Q_{(2)}$, $Q_{(3)}$, μ_3 , and β . Following [11, 20], these constraints can be solved in terms of dimensionless parameter $C \geq 3$:

$$\begin{aligned} Q_{(3)} &= \frac{4(C-1)Q_{(2)}}{C^{3/2}} \sqrt{Q_{(2)}}, \\ \mu_3 &= \frac{3\sqrt{C}}{4(2C-3)} \sqrt{\frac{1}{Q_{(2)}}}, \\ \frac{\mu_3}{\beta} &= \frac{3}{4\pi} \frac{(C-3)\sqrt{4C-3}}{(3-2C)^2}. \end{aligned} \quad (20)$$

The limit $C \rightarrow \infty$ makes the higher spin charges vanish, and we recover the BTZ case. $C = 3$ and μ_3 fixed corresponds to a zero temperature solution which defines an extremal higher spin black hole [11, 41] which is the subject of the next section.

Applying (15) to the canonical black hole (19), we get

$$S = 8k (2\beta Q_{(2)} + 3\alpha_3 Q_{(3)}), \quad (21)$$

where the thermal spin-3 source α_3 is related to the spin-3 chemical potential μ_3 as in (11). This expression is clearly compatible with a first law of thermodynamics. It is simple to generalize this expression to restore the barred variables; this gives

$$S = -8\pi i k (2\tau Q_{(2)} + 3\alpha_3 Q_{(3)}) + 8\pi i k (2\bar{\tau} \bar{Q}_{(2)} + 3\bar{\alpha}_3 \bar{Q}_{(3)}). \quad (22)$$

The entropy as a function of the charges can be achieved via (16), and for this purpose, it is convenient to trade the charges $(Q_{(2)}, Q_{(3)})$ for the eigenvalues of a_ϕ . More concretely, we parametrize

$$\text{Eigen}(a_\phi) = (\lambda_1, \lambda_2, -\lambda_1 - \lambda_2), \quad (23)$$

so that

$$Q_{(2)} = \frac{1}{4} (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2), \quad Q_{(3)} = \frac{1}{2} \lambda_1 \lambda_2 (\lambda_1 + \lambda_2), \quad (24)$$

with analogous expressions in the barred sector. In Lorentzian signature, the eigenvalues $(\lambda_i, \bar{\lambda}_i)$ are independent, and real when one chooses the connection to be valued in $\mathfrak{sl}(3; \mathbb{R})$. In Euclidean signature, we have $\lambda_i^* = -\bar{\lambda}_i$, which implies that $Q_{(2)}^* = \bar{Q}_{(2)}$ and $Q_{(3)}^* = -\bar{Q}_{(3)}$. Equation (16) then gives us immediately the entropy as a function of the charges

$$\begin{aligned} S &= 2\pi k (\lambda_1 - \lambda_3) + \text{other sector} \\ &= 2\pi k (2\lambda_1 + \lambda_2) + \text{other sector}, \end{aligned} \quad (25)$$

with λ_1 and λ_2 obtained by inverting (24) and choosing the branch of the solution that connects smoothly to the BTZ black hole as one turns off the $Q_{(3)}$ charge.

4. Extremal black holes

In general relativity, there is a wide variety of black holes which are not necessarily Euclidean. For example, there are Lorentzian black holes that do not have a real Euclidean continuation (such as five dimensional black rings), there are eternal black holes (which are the maximal extension of the Euclidean geometry in Lorentzian signature), and black holes that arise from gravitational collapse. And there are as well *extremal black holes*. Extremal black holes play undoubtedly a crucial role in string theory: due to their enhanced symmetries and their capacity to preserve supersymmetry, they have become a landpost for microstate counting and precursors to many aspects of holography. As such, it is very natural to wonder what is the definition of extremality in higher spin gravity. This is the question we will address in this section. The discussion here is a summary of the results presented in [41]. See also [42] for a discussion on related properties.

4.1. A practical definition of extremality

In conventional gravitational theories, the notion of extremality is tied to the confluence of two horizons. This feature generically implies that the Hawking temperature of the black hole is zero. We could declare that

extremality in higher spin theories is simply defined as a solution at zero temperature. However, our aim is to propose a definition that is along the lines of confluence (degeneration) of the parameters of the solution and that relies only on the topological formulation of the theory, yielding in particular the zero-temperature condition as a consequence.

In this spirit, in [41] we proposed that a 3d extremal higher spin black hole is a solution of the Chern–Simons theory corresponding to flat boundary connections a and $\bar{a}$ satisfying the following conditions:

1. They obey AAdS boundary conditions⁵;
2. Their components are constant, and therefore correspond to stationary solutions;
3. They carry charges and chemical potentials, which are manifestly real in the Lorentzian section;
4. The angular component of at least one of a and $\bar{a}$, say a_ϕ , is non-diagonalizable.

Naturally, the key point of the definition is the non-diagonalizability of the a_ϕ component. The rationale behind this requirement is as follows. Suppose both the a_ϕ and $\bar{a}_\phi$ components were diagonalizable. Since the boundary connections are assumed to be constant, by the equations of motion the (Euclidean) time components of the connection commute with the angular components, and can be diagonalized simultaneously with them. It is then possible to solve (14) and find a non-zero and well-defined temperature and chemical potentials as a function of the charges. On the other hand, if at least one of a_ϕ and $\bar{a}_\phi$ is non-diagonalizable, then a_{contract} will be non-diagonalizable as well. If we insist upon (14), then both features are compatible if we take a zero temperature limit, because the smoothness condition becomes degenerate as well. This is consistent with the usual notion that the solid torus topology of the finite-temperature black hole should change at extremality.

The role of boundary conditions is crucial for our definition. For a general connection, the degeneration of eigenvalues does not imply non-diagonalizability. However, the special form of the flat connections dictated by the AAdS boundary conditions will guarantee that if two eigenvalues of a_ϕ are degenerate, then the connection is non-diagonalizable. From this perspective, we could interpret that equating eigenvalues of a_ϕ is in a sense analogous to the confluence of horizons for extremal black holes in general relativity.

⁵ In the literature, these boundary conditions are commonly known as Drinfeld–Sokolov boundary conditions.

4.2. Example: extremal $sl(3)$ black holes

In [41], several supersymmetric and non-supersymmetric cases were studied. For brevity, here we will only look at the extremal cousin of the Euclidean black hole we considered in Subsection 3.1.

Let us write again (17) but focus on the unbarred sector for concreteness; recall that it is sufficient to impose our definition of extremality on one sector to obtain the desired features. Using canonical boundary conditions, the connections are given by

$$a_\phi = L_1 - Q_{(2)}L_{-1} - \frac{Q_{(3)}}{4}W_{-2}, \quad (26)$$

$$ia_{t_E} + a_\phi = 2a_- = 2\mu_3 \left(W_2 + 2Q_{(3)}L_{-1} + Q_{(2)}^2 W_{-2} - 2Q_{(2)}W_0 \right). \quad (27)$$

It is also instructive to re-write the solutions to (14) for the general rotating case, *i.e.* the generalization of (20). This gives

$$\tau = i \frac{2\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2}{(\lambda_1 - \lambda_2)(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)}, \quad (28)$$

$$\alpha_3 = -6i \frac{\lambda_2}{(\lambda_1 - \lambda_2)(2\lambda_1 + \lambda_2)(\lambda_1 + 2\lambda_2)}, \quad (29)$$

and

$$\mu_3 = 6(1 + \Omega) \left(\frac{\lambda_2}{2\lambda_1^2 + 2\lambda_1\lambda_2 - \lambda_2^2} \right), \quad (30)$$

$$\bar{\mu}_3 = -6(1 - \Omega) \left(\frac{\bar{\lambda}_2}{2\bar{\lambda}_1^2 + 2\bar{\lambda}_1\bar{\lambda}_2 - \bar{\lambda}_2^2} \right). \quad (31)$$

In the above expressions, we traded $Q_{(2)}$ and $Q_{(3)}$ by its eigenvalues λ_1 and λ_2 as defined in (23). With these explicit relations, we can now implement our definition of extremality. Requiring that a_ϕ should be non-diagonalizable gives as a necessary condition

$$\lambda_1 = \lambda_2 \equiv \lambda \quad \Rightarrow \quad Q_{(2)} = \frac{3}{4}\lambda^2, \quad Q_{(3)} = \lambda^3. \quad (32)$$

As a consequence, while the finite-temperature angular holonomy is diagonalizable, in the extremal limit, we obtain

$$\text{Hol}_\phi(a) \sim \begin{pmatrix} e^{-4\pi\lambda} & 0 & 0 \\ 0 & e^{2\pi\lambda} & 1 \\ 0 & 0 & e^{2\pi\lambda} \end{pmatrix}. \quad (33)$$

Turning now our attention to the potentials from (29)–(31), we see in particular that in this limit

$$\text{extremal potentials : } \beta \rightarrow \infty, \quad \mu \rightarrow 4\frac{\gamma}{\lambda}, \quad \Omega \rightarrow 1, \quad \bar{\mu} \rightarrow 0, \quad (34)$$

so the temperature is zero as expected. The spin-3 chemical potential μ remains finite and becomes a simple homogeneous function of the charges, whereas the corresponding thermal source α scales with the inverse temperature and blows up. On the other hand, the barred sector spin-3 potential $\bar{\mu}$ goes to zero because the thermal source $\bar{\alpha}$ remains unconstrained and in particular finite, as no condition is imposed on the barred charges.

Several comments are now in order.

1. *Jordan decomposition versus zero temperature:* A valid concern is to wonder if our definition of extremality implies zero temperature and *vice versa*. From (29), it is clear that there are 3 combinations of λ_1 and λ_2 that achieve $\beta \rightarrow \infty$. The additional other branches also give non-trivial Jordan forms, since they just correspond to different pairings of eigenvalues that are degenerate. For this reason, all these cases are captured by (32): any pairing $\lambda_i = \lambda_j$ with $i \neq j$ implies the extremality bound $Q_{(2)}^3 = 27/64 Q_{(3)}^2$ ⁶. At least for $N = 2, 3$, a non-trivial Jordan decomposition implies zero temperature and *vice versa*. From the heuristic argument in Section 4.1, we expect this to always be the case.
2. *Other Jordan classes:* For $\lambda \equiv \lambda_1 = \lambda_2 \neq 0$, a_ϕ has only 2 linearly independently eigenvectors. If take first $\lambda_2 = 0$ and then $\lambda_1 = 0$, the holonomy of a_ϕ belongs to a different Jordan class where there is only one eigenvector; this case corresponds to extremal BTZ within the $\mathfrak{sl}(3) \oplus \mathfrak{sl}(3)$ Chern–Simons theory.
3. *Finite entropy:* We have a continuous family of extremal $\mathcal{W}_3$ black holes parametrized by λ , and from (25) the contribution of the extremal (unbarred) sector to the total entropy is

$$S_{\text{ext}} = 2\pi k_{\text{CS}} \lambda = \frac{\pi k}{3} \sqrt{48 Q_{(2)}} = 2\pi k (Q_{(3)})^{1/3}. \quad (35)$$

The answer is clearly finite. This should be contrasted with extremal BTZ, where the contribution of the extremal sector vanishes. It would be interesting to derive such bound and residual entropy in a CFT with $\mathcal{W}_3$ symmetry.

⁶ Different pairings of eigenvalues conflict with the ordering of eigenvalues used in (25), but this is easily fixed by reordering the eigenvalues appropriately.

4. *Extremality vs. unitarity*: The extremality condition we have discussed can be thought of as a bound

$$Q_{(2)}^3 \geq \frac{27}{64} Q_{(3)}^2 \quad (36)$$

on the charges of a spin-3 black hole. On the other hand, in a theory with $\mathcal{W}_3$ symmetry, the unitary bound in the semiclassical limit is [43]⁷

$$\frac{64}{5c} \left(h^3 - \frac{c}{32} h^2 \right) \geq 9q_3^2, \quad (37)$$

where the map between the CFT variables (h, q_3) and the gravitational charges is

$$h - \frac{c}{24} = 4kQ_{(2)}, \quad q_3 = kQ_{(3)}. \quad (38)$$

It is clear that (36) and (37) do not agree. However, the $\mathcal{W}_3$ unitarity bound (37) encloses the bulk extremality bound (36), indicating that all $\text{sl}(3)$ black holes are dual to states allowed by unitarity in the dual CFT.

5. *Conformal invariance*: In two-derivative theories of gravity in $D = 4, 5$, all extremal black holes contain an AdS_2 factor in its near horizon geometry [44, 45]. The enhancement of time translations to conformal transformations is non-trivial and unexpected *a priori*; moreover, it is a key to build microscopic models of extremal black holes. Here, we have not investigated this feature explicitly, but we do expect that the connection at the extremal point is invariant a larger set of gauge transformations relative to the non-extremal connection. Some evidence was reported in [18].
6. *Entropy bounds*: The extremal limit of the spin-3 higher spin black hole was first discussed in [11]. Their bound was found as the maximal value of $Q_{(3)}$ for a given $Q_{(2)}$ such that the entropy is real, and it agrees with (32). Using the reality of entropy as a bound which enhanced symmetries of the solution was also used in [42]. It is not clear if this approach is always compatible with ours, and it will be interesting to explore potential discrepancies.

⁷ The quantum (finite- c) unitarity bound reported in [43] is

$$\frac{64}{22+5c} h^2 \left(h - \frac{1}{16} - \frac{c}{32} \right) - 9q_3^2 \geq 0.$$

7. *Supersymmetry and extremality*: As we mentioned above, extremality can be understood as the saturation of certain inequalities involving conserved charges, and it is natural to contrast these inequalities with BPS bounds that appear in supersymmetric setups. It is well-known that in two-derivative theories of supergravity, these two types of conditions are intimately related: supersymmetry always implies zero temperature and, therefore, extremality in the context of BPS black holes. In supersymmetric theories of higher spin gravity this seems not to be true! In [41], we showed that there exist non-extremal solutions in the class of diagonalizable connections that possess 4 independent Killing spinors. This is, within the $sl(3|2)$ theory, we managed to construct a smooth higher spin black hole that is both at finite temperature and BPS. Understanding why higher spin theories allow for this peculiar behaviour is an open question that needs urgent attention.

5. Wilson lines

As we have mentioned throughout, higher spin gravity does not admit a conventional geometric understanding. However, they do admit interesting higher-spin-invariant probes. In this section, we will consider the Wilson line operator constructed in [38, 39]. As we review below, this object should be thought of as the higher-spin-invariant generalization of the worldline of a massive particle moving in the bulk, carrying well-defined charges under the higher-spin symmetries. In the simplest case, when it is charged only under the spin-2 field — and thus has a mass but no other charges — its action in the bulk may thus be thought of as the higher-spin analogue of a bulk proper distance.

A Wilson loop is defined as

$$W_{\mathcal{R}}(C) = \text{Tr}_{\mathcal{R}} \left(\mathcal{P} \exp \left(\int_C A \right) \mathcal{P} \exp \left(\int_C \bar{A} \right) \right). \quad (39)$$

Here, A and $\bar{A}$ are the connections representing a higher spin background in the $SL(N, \mathbb{R})$ Chern–Simons theory. The representation $\mathcal{R}$ is the infinite-dimensional highest-weight representation of $sl(N, \mathbb{R})$, and C is a loop in the bulk. We will also be interested in line operator, where the definition is given by

$$W_{\mathcal{R}}(X_i, X_f) = \langle U_i | \mathcal{P} \exp \left(\int_{\tilde{C}_{\text{if}}} A \right) \mathcal{P} \exp \left(\int_{\tilde{C}_{\text{if}}} \bar{A} \right) | U_f \rangle, \quad (40)$$

where C_{if} is a curve with bulk endpoints (X_i, X_f) . $U(y)$ is a probe field which lives on the wordline C_{if} : its quantum numbers are governed by $\mathcal{R}$ and it satisfies suitable boundary conditions at the endpoints.

The recent developments in [34, 38, 39, 46] show that a Wilson line operator is a bulk observable that computes correlation functions of light operators in the dual CFT. More concretely, as we take the endpoints to the boundary, the Wilson line gives [34]

$$W_{\mathcal{R}}(X_i, X_j) \underset{r \rightarrow \infty}{=} \langle \Psi | \mathcal{O}(x_i) \mathcal{O}(x_j) | \Psi \rangle. \quad (41)$$

Here, (x_i, x_j) are boundary positions. $\mathcal{O}(x_i)$ is an operator with scaling dimension $\Delta_{\mathcal{O}}$ that is fixed as the central charge c goes to infinity⁸.

These Wilson lines provide us with a sensitive probe of bulk higher spin geometries. Recently, it has been used to define and characterize eternal black holes in higher spin gravity [18], which is a subject I unfortunately do not have the time to cover here. In this section, I will give a brief summary of the results in [16, 39], and give a glimpse on possible future directions which I encourage the reader to explore.

5.1. Wilson line and massive particles

As anticipated, we would like the Wilson line (39) to give information about the length of a geodesic connecting the endpoints of C_{if} . A geodesic can be understood as the trajectory followed by a massive point particle. Our Wilson line should mimic the dynamics of this massive particle, and hence as a minimal requirement it should be able to carry the data of this particle.

A point particle in the classical limit is characterized by at least one continuous parameter: the mass m . This data is stored in the representation $\mathcal{R}$ that defines the Wilson line. An infinite-dimensional representation of $\mathfrak{sl}(N, \mathbb{R}) \oplus \mathfrak{sl}(N, \mathbb{R})$ will do the trick: it allows for continuous parameters which we can identify with a mass⁹. In particular, we will work with the so-called highest-weight representation. Consider the $\mathfrak{sl}(N, \mathbb{R})$ algebra in (A.2), and we define the highest-weight state of the representation as $|\text{hw}\rangle \equiv |h, w_3, \dots, w_N\rangle$ with the following properties:

$$\begin{aligned} L_0 |\text{hw}\rangle &= h |\text{hw}\rangle, & L_1 |\text{hw}\rangle &= 0, \\ W_0^{(s)} |\text{hw}\rangle &= w_s |\text{hw}\rangle, & W_j^{(s)} |\text{hw}\rangle &= 0, \quad j = 1, \dots, s-1. \end{aligned} \quad (42)$$

⁸ Or equivalently, in gravity we would say that it is a particle with a small mass in Planck units. In a rather crude way, we can identify $\mathcal{O}$ with the probe field U . In this language, the Casimir's of the representation $\mathcal{R}$ control the quantum numbers of the dual operator.

⁹ Moreover, these infinite dimensional representations can be unitary. It can be proven that all finite-dimensional representations of $\mathfrak{sl}(N, \mathbb{R})$ are non-unitary.

The constants h and w_s with $s = 3, \dots, N$ are the parameters defining the representation. $|\text{hw}\rangle$ is annihilated by the lowering operators; a descendant state is created by acting with the raising operators: $W_{-j}^{(s)}$ and L_{-1} . With this, the Wilson line in the infinite-dimensional highest-weight representation of $\mathfrak{sl}(N, \mathbb{R}) \times \mathfrak{sl}(N, \mathbb{R})$ is labelled with two towers of quantum numbers: (h, w_s) and $(\bar{h}, \bar{w}_s)$. In particular, the mass $\hat{m}$ and orbital spin $\hat{s}$ are given by

$$\ell \hat{m} = h + \bar{h}, \quad \hat{s} = \bar{h} - h. \quad (43)$$

The choice of representation is intimately tied with the interpretation of the Wilson line in terms of the dual theory. For instance, if $h = \bar{h}$ and $w_s = \bar{w}_s = 0$, we can make contact with entanglement entropy [39]. One could also design probes that carry higher spin charge or orbital spin; the interpretation of this object in the dual CFT interpretation will be different, but still rather interesting. See [33, 34] for the case when $w_3 = \bar{w}_3 \neq 0$ in $SL(3, \mathbb{R})$ higher spin theory, and see [47, 48] for a discussion when $\hat{s} \neq 0$.

5.2. Path integral representation of the Wilson line

The more complex step is to actually evaluate the trace in (39). Following [39]¹⁰, we will interpret $\mathcal{R}$ as the Hilbert space of an auxiliary quantum mechanical system that lives on the Wilson line, and replace the trace over $\mathcal{R}$ by a path integral. This auxiliary system is described by some field U , and we will pick the dynamics of U so that upon quantization, the Hilbert space of the system will be precisely the desired representation $\mathcal{R}$. More concretely,

$$W_{\mathcal{R}}(X_i, X_f) = \int \mathcal{D}U e^{-S(U, A, \bar{A})_{C_{\text{if}}}}, \quad (44)$$

where the action $S(U, A, \bar{A})_{C_{\text{if}}}$ has $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ as a local symmetry. The auxiliary system is appropriately described by the following action¹¹:

$$\begin{aligned} S(U, A, \bar{A})_{C_{\text{if}}} &= \int_{C_{\text{if}}} dy \left(\text{Tr} (PU^{-1} D_y U) + \lambda_2(y) (\text{Tr} (P^2) - c_2) + \right. \\ &\quad \left. \dots + \lambda_N(y) (\text{Tr} (P^N) - c_N) \right). \end{aligned} \quad (45)$$

Here, P is the canonical momentum conjugate to U that lives in the Lie algebra $\mathfrak{sl}(N, \mathbb{R})$. The variable y parametrizes the curve C_{if} , and we pick

¹⁰ All conventions and further details of this section follow those in [16].

¹¹ As discussed in [39], the choice of the action $S(U, A, \bar{A})_{C_{\text{if}}}$ is not unique, it is just a useful trick. There are many auxiliary systems that will recover the trace over the representation in (39), giving the same result for the Wilson line only depending on $\mathcal{R}$ and C_{if} .

$y \in [y_i, y_f]$. The trace $\text{Tr}(\dots)$ is a short-cut notation for the contraction using the Killing forms:

$$\text{Tr}(P^m) = h_{a_1 \dots a_m} P^{a_1} \dots P^{a_m}, \quad m = 2, \dots, N, \quad (46)$$

where $P = P^a T_a$ and T_a is a generator of $\mathfrak{sl}(N, \mathbb{R})$. The functions $\lambda_m(y)$ represent Lagrange multipliers which enforce constraints on P . The elements c_m are the Casimir invariants C_m applied to the highest weight state, and contain the information of the highest-weight quantum numbers h and w_s . Note that in this action, we already implemented that $h = \bar{h}$ and $w_s = \bar{w}_s$, since there is only one momenta variable P ; this will suffice for the discussion here, and certain generalization are discussed in [48, 49].

The covariant derivative is defined as

$$D_y U \equiv \frac{d}{dy} U + A_y U - U \bar{A}_y, \quad A_y \equiv A_\mu \frac{dX^\mu}{dy}, \quad \bar{A}_y \equiv \bar{A}_\mu \frac{dX^\mu}{dy}, \quad (47)$$

where A and $\bar{A}$ are the connections that determine the background. With these definitions we have achieved our first goal: the system is invariant under the local symmetries along the curve. The transformation properties of the fields are

$$\begin{aligned} A_\mu &\rightarrow L(X^\mu(y))(A_\mu + \partial_\mu)L^{-1}(X^\mu(y)), \\ \bar{A}_\mu &\rightarrow R^{-1}(X^\mu(y))(\bar{A}_\mu + \partial_\mu)R(X^\mu(y)), \end{aligned} \quad (48)$$

and

$$\begin{aligned} U(s) &\rightarrow L(X^\mu(y))U(s)R(X^\mu(y)), \\ P(y) &\rightarrow R^{-1}(X^\mu(y))P(y)R(X^\mu(y)), \end{aligned} \quad (49)$$

with L and R being element of the group $\text{SL}(N, \mathbb{R})$.

The equations of motion are:

$$\begin{aligned} D_y P &\equiv \frac{d}{dy} P + [\bar{A}_y, P] = 0, \\ U^{-1} D_y U + 2\lambda_2(y)P + 3\lambda_3(y)P \times P + \dots + N\lambda_N(y) \underbrace{P \times \dots \times P}_{N-1} &= 0, \end{aligned} \quad (50)$$

plus the Casimirs constraints $\text{Tr}(P^m) = c_m$. The cross product is a short-cut notation for

$$\underbrace{P \times \dots \times P}_m \equiv h_{i_1 \dots i_{m+1}} P^{i_1} \dots P^{i_m} T^{i_{m+1}}. \quad (51)$$

For an open curve C_{if} , we need to choose boundary conditions for $U(y)$ at the endpoints of the curve. In the pure gravity case, it is natural to ask that the answer is invariant under Lorentz transformations (since the geodesic length shares this property). In $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, the group elements R and L that parametrize the local Lorentz subgroup is

$$R = L^{-1}. \quad (52)$$

A natural condition is to impose that $U(y_i)$ and $U(y_f)$ are invariant under a gauge transformation of the form (52); this will assure that S_{EE} is insensitive to Lorentz transformations. From (49), we see that the only boundary conditions that satisfy this condition are

$$U(y_i) = U(y_f) = \mathbb{1}. \quad (53)$$

For higher spin gravity, the symmetry group is $\text{SL}(N, \mathbb{R}) \times \text{SL}(N, \mathbb{R})$ and we cannot say that the Lorentz subgroup is described by (52); condition (52) is much bigger in this case! Still, we will impose (53) in the higher spin case since it is the natural generalization of the gravitational case.

5.3. The geodesic equation

As we have argued, this system should be equivalent to that of a massive particle moving in an AdS₃ bulk. We now demonstrate one way to see this equivalence and make contact with metric-like fields: in particular, we will see that the usual geodesic equation with respect to the metric-like fields makes a somewhat surprising appearance. The following derivations are only consistent and justified for the $\text{SL}(2)$ Chern–Simons theory; I leave it as an exercise to the reader to understand and quantify what goes wrong with these manipulations for $\text{SL}(N)$.

Consider, for example, computing an open-ended Wilson line denoted by $X^\mu(y)$: for convenience, we take $y \in [0, y_f]$, and the two endpoints are fixed at $X(y = 0) = X_i$ and $X(y = y_f) = X_f$. Note that as the bulk connections are flat, the final answer cannot depend on the actual trajectory taken by the Wilson line (provided it does not wind around a black hole in the bulk), but rather only on its endpoints. There are also boundary conditions on the probe $U(y)$, which we will discuss along the way.

For the purpose of deriving the geodesic equation, we will take $\lambda_m = 0$ for $s > 2$, *i.e.* we will only have a quadratic Casimir λ_2 (which is restricting the representation $\mathcal{R}$ to live in $\text{SL}(2)$). It is convenient to eliminate λ_2 and P from action (45). Using their classical equations of motion, we find the second order action

$$S(U; A, \bar{A})_{C_{\text{if}}} = \sqrt{c_2} \int_C dy \sqrt{\text{Tr}(U^{-1} D_y U)^2}. \quad (54)$$

Note that in this form, the action is essentially that of a gauged sigma model. The equations of motion given by varying (54) with respect to U are

$$\frac{d}{dy} \left((A^u - \bar{A})_\mu \frac{dX^\mu}{dy} \right) + [\bar{A}_\mu, A_\nu^u] \frac{dX^\mu}{dy} \frac{dX^\nu}{dy} = 0. \quad (55)$$

Here, we have made use of the gauge freedom given by reparametrizations of the worldline parameter y . In particular, we picked y to be the ‘proper distance’ of the probe, *i.e.* the integrand $\sqrt{\text{Tr}(U^{-1}D_yU)^2}$ is independent of y , which is equivalent to the choice of λ being a constant.

The actual dependence on $U(y)$ in (55) is in the definition of A^u

$$A_y^u \equiv U^{-1} \frac{d}{dy} U + U^{-1} A_y U. \quad (56)$$

In these equations, A_μ is always contracted with the tangent vector along the path, and so A_y is the only component which matters.

For reasonable choices of $A, \bar{A}$, these equations of motion are very non-trivial, and their precise form depends strongly on the choice of path $X^\mu(y)$. However, from the perspective of the equation of motion, U acts like a gauge transformation on the connection A . So it seems that a perfectly good Ansatz is to look for a solution where the particle does not move in the auxiliary space, *i.e.* $U(y) = \mathbb{1}$. This is clearly compatible with the boundary condition (53), but we are making a strong assumption: for arbitrary A and $\bar{A}$ it is not consistent, and a sufficient condition is to consider $\mathfrak{sl}(2)$ connections. In this case, we find

$$\frac{d}{dy} \left((A - \bar{A})_\mu \frac{dX^\mu}{dy} \right) + [\bar{A}_\mu, A_\nu] \frac{dX^\mu}{dy} \frac{dX^\nu}{dy} = 0. \quad (57)$$

We pause to discuss the interpretation of this equation. It appears to be a differential equation for the *path* that the Wilson line takes in the bulk. Of course, the choice of path is arbitrary: however, this equation tells us that only if the path satisfies this particular differential equation will the condition $U(y) = \mathbb{1}$ be a solution to the bulk equations of motion. For a different choice of bulk, path $U(y)$ will necessarily vary along the trajectory, but the final on-shell action will be the same.

As it turns out, (57) is actually very familiar if A and $\bar{A}$ are $\mathfrak{sl}(2)$ connections. Expressing the connections in terms of the vielbein and spin connection using (4), and further using $\omega_\mu^a \epsilon_{ab}^c = \omega_\mu^c{}_b$, we find

$$\frac{d}{dy} \left(e_\mu^a \frac{dX^\mu}{dy} \right) + \omega_\mu^a{}_b e_\nu^b \frac{dX^\mu}{dy} \frac{dX^\nu}{dy} = 0. \quad (58)$$

This is precisely the geodesic equation for the curve $x^\mu(s)$ on a spacetime with vielbein e^a and spin connection $\omega_\mu^a{}_b$. It is equivalent to the more familiar form involving the Christoffel symbols, as can be shown explicitly by relating them to the spin connection and vielbein.

Furthermore, on-shell action (54) for $U = \mathbb{1}$ reduces to

$$\begin{aligned} S_C &= \sqrt{c_2} \int_{C_{\text{if}}} dy \sqrt{\text{Tr} \left((A - \bar{A})_\mu (A - \bar{A})_\nu \frac{dX^\mu}{dy} \frac{dX^\nu}{dy} \right)} \\ &= \sqrt{2c_2} \int_{C_{\text{if}}} dy \sqrt{g_{\mu\nu}(X) \frac{dX^\mu}{dy} \frac{dX^\nu}{dy}}, \end{aligned} \quad (59)$$

which is manifestly the proper distance along the geodesic. Note that the prefactor $\sqrt{c_2}$ indicates that the value of the Casimir controls the bulk mass of the probe, as we alluded to previously.

We have shown that the calculation is simple for a particular choice of bulk path for the Wilson line. However, by the flatness of the bulk connections, the final result (59) must hold for *any* path, provided that path can be continuously deformed to a geodesic. Thus, in the classical limit, we find that in the $SL(2)$ Chern–Simons theory, the value of the Wilson line between any two points is

$$W_{\mathcal{R}}(X_i, X_f) \sim \exp \left(-\sqrt{2c_2} L(X_i, X_f) \right), \quad (60)$$

where $L(X_i, X_f)$ is the length of the bulk geodesic connecting these two points. Here, $\sim$ denotes the limit c_2 large and hence the classical saddle point approximation is valid.

The somewhat unexpected appearance of the bulk geodesic equation is interesting and (we feel) satisfying: this construction provides a way to obtain geometric data (*i.e.* a proper distance) from purely topological data (*i.e.* the flat bulk connections).

5.4. Evaluating a Wilson line

Most of the recent work show how to evaluate $W_{\mathcal{R}}(C)$ in the saddle point approximation, *i.e.* we approximate the path integral by the classical action. We will briefly summarise these results, which involve evaluating $W_{\mathcal{R}}(C)$ when C is a closed loop and a single line. Recent interesting developments related to networks of Wilson lines are studied in [50], and some exact results that do not use a saddle point approximation are discussed in [51, 52].

5.4.1. On-shell action

In this subsection, we will evaluate the classical action (45) for any background connection. The derivations will be applicable for both open and closed curves, and we will keep $w_s \neq 0$ in this subsection.

To evaluate (45), we start by eliminating the dependence of U using equation (50)

$$\begin{aligned}
 S_{\text{on-shell}} &= \int_C dy \operatorname{Tr} (PU^{-1}D_y U) \\
 &= - \int_C dy \left(2\lambda_2(y) \operatorname{Tr} (P^2) + 3\lambda_3(y) \operatorname{Tr} (P^3) + \cdots + N\lambda_N(y) \operatorname{Tr} (P^N) \right) \\
 &= - \int_C dy \left(2c_2\lambda_2(y) + 3c_3\lambda_3(y) + \cdots + Nc_N\lambda_N(y) \right), \tag{61}
 \end{aligned}$$

where in the last line we used the Casimirs constraints to eliminate P . Recall that the curve C is running from $y \in [y_i, y_f]$. It will be useful for us to define

$$\Delta\alpha_m = \alpha_m(y_f) - \alpha_m(y_i) = \int_{y_i}^{y_f} dy \lambda_m(y), \tag{62}$$

and with this simplified notation, the action becomes

$$S_{\text{on-shell}} = -2c_2\Delta\alpha_2 - 3c_3\Delta\alpha_3 + \cdots - Nc_N\Delta\alpha_N. \tag{63}$$

We need to determine $\Delta\alpha_m$ as a function of the connections A and $\bar{A}$; we will follow the method used in [39]. We start by building a solution when $A = \bar{A} = 0$: this defines for us $U_0(y)$ and $P_0(y)$ which from (50) read

$$U_0(y) = u_0 e^{-2\alpha_2(y)P_0 - 3\alpha_3(y)P_0 \times P_0 + \cdots - N\alpha_N(y)P_0 \times \cdots \times P_0}, \quad P_0(y) = P_0, \tag{64}$$

where u_0 is a constant matrix, and $\alpha_m(y)$ is defined in (62). From here, building a solution with $A \neq 0$ and $\bar{A} \neq 0$ is rather simple. As a consequence of the flatness condition (2), every connection can be expressed locally as a gauge transformation

$$A_\mu = L(x)\partial_\mu L^{-1}(x), \quad \bar{A}_\mu = R^{-1}(x)\partial_\mu R(x), \tag{65}$$

where the group elements L and R will reproduce different background connections. This means that we can build any solution to (50) for connections (65) by simply acting with L and R on (64). This gives

$$U(y) = L(x(y))U_0(y)R(x(y)), \quad P(y) = R^{-1}(x(y))P_0(y)R(x(y)). \tag{66}$$

Next, we impose the boundary condition (53); enforcing this condition on (66) gives

$$\begin{aligned} \mathbb{1} &= U(y_i) = L(y_i) \left(u_0 e^{-2\alpha_2(y_i)P_0 - 3\alpha_3(y_i)P_0 \times P_0 + \dots - N\alpha_N(y_i)P_0 \times \dots \times P_0} \right) R(y_i), \\ \mathbb{1} &= U(y_f) = L(y_f) \left(u_0 e^{-2\alpha_2(y_f)P_0 - 3\alpha_3(y_f)P_0 \times P_0 + \dots - N\alpha_N(y_f)P_0 \times \dots \times P_0} \right) R(y_f). \end{aligned} \quad (67)$$

If we combine both previous equations to eliminate u_0 , we obtain

$$e^{\mathbb{P}} = M, \quad M \equiv R(y_i)L(y_i)L^{-1}(y_f)R^{-1}(y_f), \quad (68)$$

where we define

$$\mathbb{P} \equiv -2\Delta\alpha_2 P_0 - 3\Delta\alpha_3 P_0 \times P_0 + \dots - N\Delta\alpha_N P_0 \times \dots \times P_0. \quad (69)$$

For a given P_0 , (68) determines $\Delta\alpha_m$ as a function of the background A and $\bar{A}$. Solving (68) is the most difficult task we have ahead of us.

To determine the on-shell action, we note that $\text{Tr}(\mathbb{P}P_0) = S_{\text{on-shell}}$. Hence, using (68) we find

$$-\log W_{\mathcal{R}}(C) = S_{\text{on-shell}} = \text{Tr}(\log(M)P_0). \quad (70)$$

This gives a very general expression for the on-shell value of the effective action for both open and closed curves C . The specific choice of P_0 will determine the representation $\mathcal{R}$ via the Casimirs c_m (contained in the traces of P_0).

5.4.2. Lines: correlation functions

In this portion, we will present a brief summary of the results in [16, 39] with emphasize on how to evaluate the Wilson line. To recap, the operator is defined as

$$W_{\mathcal{R}}(X_i, X_f) = \langle U_i | \mathcal{P} \exp \left(\int_{\tilde{C}_{\text{if}}} A \right) \mathcal{P} \exp \left(\int_{\tilde{C}_{\text{if}}} \bar{A} \right) | U_f \rangle. \quad (71)$$

In a saddle point approximation, the value of the Wilson line is

$$-\log W_{\mathcal{R}}(X_i, X_f) = \text{Tr}(\log(M)P_0), \quad (72)$$

where the matrix M in (72) contains the information about the background connections $(A, \bar{A})$

$$M \equiv R(y_i)L(y_i)L^{-1}(y_f)R^{-1}(y_f), \quad (73)$$

which assumes that the connections are flat, *i.e.*

$$A = L dL^{-1}, \quad \bar{A} = R^{-1} dR. \quad (74)$$

This expression makes evident that the Wilson line is only sensitive to the endpoints of C_{if} .

We will restrict now the discussion to Wilson lines in $\mathfrak{sl}(3) \times \mathfrak{sl}(3)$. As we send the endpoints of the Wilson line to one of the two boundaries, located at $r \rightarrow \pm\infty$, we only need to consider the asymptotic behaviour of the eigenvalues of M to evaluate (72). If asymptotically we have

$$b(r) = \bar{b}(r) \xrightarrow{r \rightarrow \infty} e^{rL_0}, \quad (75)$$

the eigenvalues of M will asymptote to

$$\lambda_M \sim \left(m_1 \epsilon^{-4}, \quad \frac{m_2}{m_1}, \quad \frac{\epsilon^4}{m_2} \right), \quad (76)$$

where $\epsilon = e^{-\rho}$ is the cutoff, and m_1 and m_2 are related to the coefficients of the characteristic polynomial as:

$$\text{Tr}_f(M) = m_1 \epsilon^{-4} + \dots, \quad \frac{1}{2} (\text{Tr}_f(M)^2 - \text{Tr}_f(M^2)) = m_2 \epsilon^{-4} + \dots \quad (77)$$

Note that $m_1 = m_1(y_i, y_f)$ and $m_2 = m_2(y_i, y_f)$ depend on the endpoints and the background charges carried by the connections. The asymptotic behaviour of the Wilson line close to the boundary is given by

$$-\log W_{\mathcal{R}}(X_i, X_f) = \frac{h}{2} \log \left(\frac{m_1 m_2(y_i, y_f)}{\epsilon^8} \right) + w_3 \log \left(\frac{m_1(y_i, y_f)}{m_2(y_i, y_f)} \right), \quad (78)$$

where we kept only universal terms as $\epsilon \rightarrow 0$. This result can be shown explicitly to give (41): it computes correlation functions of light operators in the dual CFT [34].

5.4.3. Loops: thermal entropy

In this subsection, we will show how to find the thermal entropy for a higher spin black hole using a Wilson loop. In this case, we consider periodic boundary conditions

$$r(y_i) = r(y_f), \quad t = 0, \quad \Delta\phi = \phi(y_f) - \phi(y_i) = 2\pi, \quad (79)$$

where $\phi \sim \phi + 2\pi$ is the compact direction. In $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$, the Wilson loop in the infinite dimensional representation computes the length

around the horizon, which is the thermal entropy of the black hole [39]. Here, we will show that we recover the thermal entropy for higher spin black holes in agreement with (16) following the results in [16].

From Section 5.4, we found a general expression for the on-shell value; however this expression simplifies greatly for a closed path. We start by noticing that the auxiliary variables of the Wilson line require

$$U(y_f) = U(y_i), \quad P(y_f) = P(y_i). \quad (80)$$

Imposing these periodic conditions for U in (66), we get

$$e^{\mathbb{P}} = u_0^{-1} (L^{-1}(y_f) L(y_i)) u_0 (R(y_i) R^{-1}(y_f)). \quad (81)$$

Using

$$R(x^\mu) = \exp \left(\int_0^\phi \bar{a} \right) b^{-1}(r), \quad L(x^\mu) = b^{-1}(r) \exp \left(- \int_0^\phi a \right), \quad (82)$$

we rewrite the previous equation as

$$e^{\mathbb{P}} = u_0^{-1} \exp(2\pi \ell a_\phi) u_0 \exp(-2\pi \ell \bar{a}_\phi). \quad (83)$$

Here, we are assuming that $(a_\phi, \bar{a}_\phi)$ are constant connections. Demanding periodicity in $P(y)$ in equation (66), we obtain the following condition

$$[P_0, R^{-1}(y_f) R(y_i)] = 0 \quad (84)$$

which says that P_0 and $\bar{a}_\phi$ simultaneously diagonalize and, therefore, the same do $\mathbb{P}$ and $\bar{a}_\phi$. If we denote V as the matrix of eigenvectors, and λ_ϕ and $\lambda_{\mathbb{P}}$ represent the eigenvalues, equation (83) reduces to

$$\exp(\lambda_{\mathbb{P}}) = (u_0 V)^{-1} \exp(2\pi a_\phi) (u_0 V) \exp(-2\pi \bar{\lambda}_\phi). \quad (85)$$

Since the left-hand side is diagonal, consistency of the previous equation requires to choose u_0 such that $u_0 V$ is the matrix which diagonalizes a_ϕ , and the right-hand side of (85) is diagonal as well. With this choice

$$\exp(\lambda_{\mathbb{P}}) = e^{2\pi(\lambda_\phi - \bar{\lambda}_\phi)}. \quad (86)$$

Analogously to Section 5.4, we use $\text{Tr}(\mathbb{P} P_0) = S_{\text{on-shell}}$ to find

$$-\log W_{\mathcal{R}}(C) = S_{\text{on-shell}} = \text{Tr}_f(2\pi(\lambda_\phi - \bar{\lambda}_\phi) P_0). \quad (87)$$

The trace here is in the fundamental representation.

To compute thermal (or entanglement) entropy, the massive particle encoded in the Wilson line needs to implement the correct type of singularity in the background solution [53]. In our language, this fixes P_0 to be

$$P_0 = \sqrt{\frac{c_2}{\text{Tr}_f(L_0 L_0)}} L_0, \quad (88)$$

where moreover, the strength of the massive particle is

$$\sqrt{\frac{c_2}{\text{Tr}_f(L_0 L_0)}} \rightarrow k. \quad (89)$$

For a more detailed discussion that justifies this choice, we refer to [39]. Gathering these expressions, the Wilson loop is given by

$$S_{\text{th}} = 2\pi k \text{Tr}_f((\lambda_\phi - \bar{\lambda}_\phi) L_0). \quad (90)$$

With this result, we have reproduced by means of our formalism the thermal entropy for higher spin black hole (16). If we choose $P_0 \sim W_0^{(3)}$, we would reproduce the thermal results for spin-3 entropy defined in [33].

These lectures are a summary of a collection of papers: primary resources are [16, 39, 41], and as well [18, 34]. I am extremely grateful to my collaborators for the many discussions and breakthroughs we shared; my view and understanding of the subject is very positively influenced by them. I would also like to thank the organizers of the LVI Cracow School of Theoretical Physics for giving me an opportunity to share my work, and the participants for their enthusiasm and interest in the subject.

Appendix

Conventions

In this appendix, we present our conventions for $\mathfrak{sl}(N, \mathbb{R})$ algebra. A convenient basis for the $\mathfrak{sl}(N, \mathbb{R})$ algebra is represented by $\{L_0, L_{\pm 1}\}$, the generators in the $\mathfrak{sl}(2, \mathbb{R})$ subalgebra, and $W_j^{(s)}$, the higher spin generators with $j = -(s-1), \dots, (s-1)$. Their commutation relations are:

$$[L_i, L_{i'}] = (i - i') L_{i+i'}, \quad (A.1)$$

$$[L_i, W_j^{(s)}] = (i(s-1) - j) W_{i+j}^{(s)}. \quad (A.2)$$

In this notation, L_0 and $W_0^{(s)}$ are elements of the Cartan subalgebra, and the rest of generators are raising and lowering operators. These commutation relations represent the principal embedding of $\mathfrak{sl}(N, \mathbb{R})$. An explicit

representation for the other $\mathfrak{sl}(N, \mathbb{R})$ generators, which is independent of the representation, is as follows:

$$W_j^{(s)} = (-1)^{s-j-1} \frac{(s+j-1)!}{(2s-2)!} \underbrace{[L_{-1}, [L_{-1}, \dots, [L_{-1}, L_1^{s-1}] \dots]]}_{s-j-1 \text{ terms}}. \quad (\text{A.3})$$

For $\mathfrak{sl}(3, \mathbb{R})$, we have 8 generators which we label $T_a = \{L_i, W_m\}$ with $i = -1, 0, 1$ and $m = -2, \dots, 2$. The algebra reads

$$\begin{aligned} [L_i, L_j] &= (i-j)L_{i+j}, \\ [L_i, W_m] &= (2i-m)W_{i+m}, \\ [W_m, W_n] &= -\frac{1}{3}(m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n}. \end{aligned} \quad (\text{A.4})$$

We work with the following matrices in the fundamental representation:

$$\begin{aligned} L_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & L_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ L_{-1} &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \\ W_2 &= 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & W_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ W_0 &= \frac{2}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ W_{-1} &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, & W_{-2} &= 2 \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.5})$$

The quadratic traces are

$$\begin{aligned} \text{Tr}_f(L_0 L_0) &= 2, & \text{Tr}_f(L_1 L_{-1}) &= -4, \\ \text{Tr}_f(W_0 W_0) &= \frac{8}{3}, & \text{Tr}_f(W_1 W_{-1}) &= -4, & \text{Tr}_f(W_2 W_{-2}) &= 16. \end{aligned} \quad (\text{A.6})$$

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