NOTES ON ANOMALY INDUCED TRANSPORT*

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Chiral anomalies give rise to dissipationless transport phenomena such as the chiral magnetic and vortical effects. In these notes, I review the theory from a quantum field theoretic, hydrodynamic and holographic perspective. A physical interpretation of the otherwise somewhat obscure concepts of *consistent* and *covariant* anomalies will be given. Vanishing of the CME in strict equilibrium will be connected to the boundary conditions in momentum space imposed by the regularization. The role of the gravitational anomaly will be explained. That it contributes to transport in an unexpectedly low order in the derivative expansion can be the most easily understood via holography. Anomalous transport is supposed to play also a key role in understanding the electronics of advanced materials, the Dirac and Weyl (semi-)metals. Anomaly related phenomena such as negative magnetoresistivity, anomalous Hall effect, thermal anomalous Hall effect and Fermi arcs can be understood via anomalous transport. Finally, I briefly review a holographic model of Weyl semi-metal which allows to infer a new phenomenon related to the gravitational anomaly: the presence of odd viscosity.

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1. Introduction

Symmetries are one of the most fundamental concepts of modern physics. The same is true for quantum theory. However, sometimes these two are in conflict with each other. More precisely, a symmetry present on the level of classical Lagrangian might not be compatible with quantum theory. When this happens, we speak of a quantum anomaly [1–3].

What shall specifically concern us here are chiral anomalies. These are intimately related to the fact that in even space-time dimensions, the Lorentz group has two unitarily inequivalent spinor representations giving rise to left- and right-handed spinors. For massless fermions, independent phase

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rotations of left- and right-handed spinors are symmetries of the classical theory. On the quantum level, at best one linear combination of these two symmetries can be preserved.

In the realm of high-energy physics, the prime example of a physical phenomenon induced by the incompatibility of chiral symmetries with quantum theory is the decay of the neutral pion into two photons. Besides explaining such otherwise forbidden (or strongly suppressed) processes in particle physics, anomalies also place very stringent consistency conditions on gauge theories. Gauging an anomalous symmetry leads to violation of unitarity. The divergence of the current couples to the longitudinal gauge degrees which normally corresponds to zero norm states. Anomalies lead to scattering of physical states into zero norm states and, therefore, destroy unitarity. Alternatively, one can allow a mass term for the gauge field, then however renormalizability is lost [4]. Even when the symmetries are not gauged, anomalies do place very stringent conditions on the strong dynamics of gauge theories. 't Hooft [5] argued that the spectrum of chiral fermions in a gauge theory is protected by this type of anomalies appearing in global symmetries. These constraints of "anomaly matching" between (weakly coupled) high-energy theories and (strongly coupled) low-energy effective theories can be exploited to get a handle on otherwise difficult to understand strong gauge dynamics. The power of anomalies lies in the fact that they are subject to a non-renormalization theorem [6] stating that the anomaly is exact as an operator relation at one loop.

In the recent years, anomalies have also emerged as the leading concept that allows to understand (and discover) unusual transport phenomena of quantum many-body physics involving chiral fermions. In high-energy physics, this is relevant to the physics of the quark–gluon plasma as created in heavy-ion collisions at RHIC and LHC. Anomalies have been invoked to predict charge asymmetries in the final state of a heavy-ion collision [7–9] and indeed charge asymmetries consistent with the prediction of anomalous transport theory have been detected in experiments at RHIC and LHC [10, 11]. In astrophysics, anomalous transport phenomena have been suggested to explain the sudden acceleration suffered by neutron stars at birth (neutron star kicks) [12] in cosmology as origin of primordial magnetic fields [13].

However (and probably somewhat surprisingly) anomalous transport phenomena are about to play also a lead role in condensed matter physics. It is already well-established that quantum Hall physics (see [14] for a recent review) can be described in a quantum field theory language via anomaly inflow [15] from bulk to boundary of a topologically non-trivial insulator. More recently, also the bulk physics of three (space) dimensional metals has been argued to be governed by chiral anomalies, *e.g.* via the phenomenon of negative magnetoresistivity. Of course, these are not ordinary metals but very special ones in which the Fermi surface lies at or very near linear band touching points [16–18]. In these cases, the effective low-energy electronic excitations near the band touching points are chiral fermions and the theory of anomalous transport can be applied to infer and describe a variety of exotic transport phenomena.

The aim of these lectures is to give an introduction to the subject with emphasis on making the underlying quantum field theoretical concepts as clear as possible. If one understands as a quantum field theory a prescription of how to compute correlation functions of (gauge invariant) operators, then the string theory derived holography needs also to be taken into account. Indeed, holography has played a major role in the modern area of anomalous transport and many subtleties arising when dealing with anomalies are most easily understood using the holographic framework [19–24].

These notes are organized as follows: in Section 2, we will review chiral triangle anomalies. Particular emphasis will be put on the ambiguities in the regularization procedure and how they can be fixed by physical constraints. This will lead to the concepts of *covariant* and *consistent* anomalies [25]. In Section 3, we will discuss the Landau level quantization of chiral fermions. We will see how the (covariant) anomaly arises as a conflict between normal-ordering and spectral flow. We will emphasize that the spectral flow needs to be supplemented with boundary conditions at a cutoff in momentum space and from this we will give a physical picture via anomaly inflow of the consistent anomaly. In Section 3, we will use the Landau level quantization to derive the anomalous transport formulas for chiral magnetic and chiral vortical effects. In Section 4, we will briefly review relativistic hydrodynamics with anomalies and the fact that the contribution of the (mixed) gravitational anomaly cannot be fixed by hydrodynamic arguments alone due to a mismatch in the number of derivatives in the transport phenomena and the anomaly. Section 5 will introduce a simple holographic model allowing to make the relation between anomalies and transport coefficients manifest. The derivative mismatch for the gravitational anomaly contribution is overcome in holography by taking derivatives in the holographic direction. Section 6 is devoted to the physics of Weyl semi-metals. After a quick introduction, we will show how almost all exotic Weyl semi-metal phenomenology can be understood from anomalous transport theory as outlined in the previous sections. These include negative magnetoresistivity in magnetic fields, in axial magnetic fields, thermal hall transport and the appearance of edge currents related to Fermi arcs. Section 7 will then briefly review a recently developed holographic model of Weyl semi-metal and show that it can be used to derive a new transport phenomenon related to the gravitational anomaly not contained in the ones discussed previously: odd viscosity.

2. Triangle anomalies

Let us start with a massless Dirac fermion

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix} . \tag{2.1}$$

In a chiral (Weyl) representation of the γ-matrices such that

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.2}$$

We define the left- and right-handed spinors via the projector $\mathcal{P}_\pm = \frac{1}{2}(1 \pm \gamma_5)$. The massless Dirac equation

$$i\gamma^\mu \partial_\mu \Psi = 0 \tag{2.3}$$

has two independent U(1) symmetries acting as $\psi_\alpha \to e^{i\varphi_+}\psi_\alpha$ and $\bar{\phi}^{\dot{\alpha}} \to e^{i\varphi_-}\bar{\phi}^{\dot{\alpha}}$ which we denote with $\mathrm{U}(1)_{\mathrm{L,R}}$. The corresponding conserved currents are $J^\mu_{\mathrm{L,R}} = \bar{\Psi}\gamma^\mu \mathcal{P}_\pm \Psi$, and on the level of classical field theory,

$$\partial_\mu J^\mu_{\mathrm{L,R}} = 0 . \tag{2.4}$$

For the future reference, let us also write down the Hamiltonians for left- and right-handed fermion in momentum space

$$\mathcal{H}_\pm = \pm \vec{p}\,\vec{\sigma} , \tag{2.5}$$

which will be convenient once we discuss Weyl semi-metals.

2.1. Chiral anomalies

Let us focus now on a single chiral fermion. We can define the generating functional that allows to compute arbitrary n-point functions of the current via gauging. We introduce an external gauge field A_μ and write the action as

$$S_\pm = \int \mathrm{d}^4x\, i\bar{\Psi}\gamma^\mu \left(\partial_\mu - iA_\mu\right) \mathcal{P}_\pm \Psi . \tag{2.6}$$

The quantum effective action (1-particle irreducible) is defined as[1]

$$e^{i\Gamma[A]} = \int D\Psi D\bar{\Psi} e^{iS_\pm} . \tag{2.7}$$

[1] More precisely, one might use the action $S_+[A] + S_-[0]$ in the exponent of the path integral to get a well-defined Dirac operator.

Since the action is invariant under the transformation $\delta A_\mu = \partial_\mu \lambda$ for arbitrary functions $\lambda(x)$, it follows seemingly that the quantum action is invariant and obeys

$$\int \mathrm{d}^4x\, \partial_\mu \varphi_\pm \frac{\delta \Gamma_\pm[A]}{\delta A_\mu} = 0\,. \tag{2.8}$$

By construction, functional variation with respect to the gauge field inserts the operator $J^\mu_\pm$. Therefore, gauge invariance suggests that $\partial_\mu J^\mu_\pm = 0$ as an operator equation, *i.e.* that arbitrary correlation functions with one insertion of the divergence of the chiral current should vanish. As it is well-known, this is not true and the obstruction of defining such a gauge invariant quantum action is the chiral anomaly.

Let us reconsider the anomaly in the elementary triangle diagram of three chiral currents. Applying the usual Feynman rules to the triangle diagrams in Fig. 1, we find the three-point amplitude $(p+q+k=0)$

$$\begin{aligned} iV^{\mu\nu\rho}(p,q,k)_\pm \;&=\; \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{tr}\left[\left(-\not{l}+\not{p}\right)\gamma^\mu(-\not{l})\gamma^\nu(-\not{l}-\not{p})\gamma^\rho \mathcal{P}_\pm\right]}{(l-p)^2\, l^2\, (l+q)^2} \\ &\quad + (\mu \leftrightarrow \nu, p \leftrightarrow q)\;. \end{aligned} \tag{2.9}$$

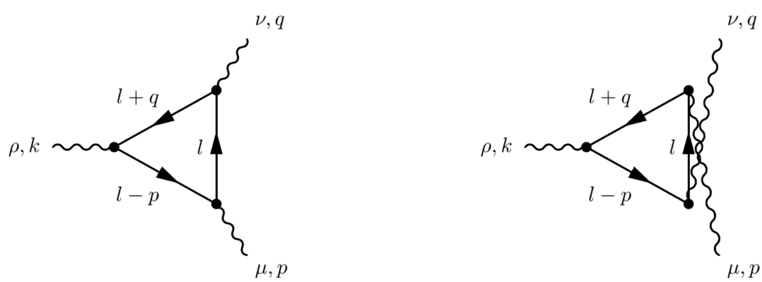

Fig. 1. Triangle diagrams with three currents at the vertices. While the diagrams are linearly divergent, the sum is actually finite but undetermined. Physical conditions such as the Bose symmetry on the external legs (chiral fermions) or covariant coupling to the external field (covariant anomaly) or conservation of the vector current (axial anomaly) have to be imposed in order to fix the ambiguity.

Details of the evaluation of this diagram are discussed in many textbooks such as [26], so we will only sketch the most important features. First, we note that the parity odd part of the projection operator $\mathcal{P}_\pm$ is relevant. So we replace it with $\frac{1}{2} \pm \gamma_5$, then use $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda\gamma_5] = -4i\epsilon^{\mu\nu\rho\lambda}$. Computing the divergence

$$k_\rho V^{\mu\nu\rho} = 2\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left[\frac{l_\alpha (l-p)_\beta}{(l-p)^2 l^2} - \frac{l_\alpha (l+q)_\beta}{(l+p)^2 l^2}\right]\epsilon^{\alpha\nu\beta\mu} + (\mu \leftrightarrow \nu, p \leftrightarrow q) = 0\,. \tag{2.10}$$

Because of the Lorentz symmetry, the integrals have to be proportional to either $p_\alpha p_\beta$ or $q_\alpha q_\beta$ and these combinations vanish once contracted with the epsilon tensor. On the other hand,

$$p_\mu V^{\mu\nu\rho} = 2 \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \left[\frac{l_\alpha q_\beta}{(l+q)^2 l^2} - \frac{(l-p)_\alpha (p+q)_\beta}{(l-p)^2 (l+q)^2} \right] \epsilon^{\alpha\nu\beta\mu} + (\mu \leftrightarrow \nu, p \leftrightarrow q) \ . \tag{2.11}$$

If the integrals were well-defined, we could make the substitution $l \to l + p$ in the second integral. The single integral is, however, linearly divergent and has to be defined properly. Vanishing of $p_\mu V^{\mu\nu\rho}$ depends now on the way we have labeled the internal loop momentum. Any other choice is just as good. The most general choice is $l \to l + c(p-q) + d(p+q)$, where c, d are arbitrary real numbers. Now, the integrals can be evaluated in a Lorentz invariant fashion. All divergences cancel but the final result is undetermined because of the ambiguity in labeling the internal loop variable (since the gamma matrix trace gives an epsilon tensor, it is only the anti-symmetric combination of external momenta that contributes). One finds

$$p_\mu V_\pm^{\mu\nu\rho} = \pm \frac{-i}{8\pi^2} (1-c) \epsilon^{\nu\rho\alpha\beta} q_\alpha k_\beta \,, \tag{2.12}$$

$$q_\nu V_\pm^{\mu\nu\rho} = \pm \frac{-i}{8\pi^2} (1-c) \epsilon^{\mu\rho\alpha\beta} k_\alpha p_\beta \,, \tag{2.13}$$

$$k_\rho V_\pm^{\mu\nu\rho} = \pm \frac{-i}{8\pi^2} 2c \, \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta \,. \tag{2.14}$$

Thus, the one-loop three-point function of three chiral currents is finite but undetermined. This poses the question *what is the correct value of* c? Not too surprisingly the answer to this question is: it depends! It does depend on the physical constraints the three-point function shall obey.

First, let us go back to the quantum effective action and demand that the three-point function of currents is

$$V^{\mu\nu\rho} = \Gamma_{(3)}^{\mu\nu\rho} = \frac{\delta^3 \Gamma}{\delta A_\mu \delta A_\nu \delta A_\rho} \,. \tag{2.15}$$

Since the order of differentiation does not play any role, we must impose the Bose symmetry on the external legs, all three vertices couple in precisely the same way to the gauge field. This imposes $c = 1/3$. If we express the anomaly now in terms of the current and the external gauge fields, we find

$$\partial_\mu \mathcal{J}^\mu_{\mathrm{L,R}} = \pm \frac{1}{96\pi^2} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \,. \tag{2.16}$$

For reasons to be explained shortly, this is called the *consistent* form of the anomaly.

On the other hand, we might be interested in defining a quantum operator $J^\mu_\pm$ that has nice properties with respect to gauge transformations. More precisely, we would like to think of the current as an object that couples covariantly (*i.e.* without anomaly) to the external gauge fields. This singles out one particular vertex and demands that the divergence on the other two vertices vanishes. The solution for this covariant definition of current is $c = 1$ and the anomaly is now

$$\partial_\mu J^\mu_{\mathrm{L,R}} = \pm \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \,. \tag{2.17}$$

This looks almost the same as before except for the overall factor of 3 in the anomaly. It is called the *covariant* anomaly. Since now we have treated the vertices in different ways, it is clear that this definition of three-point amplitude violates the Bose symmetry (2.15). This means that the covariant current obeying the covariant anomaly equation (2.17) cannot be thought of as a functional variation of a quantum effective action[2]. It might look surprising that we obtained two different answers for the divergence of "the current" by imposing two different but equally reasonable looking conditions. Later, when discussing anomalies and transport, we will suggest physical interpretations of these different quantum operators, the consistent ($\mathcal{J}^\mu$) and the covariant (J^μ) currents.

2.2. Axial anomaly

On the level of classical physics, a Dirac fermion is the direct sum of left- and a right-handed chiral fermions, $\Psi_{\mathrm{D}} = \psi_{\mathrm{L}} \oplus \psi_{\mathrm{R}}$. Anomalies pose a restriction on the possibility of defining chiral fermions in the quantum theory. Not too surprisingly, they also have implications on this direct sum. Let us proceed naively and simply define the quantum theory of a Dirac fermion as the quantum theory of a left-handed and a right-handed fermion. We want to keep the external gauge fields distinguishable, *i.e.* we introduce left- and right-handed gauge fields coupling to the chiral currents independently. We define vector and axial currents via

$$\mathcal{J}^\mu = \mathcal{J}^\mu_{\mathrm{L}} + \mathcal{J}^\mu_{\mathrm{R}} \,, \tag{2.18}$$

$$\mathcal{J}^\mu_5 = \mathcal{J}^\mu_{\mathrm{L}} - \mathcal{J}^\mu_{\mathrm{R}} \,, \tag{2.19}$$

[2] It also appears as the edge current in systems where the anomaly is localized on a co-dimension one boundary and canceled via a higher dimensional Chern–Simons term as in quantum Hall systems. See also [27] for an application this in the context of anomalous transport theory.

and a basis of vector-like and axial gauge fields

$$A_\mu = \tfrac{1}{2}\left(A^{\rm L}_\mu + A^{\rm R}_\mu\right)\,, \tag{2.20}$$
$$A^5_\mu = \tfrac{1}{2}\left(A^{\rm L}_\mu - A^{\rm R}_\mu\right)\,. \tag{2.21}$$

We can just add and subtract equations (2.16) to find

$$\partial_\mu \mathcal{J}^\mu = \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\lambda} F_{\mu\nu}F^5_{\rho\lambda}\,, \tag{2.22}$$
$$\partial_\mu \mathcal{J}^\mu_5 = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\lambda}\left(F_{\mu\nu}F_{\rho\lambda} + F^5_{\mu\nu}F^5_{\rho\lambda}\right)\,, \tag{2.23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F_{\mu\nu} = \partial_\mu A^5_\nu - \partial_\nu A^5_\mu$. We have chosen to express this in terms of the consistent currents. The result is the same up to an overall factor of 3 for the covariant currents. Equation (2.22) looks troublesome: eventually one would like the vector current to play the role of the electric current that couples to a dynamical gauge fields. Even without quantizing the gauge fields, one should expect that the electric current acts as source of Maxwell's equations

$$\mathcal{J}^\mu = \partial_\nu F^{\mu\nu}\,. \tag{2.24}$$

This is only consistent if the divergence of the vector current vanishes, since $\partial_\mu\partial_\nu F^{\mu\nu} = 0$. One might say that this is still true in the absence of axial gauge fields and that indeed in nature on a fundamental level axial gauge fields do not exist. However, one should keep in mind that equation (2.22) is just a short form for insertions of the divergence of the vector current in correlation functions. Its meaning is that there is a three-point function of a divergence of a vector current with an axial current and another vector current that does not vanish. Furthermore, as we will see later in Weyl semi-metals, such axial gauge fields do arise quite naturally in the low-energy effective description of their electronics. So we need to solve this problem of non-conservation of the vector current. Happily this has been done long time ago [28] by noticing that once gauge invariance is lost, nothing prevents us from introducing additional (non-gauge invariant) local counterterms to our quantum action. These are called Bardeen counterterms and they redefine the quantum action as follows

$$\varGamma\left[A, A^5\right] \to \varGamma\left[A, A^5\right] + \int \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\lambda} A_\mu A^5_\nu \left(c_1 F_{\rho\lambda} + c_2 F^5_{\rho\lambda}\right)\,. \tag{2.25}$$

If we now compute the (consistent) currents as variation of the effective action with respect to the gauge fields and chose $c_1 = \frac{1}{12\pi^2}$ and $c_2 = 0$, we find

$$\partial_\mu \mathcal{J}^\mu = 0\,, \tag{2.26a}$$

$$\partial_\mu \mathcal{J}^\mu = \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\lambda}\left(3F_{\mu\nu}F_{\rho\lambda} + F^5_{\mu\nu}F^5_{\rho\lambda}\right)\,. \tag{2.26b}$$

This form of the anomaly is the consistent axial anomaly. The Bardeen counterterms guarantee that a conserved vector current can be defined always, independently of the chosen regulator. We have not specified the regulator but generically a left–right symmetric regularization would not give a conserved vector current. On the other hand, manifestly gauge invariant regulators such as dimensional reduction automatically produce the Bardeen counterterms in the effective action and nothing has to be added "by hand". The particular Chern–Simons terms that are the Bardeen counterterms exist only if there are at least two independent gauge fields. That makes the nature of the axial anomaly as a mixed anomaly manifest. The precise statement of the axial anomaly is that there is no quantum theory in which both the axial- and the vector-like currents are conserved at the same time.

2.3. Wess–Zumino consistency condition

So why the anomaly is called *consistent*? To understand this, we need some more formalism. Our object of interest is the quantum effective action $\mathit{\Gamma}[A]$ (for simplicity of notation, we go back to the case of only one Abelian gauge field). An anomaly is a non-invariance of the effective action under a gauge transformation. The gauge transformation can be written as a functional differential operator $\delta_\lambda = \int \mathrm{d}^4x\,\partial_\mu\lambda\frac{\delta}{\delta A_\mu}$ and the anomaly is expressed as

$$\delta_\lambda \mathit{\Gamma}[A] = \mathcal{A}_\lambda\,. \tag{2.27}$$

The right-hand side arises because there is no regularization scheme that is compatible with the symmetry. It is a remnant of the regularization and remains even if we renormalize and take the regulator to infinity. That makes it intuitively clear that it has to be the integral of a local expression in the field A_μ. On the other hand, the quantum effective action arises by integrating out massless (chiral) fermions and is essentially a non-local expression. We further observe that the gauge transformations have to obey the gauge algebra which, in our simple example, means that two gauge transformations with different gauge parameters have to commute

$$[\delta_\lambda, \delta_\sigma] = 0\,. \tag{2.28}$$

It follows now that the anomaly has to fulfill

$$\delta_\lambda \mathcal{A}_\sigma - \delta_\sigma \mathcal{A}_\lambda = 0\,. \tag{2.29}$$

This is the Wess–Zumino consistency condition. A more geometric formulation can be given with one more piece of formalism. Let us promote the gauge parameters to a Grassmann valued field $\lambda(x) \to c(x)$ called the "ghost"[3]. It is useful to have the analogy of the exterior derivative 'd' and the formalism of differential forms in mind, *e.g.* the field strength of a gauge field (1-form) $A = A_\mu \mathrm{d}x^\mu$ is defined as the exterior derivative $F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}A$. In an analogous way, let us introduce an exterior derivative on field space

$$s = \int \mathrm{d}^4x \, \partial_\mu c \frac{\delta}{\delta A_\mu} \tag{2.30}$$

which is nothing but a gauge transformation with the Grassmann valued ghost field c as gauge parameter. It is called the BRST operator[4]. As one can easily check, it is nilpotent $s^2 = 0$. The anomaly can now be written as

$$s\Gamma[A] = \mathcal{A} \, . \tag{2.31}$$

We can think of the anomaly as a one-form on field space (a local integrated polynomial of the field A_μ, one ghost field and a finite number of derivatives, *i.e.* having ghost number one). The Wess–Zumino consistency condition is the fact that the anomaly is a closed one-form with respect to the BRST operator

$$s\mathcal{A} = 0 \, , \qquad \mathcal{A} \neq s\Gamma_{\mathrm{ct}}[A] \, . \tag{2.32}$$

Here, we have included the condition that it should not be possible to write the anomaly as the BRST variation of an integral of local term of ghost-number zero. If that were the case, we could just add Γ_{ct} as a counterterm of the effective action and get a new redefined and BRST (gauge) invariant quantum action. This maps the anomaly to a cohomology problem: the consistent anomaly is a non-trivial element of the BRST cohomology on the space of local integrated monomials in the fields at ghostnumber one. Finally, let us note that all this formalism can be extended in full generality to non-Abelian gauge algebras.

2.4. Covariant anomaly

Now we know that the consistent anomaly is a solution to the consistency condition. But what is the *covariant* anomaly? We observe that the consistent current defined as the functional derivative of the quantum action is not a gauge invariant operator if there is an anomaly. Using $[s, \delta/\delta A_\mu] = 0$, we find

$$s\mathcal{J}^\mu = \frac{\delta}{\delta A_\mu}\mathcal{A} = -\frac{\pm 1}{24\pi^2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu c F_{\rho\lambda} \, , \tag{2.33}$$

[3] In the path integral quantization of non-Abelian gauge theories, this is the Fadeev–Popov ghost that arises in defining the measure.

[4] See [29] for a recent review.

where in the expression on the right-hand side we have specialized to one chiral fermion again. We already know that in the triangle diagram we can put all the anomaly into a single vertex. Thus, it must be possible to define a current with covariant couplings to the external legs. In particular, we should demand from this quantum operator $sJ^\mu = 0$ even in the presence of an anomaly.

From (2.33), it is easy to see that by adding a Chern–Simons current to the consistent current, we can define

$$J^\mu_{\mathrm{L,R}} = \mathcal{J}^\mu_{\mathrm{L,R}} + \frac{\pm 1}{24\pi^2}\epsilon^{\mu\nu\rho\lambda}A_\nu F_{\rho\lambda}\,. \tag{2.34}$$

Adding the Chern–Simons current to the consistent current, we can construct the covariant current. The defining characteristics of this current are that it is invariant under all the gauge transformations, even the anomalous ones, and that it cannot be obtained from variation of an action with local counterterms. We emphasized this already in the analysis of the triangle diagram but now we can also see it from the Chern–Simons current in (2.34). We can compute the anomaly in the covariant current (it is a covariant object under the anomalous gauge transformations but it does have an anomaly by itself)

$$\partial_\mu J^\mu_{\mathrm{L,R}} = \frac{\pm 1}{32\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}\,. \tag{2.35}$$

As expected, this is 1/3 of the consistent anomaly[5].

We can go through the same exercise in the case of the axial anomaly and construct the covariant vector and axial currents. With our canonical choice of taking the vector current explicitly conserved, we have

$$J^\mu = \mathcal{J}^\mu + \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}A^5_\nu F_{\rho\lambda}\,, \tag{2.36a}$$

$$J^\mu_5 = \mathcal{J}^\mu_5 + \frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\lambda}A^5_\nu F^5_{\rho\lambda}\,. \tag{2.36b}$$

Note that only the axial gauge potential enters these expressions. This is a reflection of the fact that we have chosen to put all the anomaly into the axial current. For the future reference, let us also write down the covariant vector and axial anomaly

$$\partial_\mu J^\mu = \frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F^5_{\rho\lambda} = \frac{1}{2\pi^2}\left(\vec{E}\cdot\vec{B}_5 + \vec{E}_5\cdot\vec{B}\right)\,, \tag{2.37a}$$

$$\partial_\mu J^\mu_5 = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}\left(F_{\mu\nu}F_{\rho\lambda} + F^5_{\mu\nu}F^5_{\rho\lambda}\right) = \frac{1}{2\pi^2}\left(\vec{E}\cdot\vec{B} + \vec{E}_5\cdot\vec{B}_5\right)\,. \tag{2.37b}$$

[5] We could also define a conserved current by adding the Chern–Simons current with an appropriate coefficient. Such a current is then neither consistent (variation of an effective action) nor gauge invariant.

The covariant anomaly looks completely vector–axial symmetric once we express it in terms of electric and magnetic fields. Generally, there always exist Chern–Simons currents that can be added to the consistent currents making the resulting covariant current a covariant object under all gauge transformations. They are also known as Bardeen–Zumino polynomials (not to be confused with the Bardeen counterterms). The theory of covariant and consistent anomalies goes back to [25].

2.5. Gravitational anomaly

There is one more anomaly that appears in the triangle diagram of one chiral current and two energy-momentum tensors. This is the gravitational contribution to the chiral anomaly (also mixed gauge-gravitational) anomaly [30–32]. It is by nature a mixed anomaly and, therefore, one can always use Bardeen counterterms to shift the anomaly between the involved symmetries. Note that the Bardeen counterterms have the form connection $\wedge$ connection $\wedge$ field-strength. In the case of gravity, the connection (= gauge field) is the Levi-Civita connection and the field strength is the Riemann tensor. On a fundamental level, gravity is always gauged in nature and this implies that there should not be any anomaly in the diffeomorphism symmetry. So it is customary to shift the anomaly completely into the chiral current in which case it takes the form of

$$\nabla_\mu \mathcal{J}^\mu = \frac{\pm 1}{768\pi^2} \epsilon^{\mu\nu\rho\lambda} R^\alpha{}_{\beta\mu\nu} R^\beta{}_{\alpha\rho\lambda} \tag{2.38}$$

for a single chiral fermion. Again, one can find a covariant form of this anomaly applying the principles outlined in the previous subsection. While this looks a rather straightforward application of the principle that anomalies are contractions of field strength tensors with the epsilon tensor, there is at least one clear difference: the usual chiral and axial anomalies are expressions involving two derivatives, whereas the gravitational anomaly involves four derivatives.

2.6. Anomaly coefficients

Anomalies are subject to non-renormalization theorems. The anomaly coefficient is exact at the one-loop level. Therefore, one can infer the presence of an anomaly by analyzing the triangle diagram with generic currents at the vertices. Let us assume a generic symmetry group generated by matrices T_a such that $[T_a, T_b] = i f_{abc} T_c$. Chiral anomalies are present if the anomaly coefficient

$$d_{abc} = \frac{1}{2} \sum_{\mathrm{L}} \mathrm{tr}\left(\left\{ T_a^{\mathrm{L}}, T_b^{\mathrm{L}} \right\} T_c^{\mathrm{L}} \right) - \frac{1}{2} \sum_{\mathrm{R}} \mathrm{tr}\left(\left\{ T_a^{\mathrm{R}}, T_b^{\mathrm{R}} \right\} T_c^{\mathrm{R}} \right) \tag{2.39}$$

does not vanish. Here, the sums run over the species of left- and right-handed fermions and $T^{\mathrm{L,R}}$ are the representations of left- and right-handed fermions and the curly bracket is the anti-commutator. In the case when all symmetries are Abelian, this boils down to sums over triple products of charges

$$d_{abc} = \sum_{\mathrm{L}} \left(q_a^{\mathrm{L}} q_b^{\mathrm{L}} q_c^{\mathrm{L}} \right) - \sum_{\mathrm{R}} \left(q_a^{\mathrm{R}} q_b^{\mathrm{R}} q_c^{\mathrm{R}} \right) . \tag{2.40}$$

A mixed gravitational anomaly is present if

$$b_a = \sum_{\mathrm{L}} q_a^{\mathrm{L}} - \sum_{\mathrm{R}} q_a^{\mathrm{R}} \tag{2.41}$$

is different from zero. We call d_{abc} and b_a the chiral and gravitational anomaly coefficients. Note that d_{abc} is completely symmetric. By means of adding Bardeen counterterms, one cancels some of the consistent anomalies. This is precisely the case of the axial anomaly, where $d_{AVV} = d_{VAV} = d_{VVA} \neq 0$ but the consistent vector current is conserved.

3. Landau levels and anomalies

We now study chiral fermions in a magnetic field[6]. The Weyl equation is

$$i \not{D} \Psi = 0 \tag{3.1}$$

and the covariant derivative is $D_\mu = \partial_\mu - iA_\mu$ (absorbing the electric charge into the definition of the gauge field). The magnetic field is taken to point in the z-direction and the chosen gauge field is $A_y = Bx$. The Weyl equation is

$$i \left(\partial_t - \vec{\sigma} \vec{D} \right) \psi = 0 . \tag{3.2}$$

We now use the fact that the differential equation depends explicitly only on x and not on the other coordinates, so we can use the Ansatz $\psi = e^{-i(\omega t - p_y y - p_z z)} \tilde{\psi}(x)$ to find the matrix equation

$$\begin{pmatrix} \omega - p_z & i(\partial_x - Bx + p_y) \\ -i(\partial_x + Bx - p_y) & \omega + p_z \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0 . \tag{3.3}$$

For $\omega = p_z$, there is a simple solution $\psi_+ \propto \exp[\frac{-(Bx-p_y)^2}{2B}]$ and $\psi_- = 0$, whereas the corresponding solution with $\omega = -p_z$ and $\psi_+ = 0$ is non-normalizable. This is the lowest Landau with $s = +1$, where s is the eigenvalue of the spinor wave function of $\vec{\sigma} \cdot \vec{B}/|B|$. For a chiral fermion of opposite chirality, one finds that the normalizable solution has $\omega = -p_z$ and $s = -1$.

[6] A recent review on quantum field theory in magnetic field backgrounds is [33].

More generally, the whole spectrum arranges into Landau levels with energy levels given by

$$\omega_{p_z,n,s} = \sqrt{p_z^2 + B(2n+1) \mp sB}\,. \tag{3.4}$$

The Weyl equation in a magnetic field can be separated into a plane wave and a harmonic oscillator corresponding to the degrees of freedom along and transverse to the magnetic field. Accordingly, the momentum along the magnetic field is still a good quantum number but the momenta in the plane transverse to the magnetic field are replaced by just the harmonic oscillator quantum number n. In our gauge choice, the momentum in y-direction parametrizes the degeneracy of the Landau levels of $\frac{B}{2\pi}$ states per unit area. Almost all energy eigenvalues are spin degenerate, meaning that for each spin up state of a given energy and momentum, there is also a spin down state with the same energy and momentum except for the lowest one with $n = 0$, and the spin either aligned or anti-aligned with the magnetic field according to chirality. For these lowest Landau levels, the energy is simply linear in the momentum p_z. The spectrum is sketched in Fig. 2. Let us apply now the standard argument that allows to derive the chiral anomaly from spectral flow [34]. In addition to the magnetic field, we switch on a parallel electric field E_z. This field will pump momentum into the system according to Newton's law $\dot{p}_z = E_z$. In the Dirac sea of the higher Landau levels, all states are occupied and a fermion has no available state to move to[7]. For the lowest Landau level, there is something more going on. We assume, as usual in quantum field theory, that the infinite Dirac sea

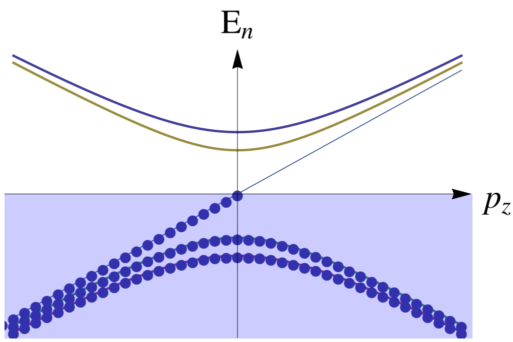

Fig. 2. Landau level spectrum of a single Weyl fermion. The higher Landau levels are spin degenerate and gapped. The lowest Landau level is chiral fermion whose motion is restricted along the magnetic field. The Dirac sea is comprised of all the states of negative energy which includes all lowest Landau level states of negative momentum. The spin is polarized along the magnetic field.

[7] In condensed matter physics, it is known that fully occupied bands do not produce an electric current (if not for topological reasons).

of negative energy states is subtracted via a normal ordering prescription. The electric field pumps momentum into the system and shifts the states in the Dirac sea of the lowest Landau level to positive momentum. Occupied states just below the normal ordered vacuum are shifted into empty states just above the vacuum. This is a particle creation out of the vacuum. We can also compute the rate of particle creation. The density of states for a one-dimensional chiral fermion (such as the fermions in the lowest Landau level) is $\mathrm{d}n = \mathrm{d}p/(2\pi)$ and their degeneracy is $B/(2\pi)$. If we combine this with the Lorentz force, we find

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\vec{E}\cdot\vec{B}}{4\pi^2}\,. \tag{3.5}$$

But this is just a Lorentz non-covariant version of the anomaly equation $\partial_\mu J^\mu = \frac{1}{32}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$. So now the anomaly has been recovered from rather elementary quantum mechanics of a single Weyl fermion without any fancy quantum field theory. Why was this so easy and where does quantum field theory hide? It hides in two aspects: first we assumed a notion of normal ordered vacuum which is our trick to subtract the infinite Dirac sea. The anomaly is then the incompatibility between the spectral flow and our normal ordering prescription. There is another aspect of it: where do all the fermions come from? In this picture, we do not really have to ask this question since the Dirac sea is infinite and any finite amount of states that we pull out of the vacuum will not be able to deplete the infinite supply of states in the Dirac sea[8]. In quantum field theory, there is another ingredient that we will eventually have to take into account: a cutoff has to be introduced at intermediate stages of the calculations. This will turn out to be an essential ingredient to the proper spectral flow picture of the axial anomaly. For the moment, we want to point out that the prefactor of the anomaly obtained via the spectral flow argument is the one of the *covariant* anomaly. In hindsight, this is not surprising, since we assumed that our chiral fermions couple covariantly to the external fields (*i.e.* we assumed the usual form the Lorentz force). The spectral flow picture of the chiral anomaly is sketched in Fig. 3.

Let us now combine left- and right-handed Weyl fermions to get an idea of the spectral flow of the axial anomaly. States are created and annihilated since for each right-handed fermion pulled out of the vacuum, there is a left-handed particle that is pushed further down into the Dirac sea. The total anomaly is

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}(n_{\mathrm{L}} - n_{\mathrm{R}})}{\mathrm{d}t} = \frac{1}{2\pi^2}\vec{E}\cdot\vec{B}\,. \tag{3.6}$$

[8] Sometimes this is compared to the Hilbert Hotel with an infinite number of rooms. Any new arrival can be accommodated by simply asking the occupants in room number n to switch to room number $n+1$ leaving the room number 1 available.

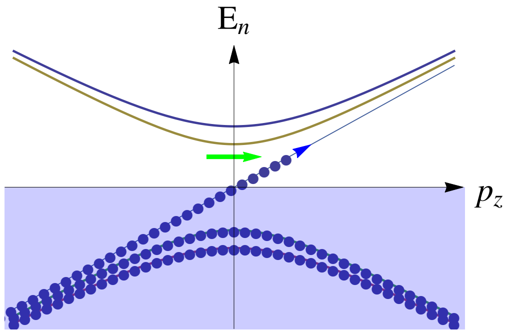

Fig. 3. (Color online) Spectral flow picture of the chiral anomaly. In parallel electric and magnetic fields, the states of the lowest Landau level are pushed across the normal ordered vacuum. The direction of the spectral flow is indicated by the black/blue arrow. The electric field is indicated by the gray/green arrow. From the quantum field theoretical perspective, particles are created out of the vacuum. Since there is an infinite supply of states in the Dirac sea, it cannot be depleted.

For this particular case, we actually cannot decide if this is the covariant or the consistent anomaly (2.22) with conserved vector current. Both anomalies have the same coefficient, which is not surprising since the vector current is actually anomaly free and trivially a covariant object under (vector-like) gauge transformations. The spectral flow picture for the axial anomaly is depicted in Fig. 4.

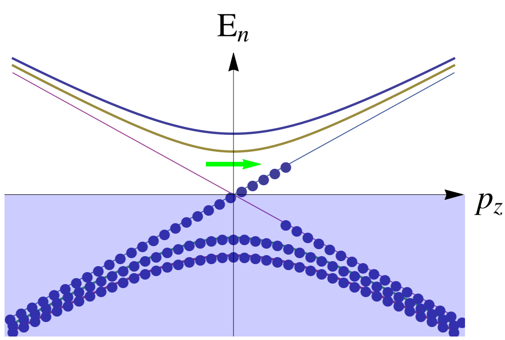

Fig. 4. Spectral flow picture of the axial anomaly. Chiral fermions of both chiralities are present. The particle creation of right-handed particles is counterbalanced by annihilation of left-handed particles. The total number of particles does not change but an imbalance in the number of right-handed and left-handed fermions is pumped into the system.

Now, let us see how we can understand the spectral flow when we switch on parallel axial electric field and usual (vector-like) magnetic field. The spectral flow picture tells us that both left-handed and right-handed particles are created out of the vacuum. This indeed looks problematic since it would mean that vector-like charge (which we eventually want to identify with the electric charge) is not conserved. To understand better what is going on, we need to keep in mind that the axial field A^5_μ is of different nature than the true gauge field A_μ because of the anomaly. Whereas the proper gauge field is not an observable itself A^5_μ is, in principle, observable. A pure gauge $A^5_\mu = \partial_\mu \lambda_5$ does not decouple from the theory, it rather couples to the anomaly. It is, therefore, natural to impose the boundary condition $A^5_\mu = 0$ at infinity, A^5_μ can be different from zero only in a compact domain. Let us imagine that $A^5_0 \neq 0$ but constant in a slab $|z| < L$. Then at $x = \pm L$, there is a strong gradient of A^5_0 which is nothing but an axial electric field $\vec{E}_5$. If a parallel (vector-like) magnetic field is present, the covariant anomaly (2.37a) gets excited and creates charge out of the vacuum at $z = -L$ but destroys charge in equal amounts at $z = +L$. This is already good since globally no net charge is created. But it is not yet good enough! Charge conservation is a local equation and tells us that charge can only leave or enter a bounded region via inflow or outflow of current. So even if with the boundary condition $A^5_0|_\infty = 0$, we still violate local charge conservation. This is where the Bardeen–Zumino polynomial comes to rescue. The relation between the covariant and conserved consistent current (2.36a) tells us that in a region with non-vanishing axial vector A^5_μ and simultaneous presence of vector-like field-strength, a Chern–Simons current is created. This current is precisely such that it provides the inflow guaranteeing local charge conservation. The spectral flow picture of this consistent picture of the "anomaly" is depicted in Fig. 5. The fermions following the spectral flow do not really just move out to arbitrary high momentum. In quantum field theory, we always have to include a finite cutoff Λ at intermediate stages of our calculations. So with the regulator in place, the fermions subject to spectral flow hit the cutoff and we need to tell them what they should do at the cutoff by imposing boundary conditions in momentum space. Of course, the correct boundary condition for a conserved electric current is that as soon as the fermions hit the cutoff, they generate a current in space transporting precisely the right amount of charge from the region of negative axial electric field to the region of positive axial electric field. This inflow guarantees local charge conservation. Of course, we could also impose some sort of leaky boundary conditions in which we allow some of the charge to vanish into to the region beyond the cutoff. On the level of the effective action Γ, the boundary conditions at the cutoff are parametrized by the Bardeen counterterms (2.25). This is basically the same mechanism that is behind the anomaly inflow mechanism of [15]. The only difference being that now the bulk is not gapped[9].

[9] In view of applications in condensed matter, we might propose the maps *Callan–Harvey* $\leftrightarrow$ *topological insulators vs. Bardeen–Zumino* $\leftrightarrow$ *topological metals.*

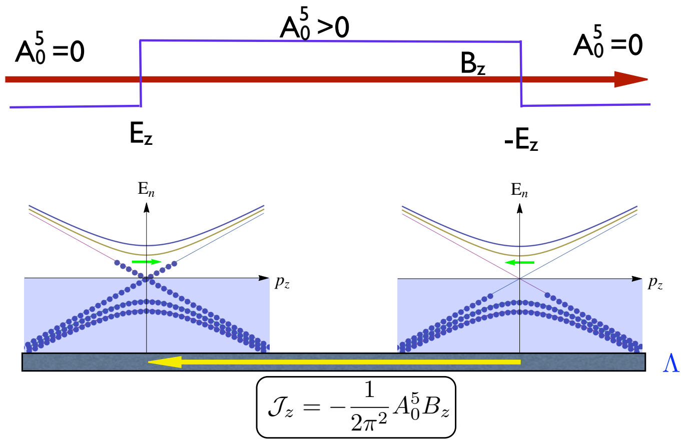

Fig. 5. Spectral flow picture of the consistent current. We assume the boundary condition $A_0^5 = 0$ at infinity and take constant but non-zero in a finite slab $|z| < L$. At $x = \pm$, strong localized axial electric fields a present. In addition, we assume a constant (vector-like) magnetic field B_z. From the perspective of the covariant anomaly, this activates the anomaly in the (covariant) vector current at $x = \pm L$ where charges of equal amounts are either created of destroyed. The boundary condition on the axial vector field A_μ^5 guarantees that no net charge is created. The spectral flow of the fermions necessarily hits the cutoff Λ in the regulated quantum field theory and is subject to boundary conditions there. Charge preserving boundary conditions can be given which creates a current whose quantum field theory implementation is the Bardeen–Zumino polynomial. This Chern–Simons current generated at the cutoff guarantees local charge conservation.

The Chern–Simons current generated to guarantee the local charge conservation is $\mathcal{J}_z = -\frac{A_0^5}{2\pi^2}B_z$. This looks precisely like the celebrated chiral magnetic effect, *i.e.* the generation of a current via magnetic field. It is only one part of the CME. So far, we have assumed that our system was in a vacuum state with respect to the normal ordered vacuum, *i.e.* no states above the vacuum are occupied. In the region where $A_0^5 \neq 0$, the vacuum energies are shifted: the left-handed vacuum is shifted down by A_0^5 and the right-handed vacuum is shifted up by A_0^5. In the next section, we will study another source of magnetic (and rotation) induced currents built up by the occupied states above the normal ordered vacuum.

4. Transport from Landau levels

For a fermion, the vacuum is the state with vanishing Fermi surface. Let us now go beyond this restriction and study what happens when there is a non-vanishing Fermi surface and once we are at it, we also introduce a finite temperature (so strictly speaking, there is sharp Fermi surface but it is smoothed by the temperature). To start, we study what happens if we have a single chiral fermion in a magnetic field with chemical potential $\mu_{\mathrm{L,R}}$ at temperature T. Of course, the charge and energy are non-vanishing, and this is described by the free-energy density[10]

$$F_{\mathrm{L,R}} = -\frac{1}{24\pi^2}\left(\mu_{\mathrm{L,R}}^2 + 2\pi^2\mu_{\mathrm{L,R}}^2 T^2 + \frac{7}{15}\pi^4 T^4\right), \tag{4.1}$$

if the magnetic field is much smaller than temperature and chemical potentials, and by

$$F_{\mathrm{L,R}} = -B\left(\frac{T^2}{24} + \frac{\mu_{\mathrm{L,R}}^2}{8\pi^2}\right) + O\left(e^{-\frac{\sqrt{B}-\mu_{\mathrm{L,R}}}{T}}\right), \tag{4.2}$$

for large magnetic field. The last expression is valid as long as the magnetic field induces a gap in the higher Landau levels $\sqrt{2B} \gg (T, \mu_{\mathrm{L,R}})$. In that case, the only states that contribute are the states in the lowest Landau level and these are effective $1+1$ dimensional chiral fermions. The contribution from the higher Landau levels in this regime is exponentially suppressed.

Let us compute the current in a magnetic field. First, we start with the higher Landau levels. Note that the current is simply the integral over the velocity of the particles weighted by the Fermi–Dirac distribution

$$J_{\mathrm{HLL}} = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi}\frac{\partial E_n}{\partial p}\left[\frac{1}{1+e^{\frac{E_n-\mu}{T}}} - \frac{1}{1+e^{\frac{E_n+\mu}{T}}}\right] = 0\,. \tag{4.3}$$

The velocity in the higher Landau levels is the group velocity $\partial E_n/\partial p$, both particles and anti-particles contribute the latter, however, with opposite sign. The integrand is a total derivative and, therefore, the integral receives contributions only from the boundaries. Since the probability of finding a particle with infinite momentum is zero, there is no current generated in the higher Landau levels.

The lowest Landau level is different. The integration region is only the positive half line and the velocity is simply ± 1 depending on chirality. So

[10] For the completely free fermions, we also could introduce independent left- and right-handed temperatures. If we understand the expressions as lowest order in a perturbative expansion of an interacting theory, only a common temperature can be defined. Local interactions can preserve separate left- and right-handed U(1) symmetries so different chemical potentials can be defined in the interacting theory. The anomaly spoils this but only in the simultaneous presence of electric and magnetic fields.

we find

$$J_{\mathrm{LLL}} = \pm \int_0^\infty \frac{\mathrm{d}p}{2\pi} \left[\frac{1}{1+e^{\frac{p-\mu_{\mathrm{L,R}}}{T}}} - \frac{1}{1+e^{\frac{p+\mu_{\mathrm{L,R}}}{T}}} \right] = \pm \frac{\mu_{\mathrm{L,R}}}{2\pi} \,. \tag{4.4}$$

Now, we remember that these states come with a multiplicity of $B/(2\pi)$ per unit area and this give the current density

$$\vec{J} = \pm \frac{\mu_{\mathrm{L,R}}}{4\pi^2} \vec{B} \,. \tag{4.5}$$

Finally here it is, the celebrated expression for the chiral magnetic effect: the generation of a current in a magnetic field. Many things are remarkable about this formula. First, we have computed it at finite temperature but the result is completely independent of it. Only the lowest Landau level contributes, but contrary to the expression for the free energy (4.1), this is not an approximation valid for large field strength but it is an exact result.

Let us go a step further, let us also compute the energy current. Since we are dealing with a relativistic theory, we can equally well call it the momentum density along the magnetic field. The calculation is very similar, only the lowest Landau level contributes and we find

$$J_{\epsilon,\mathrm{LLL}} = \pm \int_0^\infty \frac{\mathrm{d}p}{2\pi} \left[\frac{p}{1+e^{\frac{p-\mu}{T}}} + \frac{p}{1+e^{\frac{p+\mu}{T}}} \right] = \pm \left(\frac{\mu_{\mathrm{L,R}}^2}{4\pi} + \frac{\pi T^2}{12} \right) \tag{4.6}$$

giving

$$\vec{J}_\epsilon = \pm \left(\frac{\mu_{\mathrm{L,R}}^2}{8\pi^2} + \frac{T^2}{24} \right) \vec{B} \,. \tag{4.7}$$

This is the chiral magnetic effect in the energy current. Now, the temperature contributes but only in a very simple polynomial way.

It is a common lore in many-body physics that rotation has many similarities to magnetic fields. For example, the Coriolis force $\vec{F} = 2m\vec{v} \times \vec{\omega}$ is similar to the Lorentz force $\vec{F} = \vec{v} \times \vec{q}B$ if we identify $2m\vec{\omega} \sim q\vec{B}$ [11]. For relativistic, massless fermions we should replace the rest mass with the energy. Now, we can calculate the current due to rotation

$$J = \pm \int_0^\infty \frac{\mathrm{d}p}{2\pi} \left[\frac{2p}{1+e^{\frac{p-\mu}{T}}} + \frac{2p}{1+e^{\frac{p+\mu}{T}}} \right] = \pm \left(\frac{\mu_{\mathrm{L,R}}^2}{2\pi} + \frac{\pi T^2}{6} \right) \tag{4.8}$$

[11] Of course, rotation also gives rise to a centrifugal force. It is quadratic in the angular velocity $\omega = \vec{\nabla} \times \vec{v}$. We might try to ignore it on the grounds that it is higher order in derivatives.

and the energy current

$$J_\epsilon = \pm \int_0^\infty \frac{\mathrm{d}p}{2\pi} \left[\frac{2p^2}{1+e^{\frac{p-\mu}{T}}} - \frac{2p^2}{1+e^{\frac{p+\mu}{T}}} \right] = \pm \left(\frac{\mu_{\mathrm{L,R}}^3}{3\pi} + \frac{\mu_{\mathrm{L,R}} \pi T^2}{3} \right) . \tag{4.9}$$

These (admittedly somewhat hand-waving) arguments give rise to the chiral vortical effects (CVE) in the current and energy current

$$\vec{J} = \pm \left(\frac{\mu_{\mathrm{L,R}}^2}{4\pi^2} + \frac{T^2}{12} \right) \vec{\omega} \,, \tag{4.10}$$

$$\vec{J}_\epsilon = \pm \left(\frac{\mu_{\mathrm{L,R}}^3}{6\pi^2} + \frac{\mu_{\mathrm{L,R}} T^2}{6} \right) \vec{\omega} \,. \tag{4.11}$$

We get currents from the lowest Landau level. This is quite interesting by itself but it gets even a bit more interesting remembering that the lowest Landau level was also responsible for the anomaly. So is there a more direct connection? The answer is yes! As a first step, let us generalize our results for single fermions to many different species of fermions and label them with an index f (we can also switch to a basis with left-handed fermion only). Then assume that there is a bunch of U(1) symmetries labeled with an index a under which the fermions carry charges q_a^f. The chemical potential for the fermion species f is then $\sum_a q_a^f \mu_a = \mu^f$ and it sees the magnetic field $\vec{B}^f = \sum_a q_a^f \vec{B}_a$. The current corresponding to symmetry a is likewise $J_a = \sum_f q_a^f J^f$. Now, use the expression for the elementally chiral currents and obtain (summation over repeated indices is implied here)

$$\vec{J}_a = d_{abc} \frac{\mu_b}{4\pi^2} \vec{B}_c \,, \qquad \vec{J}_\epsilon = \left(d_{abc} \frac{\mu_a \mu_b}{8\pi^2} + b_c \frac{T^2}{24} \right) \vec{B}_c \,, \tag{4.12}$$

$$\vec{J}_a = \left(d_{abc} \frac{\mu_b \mu_c}{4\pi^2} + b_a \frac{T^2}{12} \right) \vec{\omega} \,, \qquad \vec{J}_\epsilon = \left(d_{abc} \frac{\mu_a \mu_b \mu_c}{6\pi^2} + b_a \mu_a \frac{T^2}{6} \right) \vec{\omega} \,, \tag{4.13}$$

where d_{abc} and $b_a = \sum_f q_a^f$ are the chiral and gravitational anomaly coefficients. This is a hint towards the deep origin of these expression. The currents are indeed induced by the chiral and gravitational anomaly as we will discuss in the next sections. We have obtained these expressions without having to worry about regularization issues. The integrals are finite because they are damped in the UV by the exponential decay of the distribution functions. Furthermore, in view of what we learned about the spectral flow, we also know that the naive counting of particles around the Fermi surface (or the normal ordered vacuum) gives us only the covariant current. To get the consistent current, additional terms can and do arise from the Bardeen–Zumino polynomials.

Let us now specialize to the case of one Dirac fermion: we have $d_{5VV} = d_{V5V} = d_{VV5} = 2$, $d_{555} = 2$, and $b_5 = 2$. Let us study the magnetic effects and include also the Chern–Simons current from the Bardeen–Zumino polynomial, *i.e.* we want to write down the magnetic field induced currents in terms of the conserved consistent current

$$\vec{\mathcal{J}} = \frac{\mu_5 - A_0^5}{2\pi^2}\vec{B}\,, \tag{4.14}$$

$$\vec{\mathcal{J}}_5 = \frac{\mu}{2\pi^2}\vec{B}\,. \tag{4.15}$$

These two are the (proper) chiral magnetic effect and the chiral separation effect (CSE). While they look very similar if written down in terms of the

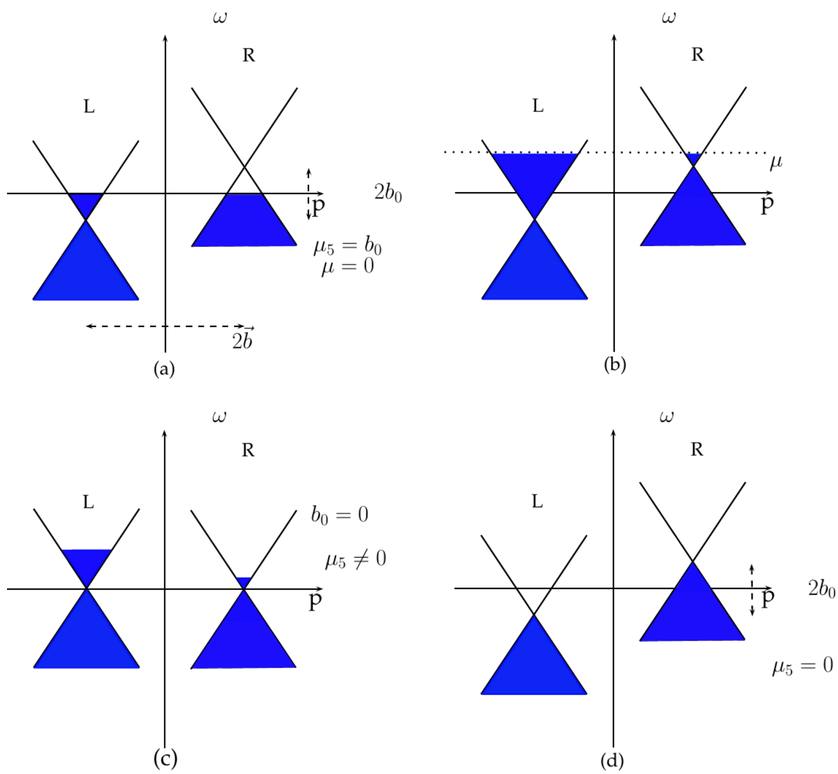

Fig. 6. The figure shows different fillings corresponding to different chemical potentials and axial vector field backgrounds (with notation $A_0^5 = b_0$, $\vec{A}_5 = \vec{b}$). In panel (a), $\mu_5 = A_0^5$ and $\mu = 0$ so neither CME nor CSE are present. In panel (b), still $\mu_5 = A_0^5$ but now $\mu > 0$ so only the CME vanishes. In panel (c), we have $\mu_5 \neq 0$ and $\mu \neq 0$ but $A_0^5 = 0$ so now both CME and CSE are non-vanishing. Finally, in panel (d), both chemical potentials are zero but $A_0^5 \neq 0$ so here only the CME is non-vanishing.

covariant current, they are quite different once written in terms of the consistent current. There is no contribution to the CSE by any Bardeen–Zumino polynomial. Ultimately, this can be traced back to the fact that it depends only on the chemical potential μ related to a truly conserved current. On the other hand, the CME receives two contributions, one from the states that fill the energy range between the vacuum and the Fermi surface, and another one that stems from the boundary conditions in momentum space and depends on the specific gauge invariant regularization. We have already mentioned that the role of A_0^5 is to shift the tips of the Weyl cones in opposite directions in energy. If this shift is equal to the imbalance in the chemical potentials, then there is a perfect cancellation between the two terms and the CME vanishes. In fact, if one defines the grand canonical ensemble for axial charge in the usual way by introducing a Hamiltonian $H - \mu_5 Q_5$, this has the effect of achieving both effects: a non-vanishing occupation number but also a shift in the locations of the tips of the Weyl cones in precisely such a way that $A_0^5 = \mu_5$. This definition of equilibrium grand canonical ensemble results in a vanishing CME. The different situations of vanishing *vs.* non-vanishing CME are sketched in Fig. 6.

5. Hydrodynamics and triangle anomalies

So far we have derived the CME and the CVE (with some hand waving) only for free fermions. We need to convince ourselves that the CVE expressions have some physical meaning. We could try to do better and study an ensemble of rotating fermions, except that this is still rather ill-defined. In a relativistic theory, an equilibrium ensemble with constant rotation cannot exist since at some large enough radius, the tangential velocity necessarily would exceed the speed of light. In addition, we need to include interactions as well. The lowest Landau level is subject to an index theorem that protects it, so we might expect that interactions do not modify the expressions for the CME. An effective way of studying an interacting gas or, more general, a fluid of chiral fermions is the formalism of hydrodynamics. Hydrodynamics is an effective field theory describing the time evolution of systems near equilibrium[12]. One assumes that locally the system equilibrates fast such that local versions of thermodynamic parameters, the temperature T and the chemical potential μ can be defined. Intuitively, thermodynamics is what is left of a system if we forget all the details that can be forgotten. That means we know nothing but the indestructible conserved charges, like energy and U(1) charges. Now, we promote these to local notions of energy density and charge density. In relativistic systems we, therefore, need a four vector J^μ

[12] Although the demanding closeness to equilibrium is probably too restricted. A modern point of view is that systems even relatively far from equilibrium can be well-described by hydrodynamics. See Heller's contribution to this volume [35].

to describe the dynamics of the U(1) charge and a symmetric second rank tensor $T^{\mu\nu}$ to describe the dynamics of the energy and momentum densities. Let us study energy-momentum conservation in quantum field theory. We assume a quantum action depending on a source field coupled to the energy-momentum tensor. This source field is the metric. If there are no diffeomorphism anomalies, this quantum action should be diffeomorphism invariant such that

$$\int \mathrm{d}^4x \sqrt{-g}\left[\frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}}\nabla_\mu\epsilon_\nu + \frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta A_\mu}\left(\partial_\mu\left(\xi^\lambda A_\lambda\right) + \xi^\lambda F_{\lambda\mu}\right)\right] = 0\,, \tag{5.1}$$

where we used that the metric and gauge field transforming under diffeomorphisms as

$$\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\nu\,, \tag{5.2}$$

$$\delta A_\mu = \partial_\mu\left(\xi^\lambda A_\lambda\right) + \xi^\lambda F_{\lambda\mu}\,, \tag{5.3}$$

where the vector field ξ^μ is the generator of the diffeomorphism. Furthermore, we identify $\frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}} = T^{\mu\nu}$ and $\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta A_\mu} = \mathcal{J}^\mu$. Since ξ^λ is arbitrary, we infer that the energy-momentum tensor has to fulfill

$$\nabla_\mu T^{\mu\nu} = F^{\nu}\;\mu\mathcal{J}^\mu - A^\nu\nabla_\mu\mathcal{J}^\mu\,. \tag{5.4}$$

So the energy-momentum tensor is naturally symmetric and conserved up to the terms on the right-hand side. The first term represents the injection of energy and momentum by the external electric and magnetic fields. The second term vanishes if the current is conserved. When an anomaly is present, it does contribute to the energy-momentum conservation. Now, we use our knowledge about anomalies gained in the previous sections. They come as consistent and as covariant ones, and the covariant one couples only to the external field strengths but not to the external vector field A_μ. Indeed, if we write the consistent current as covariant current plus a Bardeen–Zumino polynomial, the term proportional to the anomaly in (5.4) vanishes. So energy-momentum and charge conservation laws in terms of the covariant current takes the simpler form

$$\partial_\mu T^{\mu\nu} = F^{\nu}{}_{\mu}J^\mu\,, \tag{5.5}$$

$$\partial_\mu J^\mu = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}\,, \tag{5.6}$$

where we have also set the metric to the flat one. These are already the equations of motion for hydrodynamic effective field theory. The five equations can determine five dynamical variables which are commonly parametrized at the local temperature T, chemical potential μ and the fluid velocity u_μ obeying $u^2 = 1$. Energy-momentum tensor and charge current are now expressed in terms of the local fluid variables in a derivative expansion (see [36] for a recent review of relativistic hydrodynamics)

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \dots\,, \tag{5.7}$$

$$J^{\mu} = J^{\mu}_{(0)} + J^{\mu}_{(1)} + \dots \tag{5.8}$$

To zeroth order in derivatives,

$$T^{\mu\nu}_{(0)} = (\varepsilon + p) u^\mu u^\nu - p\, \eta^{\mu\nu}\,, \tag{5.9}$$

$$J^{\mu}_{(0)} = \rho\, u^\mu\,. \tag{5.10}$$

The energy density, pressure and charge density are computed via thermodynamic relations from the free energy $F(T, \mu)$ defined at equilibrium.

To first order in derivatives, one has to deal with a large set of ambiguities since redefinitions of the local temperature, chemical potential and fluid velocity of the form of $T \to T + \delta T$ *etc.* compete with the terms at higher order in derivatives. These ambiguities are fixed by a choice of frame. One of the standard frames is the so-called Landau frame, which is defined by demanding $T^{\mu\nu} u_\mu = \varepsilon u^\mu$ to all orders. It is also useful to define the projector transverse to the fluid velocity $P^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu$. Another important point is that for the covariant current, only covariant field strengths can enter the expression. It is useful to introduce electric and magnetic fields in the local rest frame $B^\nu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} u_\mu F_{\rho\lambda}$ and $E^\mu = F^{\mu\nu} u_\nu$ and the local vorticity $\omega^\nu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda} u_\mu \partial_\rho u_\lambda$. The first order terms then take the form of

$$T^{\mu\nu}_{(1)} = -\eta P^{\mu\alpha} P^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - 2/3 \eta_{\alpha\beta} \partial_\lambda u^\lambda \right) - \zeta P^{\alpha\beta} \partial_\lambda u^\lambda\,, \tag{5.11}$$

$$J^{\mu}_{(1)} = -\sigma T P^{\mu\alpha} \partial_\alpha \left(\frac{\mu}{T} \right) + \sigma E^\mu + \xi_B B^\mu + \xi_\omega \omega^\mu\,. \tag{5.12}$$

Here, η is the shear viscosity, ζ is the bulk viscosity and σ is the electric conductivity. The new coefficients ξ_B and ξ_ω give room for response to external magnetic and electric fields. One more important point about hydrodynamics is that it also includes a local form of the second law of thermodynamics. One needs to define an entropy current with a non-negative divergence. Again, the entropy current can be built up in a derivative expansion. It takes the form of

$$S^\mu = s u^\mu - \frac{\mu}{T} J^\mu_{(1)} + D_B B^\mu + D_\omega \omega^\mu\,, \tag{5.13}$$

where s is the local entropy density and D_B and D_ω new response coefficients. Now, one demands

$$\partial_\mu S^\mu \geq 0 \tag{5.14}$$

and this constrains the transport coefficients in such a way that $\eta, \zeta, \sigma \geq 0$. Surprisingly, this constraint is much stronger for response coefficients due to magnetic field or rotation. In [37, 38], it was shown that these coefficients are almost completely determined by the second law (5.14) up to some undetermined integration constant γ

$$\xi_B = \frac{1}{4\pi^2}\left(\mu - \frac{1}{2}\frac{\rho}{\varepsilon+p}\left(\mu^2+\gamma T^2\right)\right)\,, \tag{5.15}$$

$$\xi_\omega = \frac{1}{4\pi^2}\left(\mu^2 + \gamma T^2 - \frac{2}{3}\frac{\rho}{\varepsilon+p}\left(\mu^3+3\gamma\mu T^2\right)\right)\,, \tag{5.16}$$

$$D_B = \frac{1}{8\pi^2}\frac{\mu^2}{T} + \gamma T\,, \tag{5.17}$$

$$D_\omega = \frac{1}{12\pi^2}\frac{\mu^3}{T} + 2\gamma\mu T\,. \tag{5.18}$$

Furthermore, the new coefficients do not contribute to entropy production, *i.e.* they describe dissipationless transport. If we compare these with the expressions (4.12), (4.13), we see some similarity but also differences, the most striking is that there is no energy current. This can be traced back to the underlying choice of frame. We can go to another frame by redefining the four velocity such that the terms depending on charge ρ, energy ϵ and pressure p vanish. This can be done by redefining $u^\mu \to u^\mu + \delta u^\mu$, where

$$\delta u^\mu = \frac{1}{8\pi^2}\frac{1}{\varepsilon+p}\left[\left(\mu^2+\gamma T^2\right)B^\mu + \frac{4}{3}\left(\mu^3+3\gamma\mu T^2\right)\omega^\mu\right]\,. \tag{5.19}$$

In this frame, the anomaly related transport coefficients take the form of [39]

$$T^{\mu\nu} = \sigma_{\varepsilon,B}\left(u^\mu B^\nu + u^\nu B^\mu\right) + \sigma_{\varepsilon,\omega}\left(u^\mu\omega^\nu + u^\nu\omega^\mu\right)\,, \tag{5.20}$$

$$J^\mu = \sigma_B B^\mu + \sigma_\omega \omega\,. \tag{5.21}$$

If we expand for small velocities $u^\mu = (1, \vec{v})$ to first order in $\vec{v}$, we can identify the σ coefficients with (4.5), (4.7), (4.10), (4.11). This also fixes the so far undetermined constant $\gamma = \frac{\pi^2}{3}$.

The frame has a very nice physical interpretation [40]. In the presence of anomalies, the chiral magnetic and chiral vortical effects give rise to dissipationless charge and energy flows. In the new frame, the four velocity u^μ parametrizes then only the dissipative "normal" flow. If one allows momentum to relax, *e.g.* by placing an obstacle such as a heavy impurity into the flow, then on long time scales all the normal flow will vanish described by $u^\mu = 0$, while charge and momentum flow induced by CME and CVE are still present.

An alternative way of deriving the transport coefficients of a fluid subject to anomalies is the effective action approach [41]. Higher orders in derivatives have been studied in [42, 43].

The entropy argument fixes the dependence of the anomalous transport coefficients on the chemical potentials but not their temperature dependence. On the other hand, we have seen already that the temperature dependence naturally has the gravitational anomaly coefficient attached to it. Since the gravitational anomaly is fourth order in derivatives and hydrodynamics is developed only to the one derivative level here, there is no natural way it can enter here. A more general argument based on geometry was presented in [44]. Probably the most straightforward way of demonstrating a direct relationship between the gravitational anomaly and the temperature dependence of anomalous transport coefficients comes from holography as we will review below.

6. Holography

The AdS/CFT correspondence or holographic correspondence [45] has developed over the last years into a very useful and powerful tool for studying strongly coupled field theories at finite temperature and density. Recent useful reviews are [46, 47].

Before discussing how holography can be used to gain insight into anomalous transport, we very briefly review the basics of the AdS/CFT correspondence.

The origin of the AdS/CFT correspondence is the duality between type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}_5$ and $\mathcal{N} = 4$ supersymmetric gauge theory. The $\mathcal{N} = 4$ supersymmetric gauge theory is a non-Abelian, four dimensional quantum field theory whose field content consists of six scalars, four Majorana fermions and a vector field. They all transform under the adjoint representation of the gauge group $\mathrm{SU}(N)$. It features four supersymmetries and this fixes all the couplings between the different fields. As it is a gauge theory, physical observables are gauge invariant operators such as $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$. The global symmetry group $\mathrm{SO}(6)$ acts on the scalars and the fermions (in the $\mathrm{SU}(4)$ spin representation of $\mathrm{SO}(6)$).

The dual theory is a theory of gravity (this is what type IIB string theory is) but living in quite a few more dimensions, ten as opposed to the four the field theory knows about. But five of these ten are easily got rid of: the isometries of the S_5 part of the geometry form $\mathrm{SO}(6)$. The S_5 is the geometric realization of what appears as an internal, global symmetry group in the field theory.

The field theory is characterized by two parameters, the gauge coupling g_{YM} and the rank of the gauge group N. The dual string theory has a string coupling g_{s} (the amplitude for a string to split in two) and a fundamental length scale l_s, the string scale. Furthermore, the geometry is determined by a scale L determining the curvature of the AdS_5 as $R = -20/L^2$. The AdS/CFT correspondence relates these parameters in the following way:

$$g_{\mathrm{YM}}^2 N \propto \frac{L^4}{l_s^4}, \tag{6.1}$$

$$1/N \propto g_s. \tag{6.2}$$

The AdS/CFT correspondence is, therefore, a strong weak coupling duality: for weak curvature, we have large L and, therefore, also large 't Hooft coupling. In this regime of weak curvature, stringy effects are negligible and we can approximate the string theory by type IIB supergravity. If we furthermore take the rank of the gauge group N to be very large, we can also neglect quantum loop effects and end up with classical supergravity! This is the form of the correspondence most useful for the applications to many-body physics: classical (super)gravity on $(d+1)$ dimensional anti-de Sitter space.

Based on this original example, we can conjecture that every theory of gravity plus some suitably chosen matter fields on $(d+1)$ dimensional anti-de Sitter space is a dual to a certain quantum field theory in d dimensions. In fact, we might even be a bit more brave and delete the words "dual to a" in the previous phrase. This is the point of view taken in the applications of the AdS/CFT correspondence to the world of solid state physics. The additional matter fields are then chosen to reflect a particular symmetry content of the underlying quantum field theoretical system one is interested in. Having this in mind, we will forget from now on some of the seemingly non-essential[13] ingredients, such as supersymmetry, string theory and extra dimensions in form of the S_5.

For the applications to quantum field theory it is most useful to write the AdS metric in the form of

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{L^2}{r^2}\mathrm{d}r^2. \tag{6.3}$$

The space on which the dual quantum field theory lives is recovered by taking the limit $\mathrm{d}s^2_{\mathrm{QFT}} = \lim_{r\to\infty} r^{-2}\mathrm{d}s^2$. This is why sometimes it is said that the dual field theory lives on the "boundary" of AdS and why the AdS/CFT correspondence is also referred to as "holography". The physical interpretation of the holographic direction is that it represents an energy scale. We can identify the high-energy UV limit in the field theory with the $r \to \infty$ limit in the AdS geometry, whereas the low-energy IR limit is $r \to 0$.

[13] Note, however, that non-supersymmetric AdS solutions to string theory have been recently conjectured to be unstable [48].

The asymptotic behavior of the fields in AdS has the form of

$$\Phi = r^{-\Delta_-}\left(\Phi_0(x) + O\left(r^{-2}\right)\right) + r^{-\Delta_+}\left(\Phi_1(x) + O\left(r^{-2}\right)\right) . \tag{6.4}$$

The exponents $\Delta_\pm$ obey $\Delta_- < \Delta_+$ and depend on the nature of the field, *e.g.* for a scalar field of mass m, they are $\Delta_\pm = \frac{1}{2}(\mathrm{d} \pm \sqrt{\mathrm{d}^2 + 4m^2L^2})$.

We now would have to evaluate the path integral over the fields in AdS keeping the boundary values $\Phi_0(x)$ fixed. The result is a functional depending on the boundary fields $\Phi_0(x)$. Now, the boundary field $\Phi_0(x)$ is interpreted as the source $J(x)$ that couples to a (gauge invariant) operator $\mathcal{O}(x)$ of conformal dimension Δ_+ in the field theory

$$Z[J] = \int\limits_{\Phi_0 = J} \mathrm{d}\Phi\, \exp(-iS[\Phi])\,. \tag{6.5}$$

Connected Green's functions of gauge invariant operators in the quantum field theory can now be generated by functional differentiation with respect to the sources

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle = \frac{\delta^n \log Z}{\delta J_1(x_1) \cdots \delta J_n(x_n)}\,. \tag{6.6}$$

In the limit in which the gravity theory becomes classical, *i.e.* the large N and large coupling $g^2_{\mathrm{YM}}N$ limit, the path integral is dominated by the classical solutions to the field equations and $\log Z$ can be replaced by the classical action evaluated on a solution of the field equations. In this case, the coefficient $\Phi_1(x)$ of the asymptotic expansion (6.4) is the vacuum expectation value of the operator sourced by Φ_0

$$\langle \mathcal{O}(x)\rangle \propto \Phi_1(x)\,. \tag{6.7}$$

To explicitly compute $\Phi_1(x)$, we need to supply a second boundary condition, so far we have fixed only the asymptotic value Φ_0. For time-independent solutions, we demand regularity in the interior of the (possibly only asymptotically) AdS space. For time-dependent perturbation, one needs to impose the more general "infalling" boundary conditions.

The holographic dictionary relates in this way (gauge-invariant) local operators to fields in the bulk of AdS and can be summarized in Table I.

Anomalies can be incorporated rather easily: they are represented by a five-dimensional Chern–Simons term. Therefore, to study anomalous transport with the means of holography, the following model is a good starting point [24]:

$$\begin{aligned} S \;=\; \int \mathrm{d}^5x\,\sqrt{-g}\,\Big[& R + 2\Lambda - \frac{1}{4}F_{MN}F^{MN} \\ & + \epsilon^{MNPQR}A_M\left(\frac{\alpha}{3}F_{NP}F_{QR} + \lambda R^A_{BNP}R^B_{AQR}\right)\Big]\,. \end{aligned} \tag{6.8}$$

TABLE I

The holographic dictionary.

Gravity	QFT
$d+1$ dimensions	d dimensions
holographic direction	RG scale
strongly coupled	weakly coupled
scalar field	scalar operator
gauge field	conserved current
metric tensor	energy-momentum tensor

The model contains only one five-dimensional U(1) gauge field. In the dual field theory, we consider only one anomalous symmetry. The anomaly is of $\mathrm{U}(1)^3$ type, of a single chiral fermion and the mixed chiral-gravitational anomaly. We note that action (6.8) has diffeomorphism symmetry and that the U(1) symmetry is preserved up to a boundary term. This boundary term takes precisely the form of the chiral and chiral-gravitational anomaly. We can also derive the operators J^μ and $T^{\mu\nu}$ as the variations of the on-shell action with respect to the boundary values of the gauge field [43] and the metric

$$\mathcal{J}^\mu = \sqrt{-g}\left(F^{r\mu} + \frac{4\alpha}{3}\epsilon^{\mu\nu\rho\lambda}A_\nu F_{\rho\lambda}\right) , \tag{6.9}$$

$$T^{\mu\nu} = \sqrt{-g}\left(\frac{1}{2}K^{\mu\nu} - \frac{1}{2}K\gamma^{\mu\nu} + 2\lambda\epsilon^{(\mu\rho\lambda\sigma}\left(\frac{1}{2}F_{\rho\lambda}\hat{R}^{\nu)}_{\sigma} + D_\delta(A_\rho \hat{R}^{\delta\nu)}{}_{\lambda\sigma})\right)\right) . \tag{6.10}$$

Here, Greek indices are four-dimensional boundary indices, $\hat{R}$ the intrinsic four-dimensional curvature of the boundary, $\gamma_{\mu\nu}$ the boundary metric, D_μ the covariant (with respect other $\gamma_{\mu\nu}$) boundary derivative. The divergences are

$$D_\mu \mathcal{J}^\mu = \epsilon^{\mu\nu\rho\lambda}\left(\frac{\kappa}{3}F_{\mu\nu}F_{\rho\lambda} + \lambda\hat{R}^\alpha{}_{\beta\mu\nu}\hat{R}^\beta{}_{\alpha\rho\lambda}\right) , \tag{6.11}$$

$$D_\mu T^{\mu\nu} = F^{\nu\mu}\mathcal{J}_\mu + A^\nu D_\mu \mathcal{J}^\mu , \tag{6.12}$$

which are nothing but the consistent form of the chiral and chiral-gravitational anomalies.

Now, we have our holographic theory but we need a state with chemical potential μ and temperature T. This is represented by a charged black hole solution. The five-dimensional metric takes the form of

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2 f(r)} - r^2 f(r)\mathrm{d}t^2 + r^2\mathrm{d}\vec{x}^{\,2} . \tag{6.13}$$

We assume that $\lim_{r\to\infty} f(r) = 1$ (the spacetime is asymptotically AdS) and that at some finite value $f(r_{\rm H}) = 0$, there is a non-degenerate horizon with temperature

$$T = \frac{r_{\rm H}^2 f'(r_{\rm H})}{4\pi}\,. \tag{6.14}$$

The chemical potential is related to a non-trivial profile of the temporal component of the gauge field. In its most elementary definition, the chemical potential is the energy that is needed to add one unit of charge to the thermal ensemble. The thermal ensemble is represented by the black hole and to add a unit of charge is to take this charge from infinity to behind the horizon. The chemical potential can, therefore, be identified with the difference of the potential energy at the boundary and at the horizon

$$\mu = A_0(\infty) - A_0(r_{\rm H})\,. \tag{6.15}$$

A subtlety related to the value of the gauge field on the horizon [23] should be mentioned. If one defines the equilibrium state as the one with a smooth Euclidean geometry, then we need also to impose $A_0(r_{\rm H}) = 0$. This constraint comes from the fact that at the Euclidean time $t = i\tau$ the black hole geometry simply ends at $r = r_{\rm H}$, there is no interior and the geometry is smooth only if τ is periodic with periodicity $1/T$. If $A_0(r_{\rm H}) \neq 0$, an integral $\oint A_0 d\tau$ would be non-vanishing for an infinitely small circle and indicate a field strength of delta function support at $r = r_{\rm H}$.

The equations of motion for $f(r)$ and $A_0(r)$ are

$$\left(r^3 A_0'(r)\right)' = 0\,, \tag{6.16}$$

$$f'(r) + \frac{4}{r} f(r) - \frac{4}{r} + \frac{(A_0'(r))^2}{6r} = 0\,. \tag{6.17}$$

The solution for the gauge field is $A_0(r) = \alpha - \frac{\rho}{2r^2}$. The chemical potential is $\rho = 2r_{\rm H}^2\mu$. If we impose vanishing of the gauge field at the horizon, we also have $\alpha = \mu$. The solution for the metric component is $f(r) = 1 - \frac{M}{r^4} + \frac{\rho^2}{12r^6}$.

Before we go on and compute the chiral magnetic and chiral vortical effects, we note that the current (6.9) energy-momentum tensor (6.10) has a trivial part that is determined solely by the boundary values of the fields. Since we want to have a flat boundary metric, these terms vanish for the energy-momentum tensor. For the current, the corresponding Chern–Simons term is just the Bardeen–Zumino polynomial that relates covariant and consistent current. In holography, the UV origin of the Bardeen–Zumino polynomial is manifest as it is a pure boundary term. We conclude that the holographic form of the covariant current is

$$J^\mu_{\rm cov} = \sqrt{-g} F^{r\mu}|_{r\to\infty}\,. \tag{6.18}$$

Now, we are ready to compute chiral magnetic and chiral vortical effects. In order to do so we introduce appropriate sources. For the magnetic field, it is straightforward we need to impose the boundary condition

$$A_y|_{r=\infty} = Bx \tag{6.19}$$

or any equivalent gauge.

For the chiral vortical effect, we need to implement boundary conditions encoding rotation. That can be done via a metric perturbation. We remember that metric components with mixed time and space indices represent metric fields generated by rotating bodies and they induce by themselves rotation via the frame dragging effect. In fact, in the formalism of gravito-electromagnetism rotation is represented via the gravito-magnetic field $\mathcal{B}_i = \epsilon_{ijk}\partial_j g_{0k}$. A simple way of seeing this is via a fluid picture. The fluid at rest can be defined by the contra-variant four vector $u^\mu = (1,0,0,0)$, while the vorticity is defined through the co-variant components $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}u_\nu\partial_\rho u_\lambda$. Therefore, for small velocities, vorticity and gravito-magnetic field are related as $2\vec{\omega} = \vec{\mathcal{B}}$.

This motivates us to chose the metric perturbations

$$g_{ty} = \mathcal{B}xf(r)r^2\,. \tag{6.20}$$

It turns out that the gravito-magnetic field also induces a gauge field

$$A_y = \mathcal{B}x\frac{\rho}{2r^2}\,. \tag{6.21}$$

Now, the linear response due to B and $\mathcal{B}$ can be computed via the equations of motion. We will concentrate on the response in the current. The corresponding equation of motion is

$$\left(r^3 f(r)a_z'(r)\right)' - \frac{\rho}{2}h_z'^t(r) - 8\alpha B\frac{\rho}{r^3} - \alpha\mathcal{B}\frac{4\rho^2}{r^5} + \lambda\mathcal{B}\left[2r^4 f'(r)^2\right]' = 0\,, \tag{6.22}$$

where the prime denotes $\partial/\partial r$. We note that the covariant current in the z-direction is $J_z = \lim_{r\to\infty} r^3 f a_z'$. All the terms in this equation are total derivatives. This tells us that

$$\left(r^3 f(r)a_z'(r)\right) - \frac{\rho}{2}h_z^t(r) + 4\alpha B\frac{\rho}{r^2} + \alpha\mathcal{B}\frac{\rho^2}{r^4} + \lambda\mathcal{B}2r^4 f'(r)^2 = \text{const}\,. \tag{6.23}$$

Evaluating this on the horizon assuming regularity of a_z there and demanding also that the metric perturbation h_z^t vanishes at the horizon allows us to compute the constant on the right-hand side. Then we can evaluate this equation on the boundary where all terms except the constant on the right-hand side and the current vanish (the metric perturbation has to fall off at least as $1/r^4$). This gives the result

$$J_z = 8\alpha\mu B_z + \left(4\alpha\mu^2 + 32\pi^2\lambda T^2\right)\mathcal{B}_z\,. \tag{6.24}$$

Comparing (6.11) to the anomalies of a single chiral fermion, we can identify $\alpha = \frac{1}{32\pi^2}$ and $\lambda = \frac{1}{768\pi^2}$. Then these are the same results as in the free field theory! The relation between the chiral-gravitational anomaly and the temperature dependence of the chiral vortical effect is very direct in holography. It is interesting to note how holography manages to correct the mismatch in derivatives. It is still true, of course, that the gravitational anomaly and also the five-dimensional gravitational Chern–Simons term are higher order in derivatives. As can be seen in equation (6.22), the term proportional to λ is indeed of fourth order. But the trick of holography is that three of these derivatives are along the holographic direction and only one is along the actual spacetime. Non-renormalization of the CME and CVE have been discussed in holography in [49–51], using field theory in [52, 53], in a model-independent way based on a mixed hydrodynamic/geometric approach [44] and a more general relation to gravitational anomalies has also been investigated in [54].

We need to discuss one more point: we have set the horizon value of the metric perturbation $h^t_z(r)$ to zero. More generally, we can leave it arbitrary and fix it later. It turns out that this integration constant corresponds to the choice of frame that we discussed in hydrodynamics, *e.g.* we could have used it to set the response in the energy-momentum tensor to zero, then we could recover the expression of chiral vortical and chiral magnetic effect in the Landau frame.

7. Anomalous transport and Weyl semi-metals

As stated in the introduction, anomalous transport phenomena play an important role in many branches of physics. One of the most interesting and active fields of applications of anomalous transport phenomena is condensed matter theory and experiment. The electronics of a new class of materials known as Dirac and Weyl (semi-)metals [18, 55, 56] (or, more generally, topological metals) is governed by the Weyl equation. In this chapter, we will briefly review the physics of Weyl semi-metals from the point of view of anomalous transport theory.

7.1. Weyl semi-metals

Weyl semi-metals are materials in which two bands cross at isolated points in the Brillouin zone. In Fig. 7, we sketch a typical situation arising

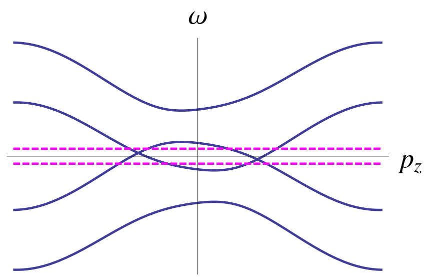

Fig. 7. The figure sketches the band structure of a Weyl semi-metal derived from a tight-binding model. The energy bands are periodic along the Brillouin zone of which a section with $p_x = p_y = 0$ is shown. If the Fermi energy lies near the band crossing points, the effective low-energy electron dynamics is well-described by the Weyl equations of opposite chirality. This makes the theory of anomalous transport applicable.

from a tight-binding model of a Weyl semi-metal. If the Fermi level is at or near the band crossing point, the effective Hamiltonian near such a crossing point is

$$H_{\mathrm{eff}} = \pm \vec{\sigma}\vec{k} \,, \tag{7.1}$$

where $\vec{k} = (k_x, k_y, k_z)$ measures the momentum relative to the position of the band crossing point. These are the Hamiltonians for a Weyl fermions of either left- or right-handed chirality. For simplicity, we have assumed rotational symmetry around the band crossing point. The Nielson–Ninomiya theorem [57] makes the band crossing points to come always in pairs with opposite choices of signs in (7.1), *i.e.* with opposite chiralities. In general, there might be many more band crossing points but clearly the simplest situation is one with only two. The effective low-energy degrees of freedom are then a left- and right-handed Weyl fermion. A more field theoretic way of describing this, is the Dirac equation

$$\left(i\partial\!\!\!/ - \gamma_5 b\!\!\!/\right) \Psi = 0 \,. \tag{7.2}$$

The four vector b_μ shifts the momenta and frequencies of left- and right-handed fermions

$$\omega_{\mathrm{L,R}} = \pm b_0 \pm \sqrt{\left(\vec{p} - \vec{b}\right)^2} \,. \tag{7.3}$$

For a start, this will serve as a model for the physics of Weyl semi-metals up to one more ingredient: in a crystal, the Brilloin zone is bounded and periodic. This has two effects we need to account for if we want to use model (7.2): one is that the linear dispersion cannot be extended to arbitrary high energies and related to it chiral symmetry is not exact even at tree level. The fact that the bands are really periodic means that there is no cutoff at infinite energy. States can only move along the band in a periodic fashion[14] but can never disappear. For our purpose, this implies that we have to supplement our field theoretical model with cutoff preserving the total number of states. More precisely, an anomaly in the total charge current stemming from both left-handed and right-handed Weyl cones is not allowed. We know already that in the quantum field theoretical description, this singles out the conserved consistent current as the unique candidate for the electric current. Furthermore, we need to take into account that the chiral symmetry is only an accidental one. Even without anomaly, it cannot be preserved for arbitrary momenta and energies along the band dispersion. Thus, the description in terms of Weyl fermions is an approximate one and

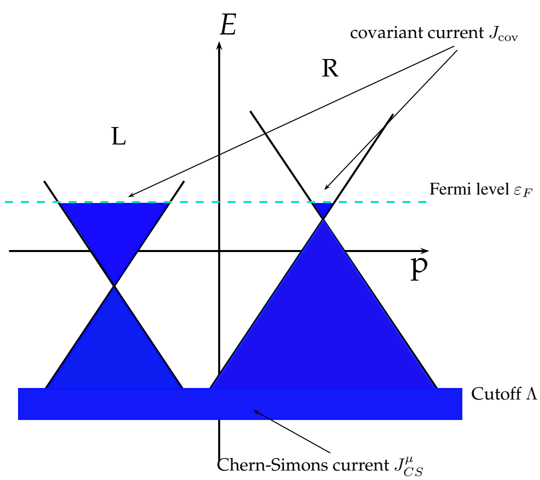

Fig. 8. Local (in the Brillouin zone) model of the electronic band structure of a Weyl semi-metal. Inter-valley scattering leads to equilibration of the Fermi levels of left- and right-handed Weyl cones. The local model has to be supplemented by a charge preserving cutoff. The covariant current is the current that stems from quasiparticles excitations near the Fermi surface. The total electric current has another component given by the Bardeen–Zumino polynomial or Chern–Simons current generated at the cutoff and guaranteeing charge conservation. From the point of view of the quantum field theoretical model, the chemical potentials have to be measured with respect to the tips of the Weyl cones. So in the equilibrium, there is a non-vanishing axial chemical potential $\mu_5 = b_0$ as long as the Weyl cones are shifted in energy. This shift in energy is given by the temporal component b_0 in the local model (7.2). On the other hand, b_0 acts precisely as temporal component of an axial vector field $A^5_0 = b_0 = \mu_5$.

[14] As electrons indeed do in very pure samples upon switching on an electric fields. The produced current shows oscillating behavior, the Bloch oscillations.

there is always a non-vanishing amplitude for an electron to scatter out of the left-handed Weyl cones and into the right-handed Weyl cone. This process is known as inter-valley scattering, it leads to an effective decay time for axial charge. On the other hand, scattering processes that leave the electron within its Weyl cone are called intra-valley scattering. It is also usual assumed that the scattering rate for inter-valley scattering is much smaller than the scattering rate for intra-valley scattering. Inter-valley scattering will always lead to equilibration of the Fermi surfaces of left-handed and right-handed Weyl cones. Therefore, the equilibrium situation can be sketched in Fig. 8.

7.2. Applying anomalous transport theory to Weyl semi-metals

7.2.1. CME

While the actual Fermi energies are equal for both Weyl cones in the local Dirac model of (7.2), we have to assign different chemical potentials to them. Intuitively, the chemical potential measures the size of the Fermi surface and it is clear that as long as the parameter b_0 is different from zero, the size of the Fermi surfaces of left- and right-handed Weyl fermions will be different in equilibrium. This means that the equilibrium state is described by a non-vanishing axial chemical potential $\mu_5 = b_0$. On the other hand, this parameter is nothing else than the temporal component of an axial vector field $A^5_0 = b_0$. Now, we can compute what happens if we switch on a magnetic field. The current due to the CME has to be calculated as

$$\vec{J} = \frac{\mu_5 - A^5_0}{2\pi^2}\vec{B} = 0\,. \tag{7.4}$$

So there is no chiral magnetic effect in equilibrium. This is in accordance with explicit numerical simulations of the CME and, more generally, with the so-called Bloch theorem [58–60] that forbids a non-vanishing equilibrium expectation value for a precisely conserved current. Instead of insisting on a description in terms of relativistic Weyl fermions, one might also stick to the convention that energies should be measured with respect to a common reference for both Weyl cones. Then there is simply no difference in the energy one needs to add an electron to the left-handed or right-handed Fermi surface. In this sense, there is no true axial chemical potential [61].

In order to activate the chiral magnetic effect, there are in principle two possibilities. One can either try to manipulate the relative occupation numbers in the Weyl cones or one can try to change the separation in energy of the tips of the Weyl cones. Both options should give a CME signal. Let us explore the first one. The key is to change the equilibrium state $\mu_5 = b_0$. A conventional way of doing this is by using the axial anomaly itself. We switch on parallel electric and magnetic fields. The change in the rate of axial charge is then [34, 62]

$$\dot{\rho}_5 = \frac{1}{2\pi^2}\vec{E}\cdot\vec{B} - \frac{1}{\tau_5}\left(\rho_5 - \rho_5^{(0)}\right) . \tag{7.5}$$

Here, we have also included the axial charge relaxation due to inter-valley scattering. The equilibrium axial charge density can be computed from the equation of state $\rho_5 = (\partial F(\mu_5, \mu, T))/(\partial\mu_5)$ by inserting $\mu_5 = b_0$. We define the axial susceptibility as $\chi_5 = \partial\rho_5/\partial\mu_5$ and express the axial charge via the axial chemical potential as $\delta\rho_5 = \chi_5\delta\mu_5$. In parallel electric and magnetic fields, a stationary non-equilibrium state will be reached with

$$\delta\mu_5 = \frac{\tau_5}{\chi_5}\frac{\vec{E}\cdot\vec{B}}{2\pi^2} . \tag{7.6}$$

The induced current is then the sum of ohmic and chiral magnetic current

$$\mathcal{J}_i = \sigma E_i + \frac{\delta\mu_5}{2\pi^2}B_i = \left(\sigma\delta_{ij} + \frac{\tau_c}{\chi_5}\frac{B_iB_j}{4\pi^4}\right)E_j , \tag{7.7}$$

where σ is the usual ohmic conductivity in absence of the magnetic field. If the axial symmetry were not explicitly broken, the induced conductivity would be infinite. This reflects the non-dissipative character of the chiral magnetic current[15]. So the chiral magnetic effect manifests itself as an enhancement of the electric conductivity along the magnetic field with a characteristic quadratic dependence on the magnetic field strength[16]. This is the anomaly-induced negative magnetoresistivity along the magnetic field. Signatures compatible with it have been measured in a variety of experiments recently [64–72].

Another option is to change the energy levels of the band touching points. Using a tight-binding model, it has recently been shown that this can be achieved by applying strain [73]. In this case, the current will not be stationary but decay over a time given by the axial charge relaxation rate of around 10^{-9} seconds.

7.2.2. Anomalous Hall effect

Another effect present in our simple model is the quantum anomalous Hall effect [74–76]. A current perpendicular to an applied electric field. In

[15] In a hydrodynamic approach, it has been shown that there are additional terms in the case of finite axial chemical potential [63].

[16] Large magnetic fields can induce anisotropies in σ, and they can also change the values of the relaxation times and the axial chemical potential. For example, for large magnetic fields, the electrons will be mostly in the lowest Landau level and then the axial susceptibility will be itself proportional to B.

our quantum field theoretical setup, it is exclusively due to the Bardeen–Zumino polynomial and takes the form of

$$\vec{\mathcal{J}} = \frac{1}{2\pi^2}\vec{E}\times\vec{b}\,. \tag{7.8}$$

It follows from the Chern–Simons current in (2.36a) by identifying $\vec{A}^5=\vec{b}$. We note that the parameter $\vec{b}$ breaks time reversal symmetry. In time reversal invariant Weyl semi-metals, there are several pairs of Weyl nodes such that the vector sum $\sum\vec{b}_i = 0$ and the axial anomaly induced Hall effect is absent. The Bardeen–Zumino polynomial also implies an induced charge in magnetic field of the form of

$$\mathcal{J}^0 = \frac{1}{2\pi^2}\vec{b}\cdot\vec{B}\,, \tag{7.9}$$

a three-dimensional generalization of what is known as the Streda formula in condensed matter theory [77].

7.2.3. Fermi arcs

From the identification $\vec{A}_5 = \vec{b}$, we can learn a lot more. First, we note that a Weyl fermion cannot be given a mass in a gauge invariant way. This is what makes the Weyl nodes stable in a Weyl semi-metal. On the other hand, we know that outside any finite piece of material, there are no low-energy fermions, the vacuum is a (trivially) gapped state in which the electrons are massive. The only way to make the Weyl fermions massive is to bring them together at the same point in momentum space and switching on a Dirac mass term. This means that the vector $\vec{b}$ must go to zero as one crosses the edges of the material. On the edge of a Weyl semi-metal, there is necessarily a gradient of $\vec{b}$. This is nothing else but an axial magnetic field $\vec{B}_5 = \nabla\times\vec{A}_5$ [78]. An axial magnetic field acts like a usual magnetic field but with opposite signs for Weyl fermions of opposite chirality. Lets make a simple model in which there is a sharp edge at say $x = 0$ and $\vec{b} = b\,\Theta(y)\,\hat{e}_z$. Thus, there is a strong axial magnetic field localized at the edge $\vec{B}_5 = b\delta(y)\hat{e}_x$. The Weyl fermions will have zero modes localized at the axial magnetic field lines (the states in the lowest Landau level). In a usual magnetic field, these zero modes are $1+1$ dimensional chiral fermions of opposite chirality but in the axial magnetic field, the zero modes which have the same chirality. The local density of states is the one of the lowest Landau level, $B_5/(2\pi)$ for each Weyl cone. So, in total, we predict localized chiral edge states with linear dispersion only along the axial magnetic field lines and degeneracy b/π. These are the famous Fermi arcs [55]. In actual materials, Fermi arcs turn out not to be just straight lines and this has been connected to boundary conditions for the Weyl fermions in [79]. It would be interesting to see how the simple picture of the axial magnetic field can be modified to catch these subtleties.

We can also apply anomalous transport formulas to the physics of Fermi arcs. Let us assume we have a doped Weyl semi-metal. Then we can employ a version of the chiral magnetic effect for axial magnetic fields

$$\vec{\mathcal{J}} = \frac{\mu}{2\pi^2}\vec{B}_5 \,. \tag{7.10}$$

This is an edge current. We can also derive the anomalous Hall effect from this. We assume that there are two edges at say $y = 0$ and at $y = L$, and in between a WSM with $b_z > 0$. In this situation, there will be no net current since the edge currents at $y = 0$ and at $y = L$ cancel each other exactly. Let us now inject additional electrons at the edge at $x = 0$. This will rise the local chemical potential there and, therefore, increase the local edge current. Now, there is net current in the form of

$$\mathcal{I}_x = \int \mathrm{d}y\, \mathcal{J}_x = \frac{\Delta_y \mu}{2\pi^2} b_z \,. \tag{7.11}$$

So we get a net current perpendicular to the gradient of the chemical potential. This is very similar to (7.8) except that now the current is localized at the edges, whereas in (7.8) it is better thought of as a bulk current. As usual for the quantum Hall effect, it appears either as edge current or as bulk current depending on whether one pumps charge from one edge to the other via a bulk current generated by an external electric field or one directly injects fermions at the edges.

7.2.4. Thermal Hall effect

Let us study now the energy transport instead of the charge transport. One of the most interesting aspects is that the thermal transport is sensitive to the gravitational anomaly. In particular, in an axial magnetic field, we have

$$\vec{J}_\varepsilon = \left(\frac{\mu^2}{4\pi^2} + \frac{1}{12}T^2\right)\vec{B}_5 \,, \tag{7.12}$$

since in Weyl semi-metals there necessarily exists a strong axial magnetic field at the edge, and we can use this to derive the thermal Hall effect. As before, suppose that there are two edges at $y = 0$ and at $y = L$. In equilibrium, the net energy current vanishes since the currents at opposite edges cancel each other exactly. Let us heat up the edge at $y = L$ to a temperature $T + \Delta T$. Now, there is a net heat current $\vec{J}_Q = \vec{J}_\epsilon - \mu\vec{\mathcal{J}}$ along the x-direction (keeping the chemical potential fixed)

$$J_{Q,x} = \tfrac{1}{6} T \Delta_y T b_z \tag{7.13}$$

from which we can infer the thermal Hall conductivity

$$\kappa_{T,\mathrm{Hall}} = \frac{T}{6}|b|\,. \tag{7.14}$$

Contrary to the Hall effect, this is a pure edge current. The bulk current in the Hall effect was related to the Bardeen–Zumino polynomial that converts the non-conserved covariant current into the conserved consistent current. There is an analogue Bardeen–Zumino current for the consistent energy-momentum tensor. It is (just as the gravitational anomaly itself) of higher order in derivatives. Thus, there is no thermal Hall current in the bulk as a response to a gradient in the temperature.

7.2.5. Axial negative magnetoresistivity

The covariant form of the anomaly also suggests that one can induce negative magnetoresistivity via an axial magnetic field since

$$\dot\rho = \frac{\vec{E}\cdot\vec{B}_5}{2\pi^2} \tag{7.15}$$

combined with the axial magnetic effect (7.10) should give a large enhancement of the conductivity. At first glance, there is seemingly a problem. In the case of the negative magnetoresistivity induced by the usually parallel electric and magnetic fields, the axial charge decays also due to tree level non-conservation. This makes the induced conductivity finite. In the case of the electric charge, this is impossible, because of gauge invariance the electric charge can never decay. This seems to give the unphysical result of an infinite conductivity in parallel electric and axial magnetic fields. The resolution comes from the nature of the axial field. As we noted, it can exist only inside the Weyl semi-metal and has to vanish necessarily in the vacuum outside the material. The net total charge induced by (7.15) is

$$\frac{\mathrm{d}}{\mathrm{d}t}Q = \int\limits_{\Omega} \mathrm{d}^3x\dot\rho = \int\limits_{\Omega} \mathrm{d}^x\epsilon_{ijk}\partial_j A^5_k E_i = \oint\limits_{\partial\Omega} \mathrm{d}S_j\epsilon_{ijk}A^5_k E_i = 0\,, \tag{7.16}$$

where Ω denotes a region of space that contains the Weyl semi-metal. This region can always be taken much larger than the space region occupied by the material. Since the axial vector vanishes on $\partial\Omega$, no net charge can be induced. Every region with a positive covariant anomaly must necessarily be offset by a region with equal amount of negative covariant anomaly. From the point of view of the consistent anomaly, everything that is happening is that a current is created transporting the charge from one region to another. Since now the charge distribution is inhomogeneous, we need to take diffusion into account

$$\dot{\rho} + \vec{\nabla} \cdot \vec{J} = \frac{\vec{E} \cdot \vec{B}_5}{2\pi^2}\,, \tag{7.17}$$

$$\vec{J} = -\sigma \vec{\nabla}\mu + \frac{\mu}{2\pi^2}\vec{B}_5\,, \tag{7.18}$$

or from the point of view of the consistent current where the covariant anomaly is interpreted as an inflow via the anomalous Hall effect

$$\dot{\rho} + \vec{\nabla} \cdot \vec{\mathcal{J}} = 0\,, \tag{7.19}$$

$$\vec{\mathcal{J}} = -\sigma \vec{\nabla}\mu + \frac{\mu}{2\pi^2}\vec{B}_5 - \frac{1}{2\pi^2}\vec{b} \times \vec{E}\,. \tag{7.20}$$

The local growth of charge due to Hall current inflow is eventually counter-balanced due to diffusive charge outflow such that in a stationary situation, $\dot{\rho} = 0$. We also note that these considerations also apply to the case where $\vec{B}_5$ is created in the bulk of the material via strain [80]. Negative axial magnetoresistivity has recently been also discussed in [81, 82].

Let us work out a simple example. We describe the Weyl semi-metal as the region with $\vec{b} = b\hat{e}_z[\Theta(y) - \Theta(y - L)]$. At the edges, there is strong axial magnetic field $\vec{B}_5 = B_5\hat{e}_x$ with $B_5 = b[\delta(y) - \delta(y - L)]$. In an electric field parallel to the axial magnetic field, the stationary solution is

$$\sigma\partial_y^2\mu = -\frac{B_5 E}{2\pi^2}\,, \tag{7.21}$$

which is solved by

$$\mu = -\frac{bE}{\sigma 2\pi^2}\left[y\Theta(y) - (y - L)\Theta(y - L) - \frac{L}{2}\right]. \tag{7.22}$$

Here, $\Delta\mu = \mu(0) - \mu(L) = \frac{LbE}{\sigma 2\pi^2}$ is the Hall voltage. The total current in direction along the electric field is

$$\mathcal{J}_x = \left[\sigma + \frac{Lb^2}{\sigma 8\pi^4}(\delta(y) + \delta(y - L))\right]E\,. \tag{7.23}$$

The conductivity is strongly enhanced at the edges due to the axial magnetic effect. For fixed width L, the enhancement is stronger as the bulk conductivity gets smaller. In the limit of vanishing bulk conductivity, the edges are perfect conductors. This is the limit in which only the edge states contribute to the conductivity. They are chiral fermions which indeed have formally infinite conductivity. On the other hand, for finite bulk conductivity σ, the charge of the edge states can diffuse into the bulk and this is what makes the anomalous enhancement finite[17].

[17] A similar conclusion has been reached in a more microscopic model in [83].

7.3. Chiral collective waves

Another direct consequence of the chiral magnetic effect is the so-called chiral magnetic wave. We combine the chiral magnetic and the chiral separation effect with current conservation (assuming absence of an electric field parallel to the magnetic field). Assuming that vector and chiral charges are linearly related to the chemical potentials and a spatially homogeneous magnetic field, we find

$$\chi_5 \dot{\mu}_5 + \frac{\vec{B}}{2\pi^2} \vec{\nabla} \mu = 0 \,, \tag{7.24}$$

$$\chi \dot{\mu} + \frac{\vec{B}}{2\pi^2} \vec{\nabla} \mu_5 = 0 \,. \tag{7.25}$$

Fourier transforming in space and time, and setting the determinant to zero gives the mode equation

$$\omega^2 4\pi^2 \chi \chi_5 - \left(\vec{k} \cdot \vec{B}\right)^2 = 0 \,. \tag{7.26}$$

This is the chiral magnetic wave. It can be understood as propagating oscillation between axial and vector charge. In the limit of large magnetic field, it follows from (4.2) that its velocity approaches the speed of light. The chiral magnetic wave should be observable as a collective mode of the electron fluid in Weyl semi-metals. It also has an important application in the physics of the quark–gluon plasma, where it leads to a quadrupole moment in the charge distribution of the final hadronic state in non-central heavy-ion collisions. This signal has been theoretically predicted in [9, 84, 85] and observations in experiment compatible with it are reported in [10, 86].

Chiral hydrodynamics has been used to find many more interesting collective modes such as appearance of non-linear Burgers solitary waves due to the chiral vortical effect [87], a chiral vortical wave [88] at non-zero chemical potentials, a chiral heat wave [89, 90], chiral Alfven waves [91].

8. The holographic Weyl semi-metal

So far, we have worked with a very simple field theory model for a (time-reversal breaking) Weyl semi-metal based on the theory of anomalous transport. Holography has been of the highest importance to gain understanding of anomaly induced transport and even to unravel some unexpected relations such as the one between the temperature dependence of the chiral vortical effect and the gravitational anomaly. This motivated work on a holographic model of Weyl semi-metal and the relation between anomalies and transport phenomena in this model [92–94].

The important feature is that the Weyl physics arises in the infrared region. This is unusual from the point of view of high-energy physics where massless fermions are usually thought of as being a good approximation for high-energy processes. A simple field theoretic model of how effective low-energy Weyl physics can arise, is the Lorentz breaking massive Dirac equation

$$\left(i\partial\!\!\!/ - M + \gamma_5 b\!\!\!/\right)\Psi = 0\,. \tag{8.1}$$

This model has been invented to investigate the consequences of Lorentz symmetry breaking in particle physics [95] and as a model for the physics of Weyl semi-metals in [75].

We will concentrate on the case in which the four vector b_μ is purely space-like and without loss of generality we take it to point in the z-direction. At high energies, the mass term is irrelevant and the theory is basically the same as (7.2). If we are interested in the behavior at low energies, there is some interesting non-trivial phase transition. It turns out that as long as $|M| < |\vec{b}|$, the low-energy theory is not gapped. Rather the low-energy theory is given again by (7.2) but with an effective low-energy parameter $b_{\text{eff}} = \sqrt{b^2 - M^2}$. So the low-energy theory is one of massless Weyl fermions despite the fact that there is a mass parameter in the fundamental Lagrangian. On the other hand, for $|M| > |b|$, the theory is gapped with a mass gap of $\Delta = \sqrt{M^2 - b^2}$. So there is a quantum phase transition at $M = b$, in fact, it is a topological phase transition since the topology of momentum space changes from a situation with band crossing points to the one with a gap (see Fig. 9). What is the signature of this topological phase transition? We can argue that it should be the anomalous Hall effect. Since the Hall effect is the response to static and homogeneous fields, it is an IR property and it should be governed by the properties of the IR theory. We already know that a Dirac fermion with a constant axial vector $\vec{A}^5 = \vec{b}$

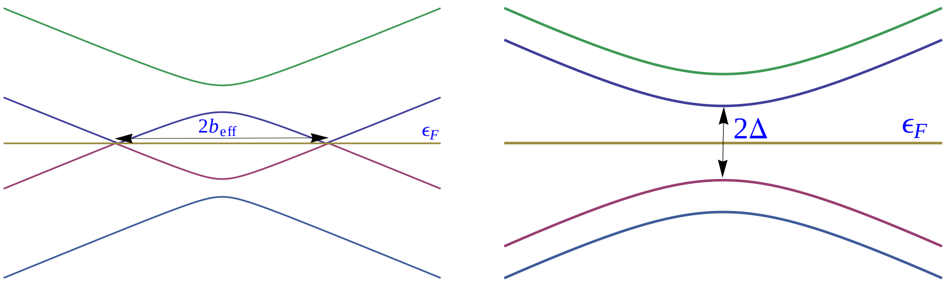

Fig. 9. Left panel: For $b^2 > M^2$, there are two Weyl nodes in the spectrum. They are separated by the distance of $2\sqrt{b^2 - M^2}$ in momentum space. Right panel: For $b^2 < M^2$, the system is gapped with a gap $2\Delta = 2\sqrt{M^2 - b^2}$.

features an anomalous Hall effect (7.8). Since in the IR theory the relevant parameter is $b_{\rm eff}$, we expect that there is an anomalous Hall conductivity of the form of

$$\sigma_{xy} = \frac{\sqrt{b^2 - M^2}}{2\pi^2}\Theta(|b| - |M|)\,. \tag{8.2}$$

The calculation of this Hall conductivity in field theory (even the free one) is not easy and subject to all the renormalization ambiguities we discussed already in the case of the triangle diagrams [96]. Instead of analyzing the field theory any further, we will take it as motivation to write down a holographic model that also has a topological quantum phase transition between a topological state with non-zero Hall conductivity and a topologically trivial state with vanishing Hall conductivity.

Now, we want to construct a holographic model based on the dictionary in Table I. We want to implement the particular symmetry and their breaking patterns based on the free fermion model (8.1). Most basically, the theory lives in the flat Minkowski space. The Lorentz symmetry is only broken by the vector b_μ which we understand as a background of an axial gauge field. There are also two U(1) symmetries, a vector-like one representing the electric charge conservation and an axial one that is broken at tree level by a mass term. The mass operator $\bar{\Psi}\Psi$ is rotated into $\bar{\Psi}\gamma_5\Psi$ under the action of the axial U(1) symmetry. The field content of our holographic model has to have a metric to represent energy and momentum conservation related to Lorentz symmetry, a completely conserved vector gauge field V_μ, an axial gauge field A_M, a complex scalar field whose real part corresponds to the mass operator. The scalar is also charged under the axial U(1). The axial anomaly has three parts, a pure axial cubed anomaly, a mixed axial vector anomaly and a mixed axial gravitational anomaly, and is represented in holography via Chern–Simons terms. This motivates the action

$$\begin{aligned} S = \int \mathrm{d}^5x\sqrt{-g}\Bigg[&\frac{1}{2\kappa^2}\left(R+\frac{12}{L^2}\right)-\frac{1}{4e^2}\mathcal{F}^2-\frac{1}{4e^2}F^2-(D_M\Phi)^*(D^M\Phi)-V(\Phi) \\ &+\epsilon^{MNPQR}A_M\left(\frac{\alpha}{3}\Big(3\mathcal{F}_{NP}\mathcal{F}_{QR}+F_{NP}F_{QR}\Big)+\zeta R^L{}_{KNP}R^K{}_{LQR}\right)\Bigg]\,. \end{aligned} \tag{8.3}$$

The covariant derivative is $D_M = \partial_M + iqA_M$ since the scalar is charged only under the axial symmetry. The scalar field potential is chosen to be $V(\Phi) = m^2|\Phi|^2 + \frac{\lambda}{2}|\Phi|^4$. The mass determines the dimension of the operator dual to Φ and we chose it to be $m^2L^2 = -3$. Here, L is the scale of the AdS space. In the following, we set $2\kappa^2 = e^2 = L = 1$. The operator dual to Φ has, therefore, dimension three just as the mass term in the Dirac equation (8.1).

The electromagnetic and axial currents are defined as

$$J^\mu = \lim_{r\to\infty} \sqrt{-g}\Big(F^{\mu r} + 4\alpha\epsilon^{r\mu\beta\rho\sigma}A^5_\beta F_{\rho\sigma}\Big)\,, \tag{8.4}$$

$$J^\mu_5 = \lim_{r\to\infty} \sqrt{-g}\Big(F^{\mu r}_5 + \frac{4\alpha}{3}\epsilon^{r\mu\beta\rho\sigma}A^5_\beta F^5_{\rho\sigma}\Big)\,. \tag{8.5}$$

We are looking for solutions that are asymptotically AdS. In addition, the holographic analogues of the mass term and the time-reversal breaking parameters in (8.1) are introduced via the boundary conditions at $r = \infty$,

$$\lim_{r\to\infty} r\varPhi = M\,, \qquad \lim_{r\to\infty} A_z = b\,. \tag{8.6}$$

Our Ansatz for the zero temperature solution is

$$\begin{aligned} \mathrm{d}s^2 &= u\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) + \frac{\mathrm{d}r^2}{u} + h\mathrm{d}z^2\,, \\ A &= A_z\mathrm{d}z\,, \qquad \varPhi = \phi\,. \end{aligned} \tag{8.7}$$

Note that due to the conformal symmetry at zero temperature, only M/b is a tunable parameter of the system.

Critical solution: The following Lifshitz-type solution is an exact solution of the system:

$$\begin{aligned} \mathrm{d}s^2 &= u_0r^2\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) + \frac{\mathrm{d}r^2}{u_0r^2} + h_0r^{2\beta}\mathrm{d}z^2\,, \\ A_z &= r^\beta\,, \qquad \phi = \phi_0\,. \end{aligned} \tag{8.8}$$

It has the anisotropic Lifshitz-type symmetry $(t, x, y, r^{-1}) \to s(t, x, y, r^{-1})$ and $z \to s^\beta z$. We need to introduce irrelevant deformations to flow to the UV and match the boundary conditions (8.6). We can use the scaling symmetry $z \to sz$ to set the coefficient in A_z to be 1. There are four constants $\{u_0, h_0, \beta, \phi_0\}$ determined by the value of λ, m and q. To flow this geometry to asymptotic AdS in the UV, we need to consider the following irrelevant perturbation around the Lifshitz fix point $u = u_0r^2\big(1 + \delta u\, r^\alpha\big)$, $h = h_0r^\beta\big(1 + \delta h\, r^\alpha\big)$, $A_z = r^\beta\big(1 + \delta a\, r^\alpha\big)$, $\phi = \phi_0\big(1 + \delta\phi\, r^\alpha\big)$. Because of the scaling symmetry, only the sign of $\delta\phi$ is a free parameter. Numerics shows that only $\delta\phi = -1$ corresponds to asymptotic AdS space at the UV. We also fix $q = 1, \lambda = 1/10$. In this case, the numerical values are $(u_0, h_0, \beta, \phi_0, \alpha) \simeq (1.468, 0.344, 0.407, 0.947, 1.315)$ and $(\delta u, \delta h, \delta a) \simeq (0.369, -2.797, 0.137)\delta\phi$. Integrating towards the UV gives the value $M/b \simeq 0.744$.

Topological non-trivial phase: The topologically non-trivial solution[18] in the IR is

$$u = r^2\,, \qquad h = r^2\,, \qquad A_z = a_1 + \frac{\pi a_1^2 \phi_1^2}{16 r} e^{-\frac{2 a_1 q}{r}}\,,$$
$$\phi = \sqrt{\pi}\phi_1 \left(\frac{a_1 q}{2r}\right)^{3/2} e^{-\frac{a_1 q}{r}}\,; \tag{8.9}$$

λ appears only at higher order terms and a_1 can be set to a numerically convenient value. Once the solution is found, it can be re-scaled to $b = 1$.

Starting from this near-horizon solution, we can numerically integrate the equations towards the UV and taking ϕ_1 as shooting parameter. One gets an AdS_5 to AdS_5 domain wall which for the chosen values of λ and q exists only for $M/b < 0.744$.

Topological trivial phase: The near-horizon expansion of the trivial solution is

$$u = \left(1 + \frac{3}{8\lambda}\right) r^2\,, \qquad h = r^2\,, \qquad A_z = a_1 r^{\beta_1}\,, \qquad \phi = \sqrt{\frac{3}{\lambda}} + \phi_1 r^{\beta_2}\,, \tag{8.10}$$

where $(\beta_1, \beta_2) = \left(\sqrt{1 + \frac{48 q^2}{3+8\lambda}} - 1, 2\sqrt{\frac{3+20\lambda}{3+8\lambda}} - 2\right)$. For the chosen λ and q numerically $(\beta_1, \beta_2) = \left(\sqrt{\frac{259}{19}} - 1, \frac{10}{\sqrt{19}} - 2\right)$. We set a_1 to 1 and take ϕ_1 as the shooting parameter. Again, one finds an AdS_5 to AdS_5 domain wall. This type of solution only exists for $M/b > 0.744$.

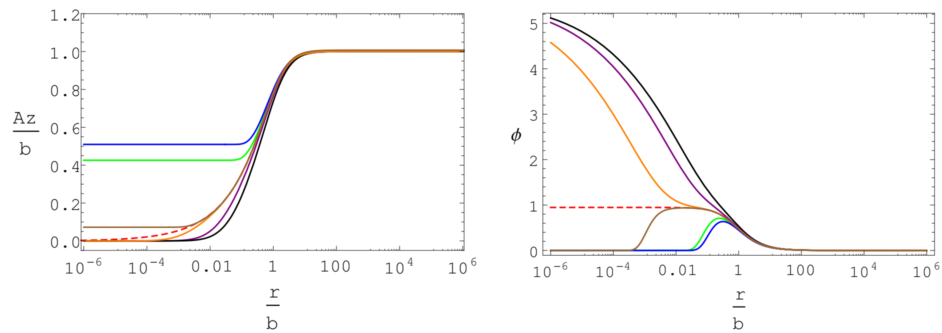

Fig. 10. (Color online) The bulk profile of background A_z and ϕ for $M/b = 0.695$ (blue), 0.719 (green), 0.743 (brown), 0.744 (dashed/red), 0.745 (orange), 0.778 (purple), 0.856 (black).

[18] Similar near-horizon geometries were found in [97, 98] in the context of holographic superconductors.

Figure 10 shows the behavior of the scalar field and the gauge field for all three phases at several different values of M/b. For a given value of M/b, only one of the three types of solutions exists. The value of the gauge field on the horizon matches continuously between the two phases, whereas the value of the scalar field on the horizon jumps discontinuously.

Finite temperature solutions: Finite temperature solutions with a regular horizon can be found with the Ansatz

$$\begin{aligned} \mathrm{d}s^2 &= -u\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{u} + f\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + h\mathrm{d}z^2\,, \\ A &= A_z \mathrm{d}z\,, \qquad \varPhi = \phi \end{aligned} \tag{8.11}$$

imposing the conditions that at $r = r_0$ $f, h, \varPhi, A_z$ are analytic and u has simple zero. Using the scaling symmetries of AdS and the constraint from the equations of motion at the horizon $r = r_0$, we are left with only two dimensionless parameters. In the UV, these are mapped to M/b and T/b.

Conductivities can now be computed with the help of Kubo formulas

$$\sigma_{mn} = \lim_{\omega\to 0} \frac{1}{i\omega} \langle J_m J_n \rangle \left(\omega, \vec{k} = 0\right)\,. \tag{8.12}$$

In holography, the retarded Green's functions can be obtained by studying the fluctuations of the gauge fields around the background with infalling boundary conditions at the horizon [99].

The anomalous Hall conductivity is the off-diagonal part of (8.12). To compute it, we need to switch on the following fluctuations $\delta V_x = v_x(r)e^{-i\omega t}$, $\delta V_y = v_y(r)e^{-i\omega t}$ and define $v_\pm = v_x + iv_y$. The equation of motion for this fluctuation is

$$v''_\pm + \left(\frac{h'}{2h} + \frac{u'}{u}\right) v'_\pm + \frac{\omega^2}{u^2} v_\pm \pm \frac{8\omega\alpha}{u\sqrt{h}} A'_z v_\pm = 0\,. \tag{8.13}$$

These are the same for the zero and finite temperature backgrounds. To solve these equations, we follow the usual near–far matching method [100]. The Green's function can be read off by normalizing the fluctuation to unity at the boundary. The response in the current is then given by

$$G_\pm = -u\sqrt{h} v'_\pm|_{r=\infty} \mp 8\alpha b\omega\,. \tag{8.14}$$

The second term stems from the Chern–Simons current in (8.4), *i.e.* it is the contribution of the Bardeen–Zumino polynomial. We only need to compute the leading order in ω. For both cases $T = 0$ and $T > 0$, we can express the result as

$$\sigma_{xy} = 8\alpha A_z(r_0)\,, \tag{8.15}$$

$$\sigma_{xx} = \sigma_{yy} = \sqrt{h(r_0)}\,. \tag{8.16}$$

For $T = 0$, we have $r_0 = 0$ and $h(0) = 0$ thus the diagonal conductivities vanish at zero temperature. The anomalous Hall effect (see Fig. 11) is determined by the IR value of the axial gauge field. We can identify $b_{\rm eff} = A_z(r = 0)$! We emphasize that this result is crucially dependent on the usage of the consistent (conserved) current. At zero temperature, it is non-vanishing only in the second type of solutions described above. It is, therefore, the topologically non-trivial solution with non-vanishing Hall conductivity. The third kind of zero-temperature solution is characterized by the restoration of time-reversal invariance at the end point of the holographic RG flow $A_z(0) = 0$ and absence of Hall conductivity. Therefore, we have a holographic model of a topological quantum phase transition between a topological and trivial semi-metal.

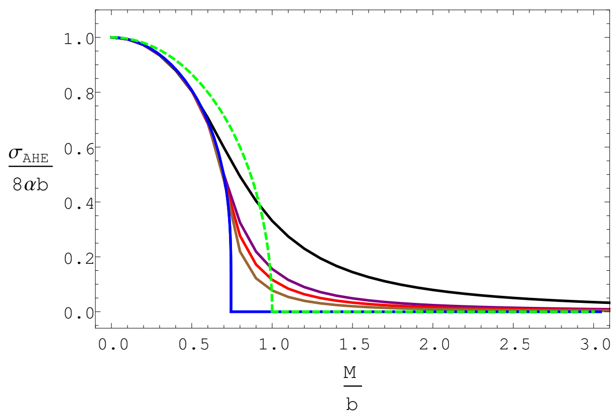

Fig. 11. (Color online) Anomalous Hall conductivity for different temperatures. The solid lines correspond to our holographic model. For $T = 0$, there is a sharp but continuous phase transition at a critical value of M/b (blue) which becomes a smooth crossover at $T > 0$. We show the curves for $T/b = 0.1$ (black), 0.05 (purple), 0.04 (red), 0.03 (brown). For comparison, we also show the result for the weak coupling model as a dashed (green) line. Near the transition, the Hall conductivity behaves as $(\sigma_{\rm AHE}/b) \propto ((M/b)_c - M/b)^\alpha$ with $\alpha \approx 0.211$ (to be contrasted with the field theory model for which $\alpha = 0.5$).

Longitudinal conductivity: The longitudinal electric conductivity at both finite and zero temperature can be computed from the fluctuation $\delta V_z = v_z e^{-i\omega t}$ with equation of motion

$$v_z'' + \left(\frac{f'}{f} - \frac{h'}{2h} + \frac{u'}{u}\right) v_z' + \frac{\omega^2}{u^2} v_z = 0\,. \tag{8.17}$$

At zero temperature, we substitute $f = u$. We find

$$\sigma_{zz} = \left.\frac{f}{\sqrt{h}}\right|_{r=r_0} . \tag{8.18}$$

The three types of background solutions can be classified according to the presence or absence of the anomalous Hall effect. There is a phase for M/b smaller than a critical value in which the axial gauge field flows along the holographic direction towards a constant but non-zero value in the IR. The end point of this holographic flow of the axial gauge field determines the Hall conductivity σ_{xy}. At $M = 0$, the flow is trivial and the Hall response is completely determined by the Chern–Simons current at the boundary of AdS space (the Bardeen–Zumino polynomial). For $M \neq 0$, a non-trivial flow develops, the Hall conductivity has now two parts, a dynamical part that can only be determined by solving equations (8.13) and the Chern–Simons part determined by the boundary values of the fields. At the critical value (for our choice of parameters this is $(M/b)_{\rm c} \simeq 0.744$), the Hall conductivity vanishes. At this value, there is a critical solution with a non-trivial scaling exponent in the z-direction. For even larger values of M/b, the solution shows no Hall effect. The axial gauge field flows to $A_z = 0$ in the far IR. In contrast, now the scalar field obtains a non-trivial IR value. This corresponds to the cosmological constant having a different value in the far IR; *i.e.*, the trivial solution is a domain wall in AdS similar to the zero-temperature superconductor solutions described in Ref. [97]. Since in holography the cosmological constant is a measure of the effective number of degrees of freedom, the trivial solution can be interpreted as one in which some of the UV degrees of freedom are gapped out along the RG flow. We have thus found a holographic zero-temperature quantum phase transition between a topological phase characterized by a non-vanishing Hall conductivity and a topological trivial phase with zero Hall conductivity. All diagonal conductivities vanish at zero temperature.

At $T \neq 0$, the quantum phase transition becomes a smooth crossover. The far IR physics is covered by a horizon at some finite value of the holographic coordinate. It is also interesting to observe the behavior of the diagonal conductivities at finite T as a function of M/b (see Fig. 12). We see that the transverse diagonal conductivities develop a peak roughly at the critical value, whereas the longitudinal one develops a minimum. The height of the peak and the depth of the minimum grow with temperature. At $M = 0$, we simply have $\sigma_{xx,yy,zz} = \pi T$ and for large M, the conductivities tend to a value of $\sigma_{xx,yy,zz} = c\pi T$ with $c < 1$ and independent of temperature. This is consistent with the interpretation that some but not all degrees of freedom are gapped out in the trivial phase and that the phase transition is between a topological semi-metal and a trivial semi-metal.

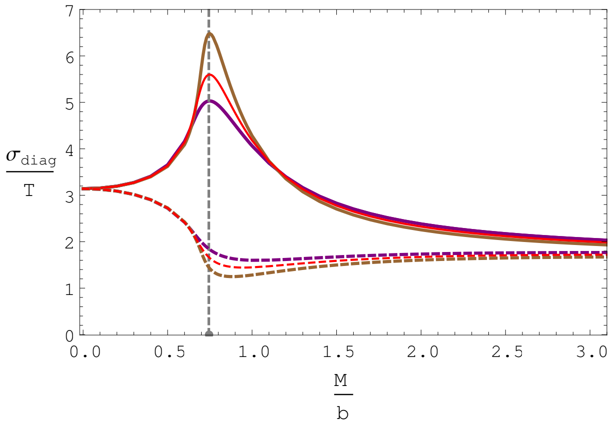

Fig. 12. (Color online) The transverse and longitudinal electric conductivities for different temperatures. The solid lines are for $\sigma_{xx} = \sigma_{yy}$ and the dashed lines are for σ_{zz} from our holographic model with $T/b = 0.05$ (black/purple), 0.04 (light gray/red), 0.03 (dark gray/brown). The dashed vertical/gray line is the critical value of M/b at the topological phase transition.

Hall viscosity: So far we have shown that our holographic model of a Weyl semi-metal has a quantum phase transition between a topological phase and a trivial phase distinguished by the presence of anomalous Hall conductivity. This is by itself interesting but so far, the model does not do anything new compared to the free fermion model (8.1). So can we use the holographic model to compute something new? It turns out the answer is yes! In three dimensions, anisotropic time reversal breaking systems have a very complicated viscosity tensor. Viscosity can be defined as a response to gradients in the fluid velocity

$$T_{\mu\nu} = -\eta_{\mu\nu\rho\lambda}\partial_\rho u_\rho \,. \tag{8.19}$$

The Lorentz invariance restricts this to two independent components, the shear and the bulk viscosity (5.11). More generally, we can define the viscosities via the Kubo formula

$$\eta_{ij,kl} = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\left[G^{\mathrm{R}}_{ij,kl}(\omega,0)\right] , \tag{8.20}$$

with the retarded Green's function of the energy-momentum tensor

$$G^{\mathrm{R}}_{ij,kl}(\omega,0) = -\int \mathrm{d}t\mathrm{d}^3x e^{i\omega t}\theta(t)\langle[T_{ij}(t,\vec{x}\,),T_{kl}(0,0)]\rangle \,. \tag{8.21}$$

This has even and odd components under the exchange of the index pairs ij and kl. We are interested in the odd components. Choosing our coordinates such that $\vec{b} = b\hat{e}_z$, the two odd viscosities related to the anti-symmetric part of the retarded Green's function under the exchange of $(ij) \leftrightarrow (kl)$ are (T denotes here the index combination xx–yy)

$$\eta_{H_\parallel} = -\eta_{xz,yz} = \eta_{yz,xz}\,, \qquad \eta_{H_\perp} = \eta_{xy,T} = -\eta_{T,xy}\,. \tag{8.22}$$

These odd viscosity components can be calculated for the holographic Weyl semi-metal. They are non-zero in the finite temperature backgrounds. The results can be expressed via the values of the bulk fields at the horizon [94]

$$\eta_{\mathrm{H}_\parallel} = 4\zeta \frac{q^2 A_z \phi^2 f^2}{h}\bigg|_{r=r_0}\,, \tag{8.23}$$

$$\eta_{\mathrm{H}_\perp} = 8\zeta q^2 \phi^2 f A_z\bigg|_{r=r_0}\,. \tag{8.24}$$

As can be seen from the plots (Fig. 13), the odd viscosities are very much suppressed in the topological phase. They rise steeply (in the chosen parametrization) and peak near the critical value of M/b. Then they fall off again. This is an indication that the odd viscosity is a property related to to the underlying quantum critical point that separates the topological from the trivial phase. In this region, scaling laws for the odd viscosities (and other transport coefficients) can be obtained from expressions (8.23), (8.24) [94], *e.g.* we find $\eta_{\mathrm{H}\perp} \propto T^{2+\beta}$ and $\eta_{\mathrm{H}\parallel} \propto T^{4-\beta}$ and β is the scaling exponent of A_z of the critical solution.

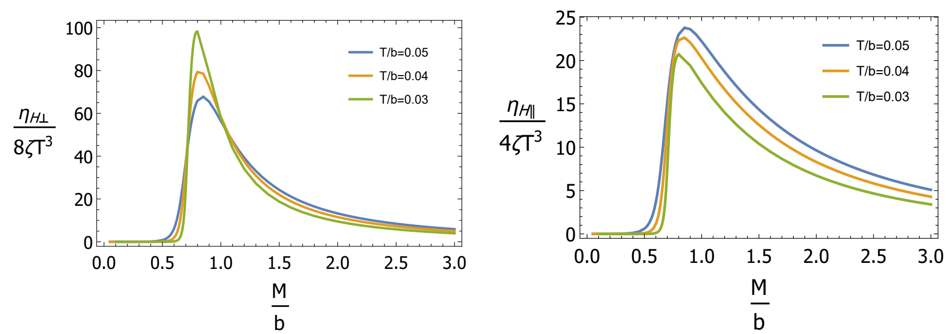

Fig. 13. Odd viscosity $\eta_{\mathrm{H}\perp}$(left panel) and $\eta_{\mathrm{H}\parallel}$ as a function of M/b at different low temperatures normalized by T^3.

9. Outlook

Anomalies are one of the cornerstones of quantum field theory. Almost 50 years after the realization that anomalies explain the decay of neutral pions, they still are a major source of progress in theoretical and also experimental physics. In these notes, I have summarized some of the story of anomalous transport phenomena emphasizing a few subtleties and hinting towards some applications. While the basic phenomenology is now rather well-understood, after a complicated history [16, 101–106] of discovery, neglect, re-discovery and final breakthrough there remain some pressing issues that need to be understood better.

First is the still somewhat mysterious way of how the mixed gravitational anomaly manages to influence transport at the one-derivative level. A hint is given by holography which allows to swallow surplus derivatives up in the extra dimension. Steps towards a holography-independent understanding have been made, *e.g.* combining hydrodynamics with geometric arguments [44], non-renormalization theorems [52, 53], considering Berry flux through Fermi surfaces [61] and links to global gravitational anomalies [54]. Beyond that, there is a pressing need of addressing the experimental side of the gravitational anomaly. In high-energy physics, a direct measurement of pion decay into gravitons seems hopeless but in condensed matter be it the quark–gluon plasma or the electron fluid of Weyl semi-metals the collective transport phenomena induced by the gravitational anomaly are, in principle, accessible. Hopefully, ingenious experimental physicists will get excited about this possibility in the near future.

While holography can probably not claim to have discovered anomalous transport, it has certainly played a major role in gaining a better understanding. But the holographic story has not yet ended: as we have reviewed, a holographic model of a Weyl semi-metal state shows very unusual viscosity properties in the quantum critical region that lies between the topological and trivial phase. Viscous flow of the electron fluid in graphene has recently been measured [107–109]. So one naturally hopes that this (string theory based) prediction of odd viscosity in the quantum critical region of Weyl semi-metals can be measured one day as well.

There are many aspects that are missing from this review. Especially, the application of anomalous transport theory to the physics of the quark–gluon plasma. Suffice it to point to the recent reviews [110–112]. Another important subject totally missing from this paper is chiral kinetic theory [113–115]. Another very systematic approach to anomalous transport has been developed in [116]. Anomalies in d dimensions are governed via the so-called descent equations by invariant polynomials in the field strengths in $d+2$ dimensions. The anomalous currents can be obtained from the invariant polynomials substitution rule $F \to \mu$, $(p_2(R) \to -T^2, p_{k>1} \to 0)$, where $p_k(R)$ is the Pontryagin classes, *i.e.* invariant polynomials in the Riemann tensor, $p_2(R)$ is the gravitational contribution to the chiral anomaly.

We have always assumed that the electric and magnetic fields are external and non-dynamical. Including dynamics of the gauge fields is, however, an important issue and leads to several new aspects. First of all, the actual QCD axial anomaly has a contribution form the gluon fields which have strong quantum dynamics in physically interesting situations such as heavy-ion collisions. This allows for processes that actually create net chiral charge and was the origin of the idea of the presence of the chiral magnetic effect in heavy-ion collisions [7]. Also the values for the anomalous transport coefficients are affected [53, 117]. In holography, anomalies with dynamical gluons can be modeled by using the Stückelberg mechanism in the bulk of AdS [49, 118]. Coupling the chiral magnetic current to Maxwells equations leads to the so-called chiral magnetic instability [119–122] converting axial chemical potentials into helical magnetic fields.

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