MULTI-SOLITON SOLUTIONS BASED ON INTERACTIONS OF BASIC TRAVELING WAVES WITH AN APPLICATION TO THE NONLOCAL BOUSSINESQ EQUATION

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It is shown that multi-waves are generated through direct or indirect nonlinear interactions of basic traveling waves. Direct and indirect nonlinear interactions are suggested via nonlinear combinations and bilinear transformations with nonlinear combinations of basic traveling wave solutions. Here, the used method is a generalization of the unified method presented by the first author in a recent work. Two- and three-soliton solutions have been obtained by the nonlocal Boussinesq equation through the simplified Hirota method very recently. Here, it is shown that they are particular cases of those found in this work. Multi-soliton waves are shown to be super-diffracted notably for higher speed of waves spreading. Further, a giant wave is formed in the region of interaction of the soliton waves.

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1. Introduction

Nonlinear interactions of waves may deserve to interpret phenomena complexity. Here, they are classified as direct or indirect nonlinear interactions (DNLI or IDNLI) respectively.

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Attention is focused here on traveling waves (or wave-like patterns). DNLI is suggested to occur via nonlinear combinations of waves, while IDNLI is taken to occur via a bilinear transformation with nonlinear combinations of waves or wave-like patterns.

Different single traveling waves are well-known, namely solitons, solitary, trigonometric or elliptic waves. Wave-like patterns were also suggested in the literature, namely, peakon, kink, anti-kink, compacton, or double hump configurations. In this work, we suggest the "jet stream" configuration, which is expressed by an exponential function in the similarity variable. It is worth to point out that a soliton wave may be considered as an elastic collision of two (front and back) solitary waves, via DNLI, while it can be considered as an elastic collision of two "jet streams" via IDNLI with linear combinations. The peakon may be considered as inelastic collision of two "jet streams" via IDNLI.

Complexity phenomena revealed in the dynamical evolution of many systems in physics and mechanics are governed by a vector PDE in $(q+1)$-dimensions. Exact solutions to these equations are the most relevant to interpret the phenomena complexity through searching for multi-wave solutions. Attention is focused here to the completely integrable equations. The integrability of PDE is tested via the Painleve' analysis [1].

A variety of methods for studying the integrability of nonlinear partial differential equations and for constructing multiple-soliton solutions have been developed. Among these methods, the inverse scattering method [1–3], Hirota's bilinear method and its simplified form [4–7] are the well-known ones.

The inverse scattering method, which may serve to find multi-wave solutions to the completely integrable PDE, had been presented in [8–10]. It suggests that a nonlinear partial differential equation may be expressed as a condition of compatibility between two linear operators via the so-called Lax pairs [11]. Equivalently, a PDE may be expressed as a combination of the Ablowitz–Kaup–Newell–Segur formulation [12] together with the Lax pairs. In fact, this method requires a hard calculation work.

The Hirota's method leads to the bilinear transformation equation where soliton solutions can be constructed by using the exponential functions. Equivalently, this method may be thought of as a rational function solution with nonlinear combination of exponential functions. That is, in some sense, it is a generalization of the well-known exponential function method. The simplified Hirota method does not depend on the construction of bilinear forms. Instead, it assumes that the multi-soliton solutions are constructed via a potential function (whenever it exists). This function is expressed as polynomials in exponential functions. The required solution may be given as space-partial derivative of the logarithmic of the potential function. On the other hand, the linear dispersion equation is imposed on to the use of this method. The difficulty of using this method arise in cases of PDEs where terms of high differentiability and high linearity are mixed.

The Hirota method, although mathematically well-structured, is not straightforward for handling a vector of PDEs. On the other hand, by the Hirota and its simplified form, one can only find multi-soliton solutions and rather in terms of exponential functions (or "jet streams").

The main aim of this work is to present a more general method compared with those stated above. On the other hand, it is based on the work done in [8, 11] by accounting for multi-auxiliary equations. By this method, we have been able to find not only multi-solitons solutions but also multi-periodic or multi-elliptic (or a lattice of) wave solutions to PDEs.

To this end, by basic traveling wave solutions it is meant that they are the solutions of $q+1$ auxiliary equations. In the case of N-wave solutions, $q+N$ auxiliary equations are used. To make clear, in the present method $q+N$-wave solutions are expressed by a polynomial of degree $q+N$ (via DNLI), while IDNL of $(q+N)$-wave solutions are expressed by a rational function as quotient of two polynomials, both of degree of $q+N$.

2. The generalized unified method

In this section, we present the outlines of the generalized unified method. Let us consider the PDE's equations of the $(q+1)$-dimension

$$\begin{aligned} &F_i(u_j,\, (u_j)_t,\, (u_j)_{x_1}, \ldots,\, (u_j)_{x_q},\, (u_j)_{x_1\,x_2},\, (u_j)_{x_1\,x_3}, \ldots\,) = 0\,, \\ &i,\, j = 1,\, 2, \ldots\, m\,, \end{aligned} \tag{1}$$

where $u_j = u_j(t, x_1, \ldots, x_q)$. Each physical observable u_j possesses $(q+1)$ basic traveling wave solutions that satisfy the equation

$$\begin{aligned} &H_i(U_j,\, (U_j)_{z_1}, \ldots,\, (U_j)_{z_q},\, (U_j)_{z_1\,z_2},\, (U_j)_{z_1\,z_3}, \ldots\,) = 0\,, \\ &z_j = \alpha_j\, t + \sum_{s=1}^{q} \alpha_{j,s}\, x_s\,, \end{aligned} \tag{2}$$

where $U_j = U_j(z_1, \ldots, z_{q+1})$, α_j and $\alpha_{j,s}$ are arbitrary constants. We mention that a basic traveling wave solution may be solitary or of wave-like pattern.

The solutions (2) express the physical aspects associated with the terms of higher order derivatives and nonlinearity in (1).

The fundamental rules and objectives of the unified method are also used here. For details, see [7]. The only distinction is that the main aim in [7, 13] is to search for a single traveling wave solution, namely $U_j = U_j(z)$, $z = \alpha_0\, t + \sum_{j=1}^{q} \alpha_j\, x_j$.

For N-soliton (periodic or elliptic) wave solutions of (1), we have to construct solutions of (1) in the form of

$$u(x_1, \ldots, x_q, t) = U(z_1, \ldots, z_{N+q})\,. \tag{3}$$

As done in the unified method, we search for solutions in the form of

(i) Polynomial function solutions;

(ii) Rational function solutions.

In the present work, we confine ourselves to consider rational function solutions. To find multi-periodic (or elliptic) ones, the case *(i)* is used but the functions $\phi_i(z_i)$ in the auxiliary equation (5) are replaced by $\phi_{ij}(z_i + \delta_{ijj})$.

2.1. The rational function solutions

In this section, we introduce the steps of computations of N-wave rational functions solutions as follows:

Step 1. The method asserts that the N-wave solutions of (2)

$$\begin{aligned} U(z_1, z_2, \ldots, z_{N+q-1}) \;&=\; \frac{P_n(\phi_1(z_1),\, \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))}{Q_r(\phi_1(z_1),\, \phi_2(z_2), \ldots, \phi_{N+q-1}(z_{N+q-1}))}\,, \\ & n > r\,, \qquad k > 1\,, \end{aligned} \tag{4}$$

where $n = r$, then $k = 1$ and $\phi_{i_j}(z_{i_j})$, $i_j = 1, 2, \ldots, N+q-1$ satisfy the auxiliary equations

$$\begin{aligned} (\phi'_{i_j}(z_{i_j}))^p \;&=\; \sum_{r=0}^{p\,k} b_{j,r}\, \phi^r_{i_j}(z_{i_j})\,, \qquad & z_{i_j} = \alpha_{j,0}\, t + \sum_{s=1}^{q} \alpha_{j,s}\, x_s\,, \\ p \;&=\; 1, 2\,, & i_j = 1, 2, \ldots N+q-1\,, \end{aligned} \tag{5}$$

where $\alpha_{j,s}$ and $\alpha_{j,0}$ are constants. It is worth to notice that n and k are determined from the balance equation by the criteria given in [7].

Step 2. By inserting (4) together with (5) into (2), we get an equation which is splitting to a set of nonlinear algebraic equations namely "the principle equations". They are solved by any computer algebra system.

Step 3. Solving the auxiliary equations in (5).
Step 4. Finding the formal exact solutions which is given in (4).

To simplify the computations notably in the case of rational function solutions, we may use the following proposition:

Proposition 2.1. *Nonlinear interaction (DNLI or INLI) via self-wave (or wave-like pattern) interactions is excluded.*

By this proposition, the polynomial in the numerator of the rational function solutions when $n = r$, $k = 1$ (that is when the solutions of the auxiliary equations are "jet streams") takes the form

$$\begin{aligned} &P_n(\phi_1(z_1),\, \phi_2(z_2), \dots, \phi_{N+q-1}(z_{N+q-1})) = a_0 + \sum_{i_1=1}^{n} a_{i_1}\, \phi_{i_1}(z_{i_1}) \\ &+ \sum_{i_1 \neq i_2}^{n} a_{i_1,i_2}\, \phi_{i_1}(z_{i_1})\, \phi_{i_2}(z_{i_2}) + \dots \\ &+ \sum_{i_1 \neq i_2 \neq \cdots \neq i_{N+q-1}=1}^{n} a_{i_1,i_2,\dots,i_{N+q-1}}\, \phi_{i_1}(z_{i_1})\, \phi_{i_2}(z_{i_2}) \dots \phi_{i_{N+q}}(z_{i_{N+q-1}})\,, \\ &N \geq 2\,. \end{aligned} \tag{6}$$

A similar result for $Q_r(\phi_1(z_1),\, \phi_2(z_2), \dots, \phi_{N+q-1}(z_{N+q-1}))$ holds.

Now, we introduce the following theorem:

Theorem 2.2. *The N-soliton solutions via rational function solutions are given by* ($p = 1$, $k = 1$)

$$U(z_1, z_2, \dots, z_{N+q-1}) = \frac{P_n(\phi_1(z_1),\, \phi_2(z_2), \dots, \phi_{N+q-1}(z_{N+q-1}))}{Q_n(\phi_1(z_1),\, \phi_2(z_2), \dots, \phi_{N+q-1}(z_{N+q-1}))}\,, \tag{7}$$

where $P_n(\phi_1(z_1),\, \phi_2(z_2), \dots, \phi_{N+q-1}(z_{N+q-1}))$ *is given by* (6). *The auxiliary equations are* $\phi_i'(z_i) = \phi_i(z_i) + c_i$, $i = 1, 2, \dots, N + q - 1$.

3. Multi-soliton solutions of the nonlocal Boussinesq equation

The nonlocal Boussinesq equation (nlBq) had been derived as a direct bilinearization of the Kaup's higher order water wave equation [10, 12, 14–21], which is given by

$$\begin{aligned} u_t &= p_x\,, \\ p_t &= \alpha\, u_x - u_{xxx} - \left(u^2\right)_x + \left(\frac{p^2 + u_x^2}{u - \frac{\alpha}{2}}\right)\,. \end{aligned} \tag{8}$$

Very recently, by using the simplified Hirota method, multi-soliton solutions to the closed form of the equation (8), namely to

$$v_{tt} = \alpha\, v_{xx} - v_{xxxx} - \left(v_x^2\right)_x + \left(\frac{v_t^2 + v_{xx}^2}{v_x - \frac{\alpha}{2}}\right)_x , \tag{9}$$

have been obtained in [22, 23]. We notice that in (9), $\sqrt{\alpha}$ is the speed of wave spreading. In the absence of nonlinear terms, it can be shown that each wave splits into two waves that are moving at speeds $\pm\sqrt{\alpha}$ along the characteristics lines.

We realize that, after the figures shown in [22], the effects of super-diffraction (that may be argued to the presence of v_{xxxx}) have not been observed.

Here, to find multi-wave solutions of the nlBq equation, we write (9) in the conservative form

$$\begin{aligned} v_t &= q_x\,, \\ q_t &= \alpha\, v_x - v_{xxx} - v_x^2 + \left(\frac{q_x^2 + v_{xx}^2}{v_x - \frac{\alpha}{2}}\right) \end{aligned} \tag{10}$$

and the required solution is $u = v_x$.

For instance, by using Theorem 2.2, the two-soliton solutions is

$$v(x,t) := v_0(z_1, z_2) = \frac{\sum_{i=0}^{2} d_i\, g_i(z_i) + d_3\, g_1(z_1)\, g_2(z_2)}{\sum_{i=0}^{2} b_i\, g_i(z_i) + b_3\, g_1(z_1)\, g_2(z_2)}\,, \tag{11}$$

and a similar equation holds for $q(x,t) := q_0(z_1, z_2)$ as in (11) but $d_i \mapsto a_i$, $i = 0, 1, 2$.

In (11), the auxiliary equations are

$$\begin{aligned} g_1'(z_1) &= g_1(z_1) + c_1\,, \qquad & g_2'(z_2) = g_2(z_2) + c_2\,, \\ z_1 &= \alpha_1\, x + \alpha_0\, t\,, & z_2 = \beta_1\, x + \beta_0\, t\,. \end{aligned} \tag{12}$$

By using a computer-algebra system and by following the steps of computations (Section 2.1), we get the solutions $q_0(z_1, z_2)$ and $v_0(z_1, z_2)$. We focus our attention here, to show the relevant results for $v_0(z_1, z_2)$. We have

$$v_0(z_1, z_2) = d_2 + 2(1 - b_3\, c_1) + \frac{2(-1 + b_3\, c_1)\, \alpha_1}{1 + b_3(-c_1 + e^{z_1})} - \frac{2(d_2 - b_3\, d_0\, \beta_1)}{d_2 - b_3\, d_0 + 2\beta_1\, b_3\, e^{z_2}}\,, \tag{13}$$

where $\alpha_0 = \pm\alpha_1 \sqrt{\alpha - \alpha_1^2}$, $\beta_0 = \pm\beta_1 \sqrt{\alpha - \beta_1^2}$.

We observe that the arbitrary parameters in (13) are α_1, β_1, $b_3\, c_1$, and $d_2 - b_3\, d_0$, while the arbitrary parameters in [22] (*cf.* equations (18) and (14)) are only two parameters namely $k_1\, (= \alpha_1)$ and $k_2\, (= \beta_1)$. Thus, the solutions obtained there are a particular case of (13).

By the same reasoning, we think that the generalized unified method generalizes also the results found by using the inverse scattering method. This may be inspired from different works done via that method (see [12, 24–33]).

We notice that in (13), $c_1 \neq 0$ and $c_2 \neq 0$ are taken *a priori*.

The results given in (13) are displayed against x and t in figure 1.

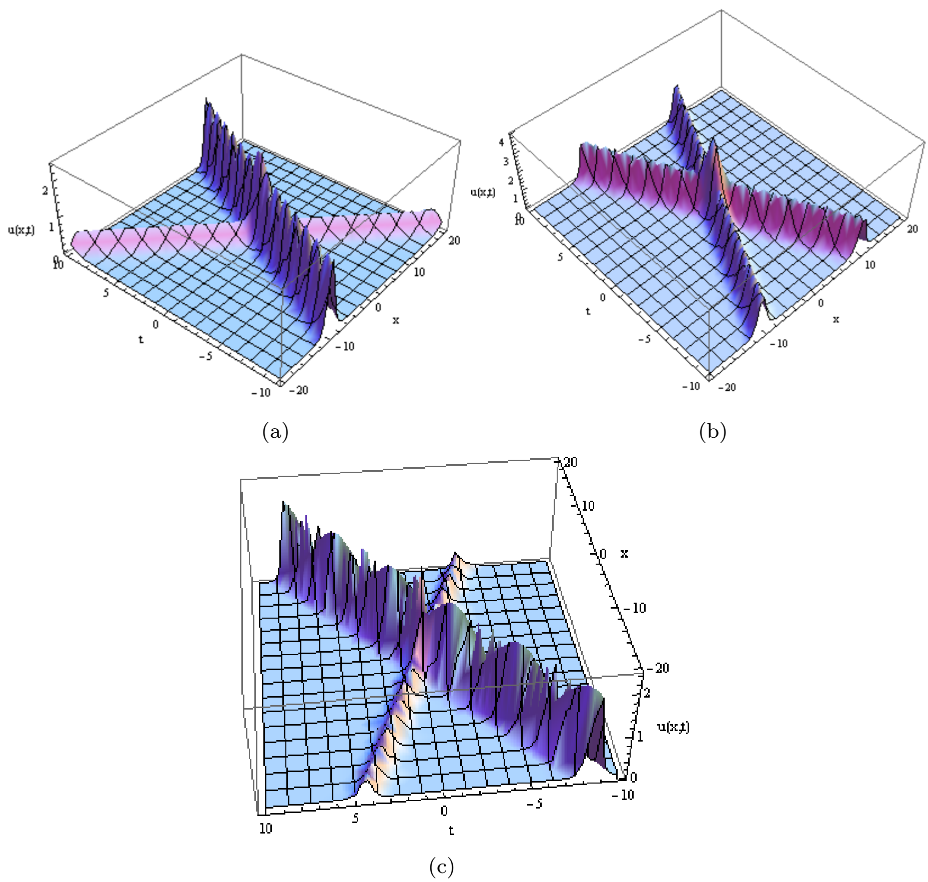


Fig. 1. (a) $b_3 = 0.9$, $\alpha = 5$, $\beta_1 = 2$, $\alpha_1 = 2$, $d_0 = 5$, $c_1 = -0.5$, $d_2 = 5$. (b) $b_3 = 0.2$, $\alpha = 25$, $\beta_1 = 2$, $\alpha_1 = 1$, $d_0 = 2$, $c_1 = 0$, $d_2 = 4$. (c) $b_3 = 10$, $\alpha = 5$, $\beta_1 = 1$, $\alpha_1 = 2$, $d_0 = 2$, $\alpha_0 = 2$, $\beta_0 = 2$.

Figure 1 shows two-soliton solutions that exhibit super-diffraction in the main wave. On the other hand, a giant wave is generated in the region of interaction of the two solitons. This may be argued to the interaction of the jet streams as far as super diffraction dominates when $\alpha_1 \gg \beta_1$. We observed that by increasing the factor of the nonlinear interactions of the jet streams, namely b_3, the diffraction is reduced significantly. Also, the diffraction that effects one soliton is more significant than the other one. That is, the diffraction varies on the two characteristic lines.

Now, we consider the case when $c_1 = c_2 = 0$ are taken *a priori* in (16). The results for the two-soliton solutions in this case are shown in figure 2.

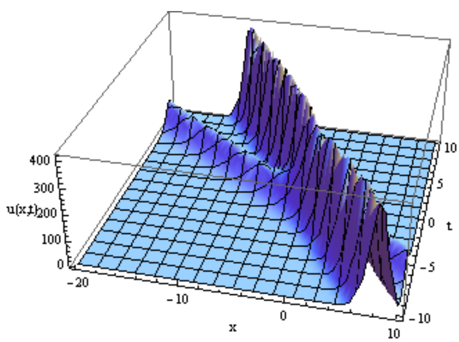


Fig. 2. Profile of two-soliton solution.

It is worth noticing that figures 1 and 2 show multi-anti-soliton solutions but by changing the signs of α_1, and/or β_1.

For three-soliton solutions, by using Theorem 2.2, we have

$$\begin{aligned} v(x,t) &:= v_0(z_1,z_2,z_3) = \frac{P_3(z_1,z_2,z_3)}{Q_3(z_1,z_2,z_3)}\,, \\ P_3(z_1,z_2,z_3) &= \sum_{i=0}^{3} d_i\, g_i(z_i) + g_1(z_1)(d_4\, g_2(z_2) + d_6\, g_3(z_3)) \\ &\quad + d_5\, g_2(z_2)\, g_3(z_3) + d_7\, g_1(z_1)\, g_2(z_2)\, g_3(z_3) \end{aligned} \tag{14}$$

and $Q_3(z_1,z_2,z_3)$ is given as $P_3(z_1,z_2,z_3)$ in $(14)_2$ but $d_i \mapsto b_i$.

Also,

$$q(x,t) := q_0(z_1,z_2,z_3) = \frac{\widetilde{P_3}(z_1,z_2,z_3)}{Q_3(z_1,z_2,z_3)}\,, \tag{15}$$

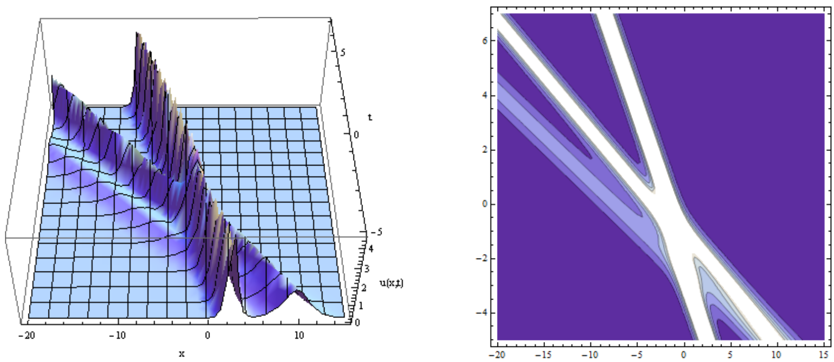
where $\widetilde{P_3}(z_1,z_2,z_3)$ is given as $P_3(z_1,z_2,z_3)$ in $(14)_2$ but $d_i \mapsto a_i$.

In (15), the auxiliary equations are

$$g_1'(z_1) = g_1(z_1) + c_1\,, \qquad g_2'(z_2) = g_2(z_2) + c_2\,, \qquad g_3'(z_3) = g_3(z_3) + c_3\,, \tag{16}$$

where $z_1 = \alpha_1\, x + \alpha_0\, t$, $z_2 = \beta_1\, x + \beta_0\, t$, $z_3 = \gamma_1\, x + \gamma_0\, t$.

Here, for simplicity, we take $c_1 = c_2 = c_3 = 0$. The details of the results for the determined parameters are given in Appendix. The results for three-soliton solutions are shown in figure 3.

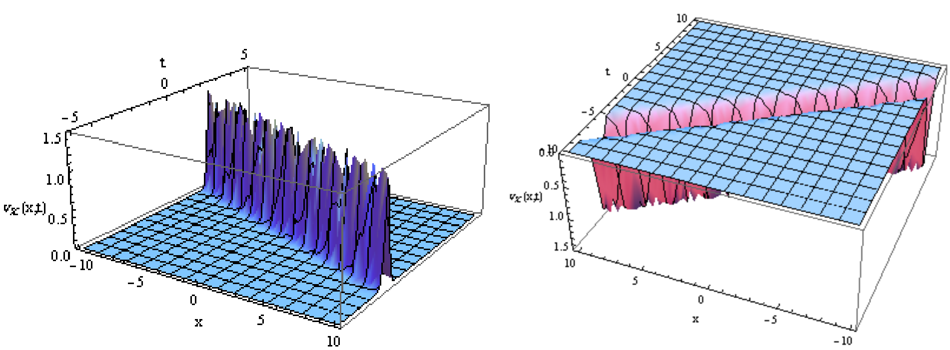


(a) (b)

Fig. 3. (a) $b_7 = 100$, $\alpha = 10$, $\alpha_1 = 1$, $\beta_1 = 2$, $\gamma_1 = 3$. (b) $b_7 = 0.5$, $\alpha = 10$, $\alpha_1 = 1$, $\beta_1 = 2$, $\gamma_1 = 3$.

By changing the signs of α_1, β_1 or γ_1 multi-anti-solitons are produced. The same discussion as in the case of the two-soliton solutions holds in the case of three-soliton solutions, but they will not be repeated here.

For a single-soliton solution, the results are shown in figure 4.



(a) (b)

Fig. 4. (a) $b_1 = -1$, $b_2 = -1$, $\alpha = 5$, $\alpha_1 = 1.92$, $\beta_1 = -0.5$, $\beta_0 = 0.5$. (b) $b_1 = -1$, $b_2 = -1$, $\alpha = 5$, $\alpha_1 = 1.92$, $\beta_1 = -0.5$, $\beta_0 = 0.5$.

We need to state that anti-soliton solutions are obtained by changing the signs of b_1 and b_2 in the denominator.

By the aid of lemmas 2 and 4, that are listed in [7], it is possible to conclude that no periodic or elliptic wave solutions are available to equation (9) listed above.

4. Conclusions

Multi-wave solutions to nonlinear evolution equations via nonlinear interactions of basic traveling waves or wave-like patterns were suggested. This is based on the physical insight that nonlinear waves may be generated via nonlinear interactions of basic traveling waves (or wave patterns).

The method used generalizes the unified version that have been suggested very recently by one of the authors. Multi-soliton solutions to the nonlocal Boussinesq equation that reveal the super-diffraction effects and formation of a generation of giant wave in the region of interference of solitons were found. It was also shown that these results generalize those obtained by the Hirota method and its simplified version.

Appendix

By using a computer algebra system and by carrying out the first step of computation, we get

$$
\begin{aligned}
\alpha_0 &= \pm\alpha_1\sqrt{\alpha-\alpha_1^2}\,, \qquad \beta_0=\pm\beta_1\sqrt{\alpha-\beta_1^2}\,, \qquad \gamma_0=\pm\gamma_1\sqrt{\alpha-\gamma_1^2}\,,\\
b_3 &= \frac{b_5\,b_6\left(\alpha\left(\alpha_1^2+\beta_1^2\right)-2\left(\alpha_1^2\,\beta_1^2+\alpha_0\,\beta_0\right)\right)}{b_7\,\alpha\left(\alpha_1+\beta_1\right)^2}\,,\\
b_2 &= \frac{b_4\,b_6\left(\alpha\left(\alpha_1^2+\gamma_1^2\right)-2\left(\alpha_1^2\,\gamma_1^2+\alpha_0\,\gamma_0\right)\right)}{b_7\,\alpha\left(\alpha_1+\gamma_1\right)^2}\,,\\
b_1 &= \frac{b_5\,b_4\left(\alpha\left(\gamma_1^2+\beta_1^2\right)-2\left(\gamma_1^2\,\beta_1^2+\gamma_0\,\beta_0\right)\right)}{b_7\,\alpha\left(\gamma_1+\beta_1\right)^2}\,,\\
b_0 &= -\frac{b_2b_3\left(\alpha^2\beta_1\gamma_1+4\left(\beta_1^2\gamma_1^2+\beta_0\gamma_0\right)-\alpha\left(2\beta_1^2+\beta_1\gamma_1+2\gamma_1^2-\beta_1\gamma_1+\beta_0\gamma_0\right)\right)}{\left(\beta_1+\gamma_1\right)^2\left(3\alpha-4\left(\beta_1^2+\beta_1\gamma_1+\gamma_1^2\right)\right)}\,,\\
d_7 &= 0\,, \qquad d_6=\frac{b_6\,d_0\,\alpha_1}{k}\,, \qquad d_5=\frac{b_5\,d_0\,\beta_1}{k}\,, \qquad d_4=\frac{b_4\,d_0\,\gamma_1}{k}\,,\\
d_3 &= \frac{b_3\,d_0\left(\alpha_1+\beta_1\right)}{k}\,, \qquad d_2=\frac{b_2\,d_0\left(\alpha_1+\gamma_1\right)}{k}\,, \qquad d_1=\frac{b_1\,d_0\left(\beta_1+\gamma_1\right)}{k}\,,\\
d_0 &= -2b_0(\alpha_1+\beta_1+\gamma_1)\,, \qquad k=b_0\left(\alpha_1+\beta_1+\gamma_1\right). \qquad (17)
\end{aligned}
$$

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