RELATIVISTIC QUANTUM PSEUDO-TELEPATHY

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We analyze the impact of the Unruh effect on quantum Magic Square game. We find the values of acceleration parameter for which quantum strategy yields higher players' winning probability than classical strategy.

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1. Introduction

Quantum game theory is an interdisciplinary field that combines game theory and quantum information. It lies at the crossroads of physics, quantum information processing, computer and natural sciences. Various quantizations of games were presented by different authors [1–3].

Quantum pseudo-telepathy games [4] form a subclass of quantum games. A game belongs to the pseudo-telepathy class providing that there are no winning strategies for classical players, but a winning strategy can be found if the players share a sufficient amount of entanglement. In these games, quantum players can accomplish tasks that are unfeasible for their classical counterparts.

Given a pseudo-telepathy game, one can implement a quantum winning strategy for this game [4]. In an ideal case, the experiment should involve a significant number of rounds of the game. The experiment should be continued until either the players lose a single round or the players win such a great number of rounds, that it would be nearly impossible if they were using a classical strategy. Unfortunately, it is impossible to achieve an ideal noiseless setup of a quantum game.

The motivation to study the Magic Square game and pseudo-telepathy games, in general, is that their physical implementation could provide convincing, even to a layperson, demonstration that the physical world is not local realistic. By *local* we mean that no action performed at some location

X can have an effect on some remote location Y in a time shorter then that required by light to travel from X to Y. *Realistic* means that a measurement can only reveal elements of reality that are already present in the system [4].

It has been shown [5] that noise in a quantum channel can decrease the probability of winning the Magic Square game even below the classical threshold, although the players can counteract this effect for lower noise [6].

The main aim of this paper is to study the Magic Square game in a relativistic setup. As Bob accelerates away from Alice, he observes a black body radiation due to the Unruh effect [7]. In order to accommodate this effect, we use the notion of Rindler coordinates and arrive at an equation which gives us the probability of winning the game as a function of Bob's acceleration.

This paper is organized as follows. In Sec. 2, we introduce the Magic Square game and discuss its classical and quantum versions. Next, in Sec. 3, we introduce a non-inertial reference and introduce the Rindler coordinates. This allows us to rewrite the game setup to the relativistic case. In Sec. 4, we discuss the implications of the relativistic setup on the outcomes of the Magic Square game. Finally, in Sec. 5, we summarize our model and results.

2. Magic Square game

Let us consider a 3×3 square consisting of nine variables $x_1, \ldots, x_9$ (see Fig. 1). Each of these variables can have the value either 0 or 1. The task is to assign values to the variables, so that the sum in each row is even and the sum in each column is odd. This is impossible and can be shown by the following argument. Let us explicitly write all of the constraints. We get

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \mod 2\,, & x_1 + x_4 + x_7 &= 1 \mod 2\,, \\ x_4 + x_5 + x_6 &= 0 \mod 2\,, & x_2 + x_5 + x_8 &= 1 \mod 2\,, \\ x_7 + x_8 + x_9 &= 0 \mod 2\,, & x_3 + x_6 + x_9 &= 1 \mod 2\,. \end{aligned} \tag{1}$$

Summing the columns of Eq. (1), we get two contradictory conditions, namely

$$\sum_{i=1}^{9} x_i = 0 \mod 2\,, \qquad \sum_{i=1}^{9} x_i = 1 \mod 2\,. \tag{2}$$

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

Fig. 1. Magic Square.

Hence, the Magic Square as described above cannot exist. Despite this, we can consider the following game.

The game setup is as follows. There are two players: Alice and Bob. Alice is given a row, Bob is given a column. Alice has to give the entries for a row and Bob has to give the entries for a column so that the parity conditions are met. Winning condition is that the players' entries at the intersection must agree. Alice and Bob can prepare a strategy but they are not allowed to communicate during the game.

In the classical setup, Alice and Bob can prepare correct answers beforehand for 8 of 9 of the possible referee inputs. Assuming these inputs are random and uniformly distributed, they can achieve a maximal winning probability of 8/9. If the parties are allowed to share a quantum state they can achieve probability of success equal to one [4].

In the quantum version of this game [8, 9], Alice and Bob are allowed to share an entangled quantum state. The winning strategy is as follows. Alice and Bob share entangled state being a tensor product of two singlet states that can be effectively written as

$$\begin{aligned} |\psi\rangle &= U_{\mathrm{SWAP}}\left[\tfrac{1}{2}(|01\rangle - |10\rangle)\otimes(|01\rangle - |10\rangle)\right] \\ &= \tfrac{1}{2}\left(|0011\rangle + |1100\rangle - |0110\rangle - |1001\rangle\right)\,, \end{aligned} \tag{3}$$

where U_{SWAP} denotes the two-qubit swap operation and is given by

$$U_{\mathrm{SWAP}} = \mathbb{1}_2 \otimes \left(\sum_{(i,j)\in\{0,1\}^{\times 2}} |ji\rangle\langle ij|\right) \otimes \mathbb{1}_2\,. \tag{4}$$

The first two qubits in the final equality belong to Alice and the last two belong to Bob. Next, Alice and Bob apply local unitary operators $A_i \otimes B_j$ [4], where

$$A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}, \qquad A_2 = \frac{1}{2}\begin{pmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{pmatrix},$$

$$A_3 = \frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \qquad B_1 = \frac{1}{2}\begin{pmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix},$$

$$B_2 = \frac{1}{2}\begin{pmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{pmatrix}, \qquad B_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Indices i and j label rows and columns of the Magic Square. Therefore, the state of this scheme before measurement is

$$\rho_f = (A_i \otimes B_j)\ |\psi\rangle\langle\psi|\ \left(A_i^\dagger \otimes B_j^\dagger\right)\,. \tag{5}$$

The final step of the game consists of the measurement in the computational basis. This gives each of Alice and Bob a two-bit string. Next, they can calculate the third bit from the parity constraints.

We are interested in the mean probability of Alice and Bob winning the game. This probability is given by

$$p = \frac{1}{9} \sum_{i,j=1}^{3} \sum_{\xi \in \mathcal{S}_{ij}} \operatorname{Tr} \rho_f |\xi\rangle\langle\xi| \,, \tag{6}$$

where $\mathcal{S}_{ij}$ is the set of right answers for the column and row ij (Table I). In the case of perfect realization of this protocol, we get $p = 1$.

TABLE I

Sets $\mathcal{S}_{ij}$ — plus sign $(+)$ indicates that the given element belongs to the set, dot $(\cdot)$ sign indicates that the element does not belong to the set.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathcal{S}_{11}$	+	+	·	·	+	+	·	·	·	·	+	+	·	·	+	+
$\mathcal{S}_{12}$	+	+	·	·	·	·	+	+	+	+	·	·	·	·	+	+
$\mathcal{S}_{13}$	+	+	·	·	·	·	+	+	·	·	+	+	+	+	·	·
$\mathcal{S}_{21}$	+	·	+	·	+	·	+	·	·	+	·	+	·	+	·	+
$\mathcal{S}_{22}$	+	·	+	·	·	+	·	+	+	·	+	·	·	+	·	+
$\mathcal{S}_{23}$	+	·	+	·	·	+	·	+	·	+	·	+	+	·	+	·
$\mathcal{S}_{31}$	·	+	+	·	·	+	+	·	+	·	·	+	+	·	·	+
$\mathcal{S}_{32}$	·	+	+	·	+	·	·	+	·	+	+	·	+	·	·	+
$\mathcal{S}_{33}$	·	+	+	·	+	·	·	+	+	·	·	+	·	+	+	·

3. Magic Square game in non-inertial reference frames

In order to derive expressions for the Magic Square game in a non-inertial reference frame, let us consider the initial state of the game as an entangled state of four fermionic qubits of mode frequencies ω_{A} and ω_{B} in a flat Minkowski space-time

$$|\psi\rangle = \tfrac{1}{2} \left(|0_{\omega_{\mathrm{A}}}\rangle |1_{\omega_{\mathrm{B}}}\rangle - |1_{\omega_{\mathrm{A}}}\rangle |0_{\omega_{\mathrm{B}}}\rangle \right) \otimes \left(|0_{\omega_{\mathrm{A}}}\rangle |1_{\omega_{\mathrm{B}}}\rangle - |1_{\omega_{\mathrm{A}}}\rangle |0_{\omega_{\mathrm{B}}}\rangle \right) , \tag{7}$$

where kets $|0_{\omega_{\mathrm{x}}}\rangle$ and $|1_{\omega_{\mathrm{x}}}\rangle$ ($\mathrm{x} \in \{\mathrm{A}, \mathrm{B}\}$) represent the vacuum and excited states from the perspective of an inertial observer. The state in Eq. (7) is equivalent to the one defined in Eq. (3) up to the operation: U_{SWAP} which swaps the middle qubits.

In order to describe accelerated observers, we use Rindler coordinates, which define two causally disconnected regions $(\mathrm{I}, \mathrm{II})$, *i.e.* a uniformly accelerated observer in region I has no access to information in region II and *vice versa* [10]. In order to simplify our scheme, we consider a single mode in the Rindler region I. This approach is justified if the observers' detectors are highly monochromatic and detect the frequency $\omega_{\mathrm{A}} \approx \omega_{\mathrm{B}} = \omega$. Hereafter, due to this approximation, we will omit the subscript of ω.

In an accelerated reference frame, the vacuum state becomes a two-mode squeezed state given by [10–13]

$$|0\rangle = \cos(r)|0\rangle_{\mathrm{I}}|0\rangle_{\mathrm{II}} + \sin(r)|1\rangle_{\mathrm{I}}|1\rangle_{\mathrm{II}}\,, \tag{8}$$

and the excited state becomes

$$|1\rangle = |1\rangle_{\mathrm{I}}|0\rangle_{\mathrm{II}}\,, \tag{9}$$

where I and II represent modes in the two Rindler regions. The parameter r is a dimensionless acceleration given by

$$\cos r = \sqrt{\exp\left(\frac{-2\pi\omega c}{a}\right) + 1}\,, \tag{10}$$

where a is Bob's acceleration. The value $r = 0$ corresponds to no acceleration and the value $r = \frac{\pi}{4}$ corresponds to infinite acceleration.

Equation (8) shows that an observer in an non-inertial reference frame, moving with a constant acceleration, sees a thermal state instead of a vacuum state, *i.e.* observes a black body radiation. This phenomenon is called the Unruh effect [7, 14, 15].

In the case of stationary Alice and accelerating Bob, using the Rindler coordinates, we obtain the following initial state

$$|\psi\rangle = \left|\psi_{(1)}\right\rangle \otimes \left|\psi_{(2)}\right\rangle\,, \tag{11}$$

where

$$\begin{aligned} \left|\psi_{(i)}\right\rangle = \frac{1}{\sqrt{2}}\Big(& \left|0_{\mathrm{IA}_{(i)}}\right\rangle \left|0_{\mathrm{IIA}_{(i)}}\right\rangle \left|1_{I\mathrm{B}_{(i)}}\right\rangle \left|0_{\mathrm{IIB}_{(i)}}\right\rangle \\ & - \cos(r) \left|1_{\mathrm{IA}_{(i)}}\right\rangle \left|0_{\mathrm{IIA}_{(i)}}\right\rangle \left|0_{\mathrm{IB}_{(i)}}\right\rangle \left|0_{\mathrm{IIB}_{(i)}}\right\rangle \\ & - \sin(r) \left|1_{\mathrm{IA}_{(i)}}\right\rangle \left|0_{\mathrm{IIA}_{(i)}}\right\rangle \left|1_{\mathrm{IB}_{(i)}}\right\rangle \left|1_{\mathrm{IIB}_{(i)}}\right\rangle \Big)\,, \end{aligned} \tag{12}$$

which follows from using $r = 0$ for Alice's qubits.

As there is no access to information in region II, we trace over the modes in this region. Then, we apply partial swap operation U_{SWAP}. We get the following density matrix

$$\rho_r = U_{\mathrm{SWAP}}\,\mathrm{Tr}_{\mathrm{IIA}_{(1)}\mathrm{IIB}_{(1)}\mathrm{IIA}_{(2)}\mathrm{IIA}_{(2)}}(|\psi\rangle\langle\psi|)U_{\mathrm{SWAP}}^{\dagger}\,. \tag{13}$$

Note that for $r = 0$, we recover the state from Eq. (3).

The final state of the game depending on the question pair (i, j) is

$$\rho_f = (A_i \otimes B_j)\ \rho_r\ \left(A_i^{\dagger} \otimes B_j^{\dagger}\right)\,. \tag{14}$$

The mean success probability can be calculated in accordance with Eq. (6)

$$p(r) = \frac{1}{9}\sum_{i,j=1}^{3}\sum_{\xi\in\mathcal{S}_{ij}} \mathrm{Tr}\,\rho_f|\xi\rangle\langle\xi|\,. \tag{15}$$

Since ρ_f depends on r, mean success probability also depends on r.

4. Results and discussion

Using Eqs. (13), (14) and (15), we obtain the following formula for the probability of winning the game as a function of the dimensionless acceleration:

$$p(r) = \tfrac{1}{9}\left(6 + 2\cos 2r + \cos 4r\right)\,. \tag{16}$$

The impact of constant acceleration on the probability of winning the game $p(r)$ is shown in Fig. 2. It shows that we can achieve probability of winning higher than the classical threshold for non-zero accelerations. For

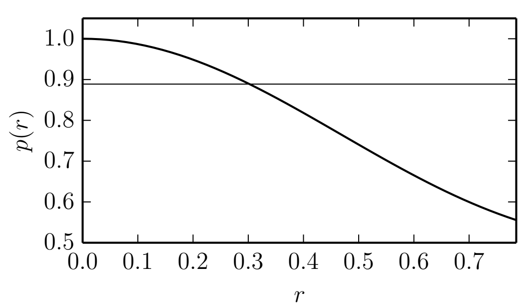

Fig. 2. Expected pay-off of Alice and Bob as a function of Bob's acceleration. The thin horizontal line is the classical threshold. The lines intersect for $r = \tan^{-1}\left(\sqrt{\frac{1}{3}\left(2\sqrt{7}-5\right)}\right) \approx 0.302171$.

$r = \tan^{-1}\left(\sqrt{\frac{1}{3}\left(2\sqrt{7}-5\right)}\right) \approx 0.302171$, players' probability of winning achieves the classical threshold of 8/9, therefore, the quantum strategy is indistinguishable from the classical one.

5. Summary

We studied the quantum Magic Square game in a relativistic setup, in the case in which one of the parties is accelerated with respect to the other. Due to this, the party experiences the Unruh effect. We introduced a formalism which takes this effect into account. This allowed us to show how the relative acceleration impacts the probability of winning the game. We obtained an analytical expression for the acceleration at which the quantum and classical probabilities of winning the game are equal.

It remains an open question if the parties are able to alter their respective strategies in order to counteract the acceleration. There are two possible approaches to solving this problem. The first one is to consider only local unitary operations which can be better suited for the accelerated case. This would result in a family of unitaries of the form $A_i(r)$, $B_j(r)$. Another approach could be to broaden the set of allowed operations and allow the parties to perform local, with respect to their qubits pairs, quantum channels. This approach can provide better strategies for the players. One way to find such local channels is by numerical optimization using semidefinite programming as it was shown in [6].

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