STOCHASTIC DYNAMICS FOR SYSTEMS WITH LÉVY FLIGHTS AND NONHOMOGENEOUSLY DISTRIBUTED TRAPS*

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(Received January 11, 2016; revised version received February 19, 2016)

The stochastic transport in a medium containing traps is described in terms of a subordination technique in which the physical time is regarded as a random quantity given by a density distribution. The traps are assumed to be nonhomogeneously distributed and the subordination method is modified by introducing a position-dependent intensity of a random time distribution. The problem resolves itself to a Langevin equation with a multiplicative noise which defines a process subsequently subordinated to the random time. Moreover, the random stimulation in a form of the Lévy stable distribution is assumed. In the absence of an external potential, the diffusion process is described by the variance which can be finite because an additional multiplicative noise is introduced at some position and effectively makes the system bounded. The diffusion exponent is evaluated and it is demonstrated that it varies with the stability index only if traps are nonhomogeneously distributed. The density distribution converges to a stationary state when a potential is introduced and the relaxation process is analysed for the linear case. The relaxation pattern for the long time always corresponds to the asymptotics of the Mittag–Leffler function but the effective relaxation time strongly depends on the nonhomogeneity parameter.

DOI:10.5506/APhysPolB.47.1313

1. Introduction

The traditional approach assumes that stochastic quantities possess a finite variance and the processes are Markovian which corresponds to a locality in time of the corresponding Fokker–Planck equation. Then, the central

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* Presented at the XXVIII Marian Smoluchowski Symposium on Statistical Physics, "Thermodynamics in a Nanoscale", Kraków, Poland, 14–17 September, 2015.

limit theorem is satisfied: the variables are normally distributed and the variance for the diffusion processes rises linearly with time. In contrast to that simple picture, the central limit theorem is violated in the presence of the memory effects which results in a nonlinear variance growth (anomalous transport). The jumping processes with memory are described by a continuous time random walk theory (CTRW) where the waiting time distribution has long tails and we observe a subdiffusion [1].

However, in the nonhomogeneous media, the time properties of the system may depend on the environment structure. This may be the case, for example, in disordered systems [2] where heterogeneously distributed traps and defects are present. In particular, for a simple version of CTRW when the waiting time distribution is Poissonian, $w(t) = \nu(x)e^{-\nu(x)t}$, the rate $\nu(x)$ is position-dependent because of the medium nonuniformity [3]. Then, CTRW predicts the anomalous transport, in a form of both the subdiffusion and the enhanced diffusion, even in the absence of any memory effects.

In nature, not only phenomena characterised by the normal distribution are present, also distributions with long, power-law tails are observed [4] and the variance diverges. Such processes can still be characterised by the stable distributions but in a generalised form. They are defined, for a symmetric case, by a characteristic function $e^{-K^\alpha|k|^\alpha}$ $(0 < \alpha \leq 2)$, where α is a stability index and $K = \text{const}$; the asymptotics is algebraic, $\propto |x|^{-1-\alpha}$. The folded polymers constitute an example of a complex system where both long jumps and a nonhomogeneous medium structure must be taken into account [5].

In this paper, we consider the stochastic processes characterised by the memory effects, long distribution tails and a nonhomogeneous medium structure. We demonstrate that those spacial dependences, related to the trap distribution, essentially influence the time characteristics of the system: time dependence of the variance in the diffusion case, as well as that of the relaxation function when an external potential is included.

2. Stochastic equations

The convenient method to describe dynamics of a system characterised by strong memory effects is a subordination technique [6]. Two times are introduced. The dynamics, given by a Langevin equation, is determined in terms of an operational, auxiliary time τ. During that evolution, the particle is brought — from time to time — to a standstill due to the traps. The resting time is random and the relation of τ to a physical time t is defined by an adjoint Langevin equation. Therefore, the problem is formulated by a set of two Langevin equations

$$\begin{aligned} \mathrm{d}x(\tau) &= F_\mathrm{d}(x)\mathrm{d}\tau + \eta(\mathrm{d}\tau)\,, \\ \mathrm{d}t(\tau) &= \xi(\mathrm{d}\tau)\,. \end{aligned} \tag{1}$$

In the first equation, the random driving $\eta(\mathrm{d}\tau)$ has a symmetric Lévy stable distribution and $F_{\mathrm{d}}(x)$ stands for a deterministic force. The random time generator also involves a Lévy distributed quantity: $\xi(\mathrm{d}\tau)$ means a one-sided, completely asymmetric process with the infinite mean; it is defined by a subordinator $\bar{h}(t,\tau)$ which has a Laplace transform $\mathcal{L}_u[\bar{h}(t,\tau)] = e^{-\tau u^\beta}$ $(0 < \beta < 1)$. Then, $\bar{h}(t,\tau)$ has long tails $\bar{h}(t,\tau) \propto t^{-\beta-1}$.

We would like to generalise the above formalism to the case when, in some complex medium, traps are nonuniformly distributed [7]. We introduce a function $g(x)$ which serves as a variable intensity of the random time generator. For this generalised case, Eq. (1) takes the form of

$$\begin{aligned} \mathrm{d}x(\tau) &= F_{\mathrm{d}}(x)\mathrm{d}\tau + \eta(\mathrm{d}\tau)\,, \\ \mathrm{d}t(\tau) &= g(x)\xi(\mathrm{d}\tau)\,, \end{aligned} \tag{2}$$

and $g(x) \geq 0$. $g(x)$ estimates a density of traps: the particle remains longer in the area where this density is large. Since $g(x)$ does not affect the dynamics in the first equation and is responsible only for the rests, there is no correlation between the trapping time properties and the trajectory $x(\tau)$, in contrast to the Lévy walk approach. In the latter case, the medium heterogeneity may be taken into account by introducing a variable $\beta(x)$ [8]. Two limiting cases can be distinguished. If $g(x)$ is very small, t rises very slowly with τ and, as a result, the particle instantly (in a sense of the physical time t) leaves the region near x. On the other hand, a large $g(x)$ means a long trapping time.

The dynamics given by Eq. (2) can be expressed by an alternative system of equations which involves a multiplicative noise in the Itô interpretation. It reads

$$\begin{aligned} \mathrm{d}x(\tau) &= F_{\mathrm{d}}(x)\nu(x)\mathrm{d}\tau + \nu(x)^{1/\alpha}\eta(\mathrm{d}\tau)\,, \\ \mathrm{d}t(\tau) &= \xi(\mathrm{d}\tau)\,, \end{aligned} \tag{3}$$

where $\nu(x)$ is a positive function. Connection of the above sets of equations, (2) and (3), can be demonstrated in the following way [7]. Let us assume for a moment that there is no memory, *i.e.* the distribution of increments $\mathrm{d}\xi(\tau) = \xi(\mathrm{d}\tau)$ does not have long tails and the mean value $\langle\xi\rangle$ exists. The approximation of ξ by this mean allows us to evaluate the operational time increment from the second equation (2). Inserting this result into the first equation produces the first equation (3), where $\nu(x) = (\langle\xi\rangle g(x))^{-1}$ and τ is substituted by the physical time t. That equation includes the information about the (nonuniform) trap distribution but neglects memory effects which, in turn, can be approximately taken into account by a subsequent subordination, according to the procedure (3). Equations (2) and (3) are, in fact, equivalent when one takes

$$\nu(x) = g(x)^{-\beta}\,, \tag{4}$$

which has been recently proved for the Gaussian case [9]. The above result can be easily generalised to the general stable distributions and, in the following, we will use the relation between $g(x)$ and $\nu(x)$ in the form (4).

3. Diffusion

In the absence of any external potential, the memory in stochastic systems invokes the anomalous transport: the variance, if exists, rises nonlinearly with time. On the other hand, if the random driving has long tails, the variance is divergent (the accelerated diffusion). In this section, we discuss the diffusion properties of such a system for the case when the medium is characterised, in addition, by a heterogeneous trap structure. We put $F_\mathrm{d} = 0$ in Eq. (3) and restrict our considerations to a power-law form of the trap density

$$g(x) = |x|^\theta\,. \tag{5}$$

The scaling dependence is natural in complex systems; it corresponds to fractal structures [10] and is frequently observed in nature, *e.g.* in geology where it describes the distribution of fracture lengths responsible for the transport of a liquid in a rock [11].

The first equation (3) corresponds, in the diffusion case, to the fractional Fokker–Planck equation with a variable diffusion coefficient [12],

$$\frac{\partial p_0(x,\tau)}{\partial\tau} = \frac{\partial^\alpha[\nu(x)p_0(x,\tau)]}{\partial|x|^\alpha}\,. \tag{6}$$

On the other hand, Eq. (6) directly follows from CTRW [13], mentioned in Introduction, which is defined by the Poissonian waiting time and a jump-size distribution $Q(x)$ in a form of the Lévy α-stable symmetric distribution; the process corresponds to an infinitesimal transition probability

$$p_\mathrm{tr}(x,\Delta\tau|x',0) = \Big[1-\nu(x')\Big]\Delta\tau\delta(x-x') + Q(x-x')\nu(x')\Delta\tau \qquad (\Delta\tau \ll 1)\,. \tag{7}$$

The density as a function of the physical time can be obtained from $p_0(x,\tau)$ by the integration over τ

$$p(x,t) = \int_0^\infty p_0(x,\tau)h(\tau,t)\mathrm{d}\tau\,, \tag{8}$$

where $h(\tau,t)$ is an inverse subordinator. It determines a distribution of τ (t serves as a distribution width) and is given by a Laplace transform $h(\tau,u) = u^{\beta-1}\exp(-\tau u^{\beta})$. The evaluation of $p_0(x,\tau)$ requires a different procedure for $\alpha = 2$ and $\alpha < 2$. In the former case, it can be exactly derived and the direct calculation of the integral (8) yields [14]

$$p(x,t) = -\frac{2}{\beta t}\frac{(1+\theta\beta/2)^{\nu_c+2c/\beta}}{\Gamma(-\nu_c)}|x|^{(2+\theta\beta)/\beta-1} \times H^{2,0}_{1,2}\left[\frac{|x|^{(2+\theta\beta)/\beta}}{(2+\theta\beta)^{2/\beta}t}\left|\begin{array}{c}(0,1)\\ (c/\beta-\nu_c/2,1/\beta),(c/\beta+\nu_c/2,1/\beta)\end{array}\right.\right], \quad (9)$$

where $\nu_c = 1/(2+\theta\beta)$ and $c = \beta - \beta/(4+2\theta\beta) - 1$; the normalization condition requires $\theta > -1/\beta$. Equation (9) involves a Fox H-function and is not transparent but $p(x,t)$ has a simple stretched-Gaussian asymptotics,

$$p(x,t) \sim |x|^{\theta} t^{-\frac{\beta(1+\theta\beta)}{(2+\theta\beta)(2-\beta)}} \exp\left[-A|x|^{(2+\theta\beta)/(2-\beta)}/t^{\frac{\beta}{2-\beta}}\right], \quad (10)$$

where $A = (2/\beta - 1)/[\beta^{3/(2-\beta)}(2+\theta\beta)^{2/(2-\beta)}]$. The variance, $\langle x^2\rangle(t) \propto t^{2\beta/(2+\theta\beta)}$, may rise with time not only linearly but also slower and faster than linearly, implying all kinds of the diffusion: we observe the normal diffusion, subdiffusion and enhanced diffusion if θ is equal, larger or smaller than $-2(1-\beta)/\beta$, respectively. For small (negative) θ and β close to unity, diffusion approaches a ballistic limit. If $\beta = 1$, the above expression for the variance does not apply; for $\theta = 0$ a logarithmic correction emerges: $\langle x^2\rangle(t) \propto t/\ln t$ [2].

In the case of $\alpha < 2$, the exact evaluation of $p_0(x,\tau)$ is not possible and we will restrict our considerations to the asymptotic limit of large $|x|$. Due to the long tails of the distributions for Lévy flights, the variance is infinite for any time which property leads to unphysical consequences when we are dealing with a massive particle moving in the ordinary space. This difficulty can be avoided when we introduce a truncation: the far tail of the distribution is removed or substituted by some fast falling function [15]. Then, the features of the Lévy flights — such as a power-law shape of the distribution — can still be preserved though the distributions actually converge (typically very slowly) to the normal distribution, according to the central limit theorem. This convergence does not take place when, instead of the truncation procedure, a multiplicative noise is introduced. Let us consider a set of equations

$$\begin{aligned} \mathrm{d}x(\tau) &= f(x)\eta(\mathrm{d}\tau)\,, \\ \mathrm{d}t(\tau) &= g(x)\xi(\mathrm{d}\tau)\,, \end{aligned} \quad (11)$$

where $f(x)$ is a decreasing function. Then, the finiteness of variance follows from the suppressing of the random driving by a variable intensity and the statistics remains intact. However, the distribution shape qualitatively depends on the interpretation of the stochastic integral corresponding to the multiplicative noise $f(x)$ and, in fact, Eq. (11) is meaningless as long as we do not determine this interpretation. The question is whether $f(x)$ should be evaluated at a time before the random force acts (Itô), after that or somewhere in-between, *e.g.* in the middle point (Stratonovich). The interpretation dilemma is well-known for the Gaussian processes. Then, the Stratonovich interpretation constitutes a white noise limit of the coloured noises and the standard calculus is applicable: in one dimension, the Langevin equation can be transformed to an equation with the additive noise. Those properties are not obvious for the Lévy flights, and in this case, employing a Marcus interpretation has been suggested [16]. However, in practice, applying the standard calculus seems to work well and comparison of the density distributions obtained by a variable change with the numerical simulations by means of the Stratonovich prescription exhibits a perfect agreement [17]. On the other hand, the ordinary rules of the calculus are valid if one regards the white noise η as the limit of a coloured noise [18].

To demonstrate the influence of the multiplicative noise in the Stratonovich interpretation on the density slope, first, we assume $f(x) = |x|^{-\gamma_m}$. Then, the transformation $x \to y = \frac{1}{1+\gamma_m}|x|^{1+\gamma_m}\operatorname{sign} x$ results in a Langevin equation (for $y(\tau)$) with the multiplicative noise in the Itô interpretation and the corresponding Fokker–Planck equation can be solved in a limit of small wave numbers [14]. The integrating over τ and transforming back to the variable x yields the final density $p(x,t)$ and the asymptotics reads

$$p(x,t) \propto t^{\beta c_\theta}|x|^{-1-\alpha-\alpha\gamma_m}\,, \tag{12}$$

where $c_\theta = 1/(\alpha + \theta\beta/(1+\gamma_m))$. Then, the variance is finite if $\gamma_m > 2/\alpha - 1$ and its direct evaluation yields $\langle x^2\rangle(t) \propto t^{2\beta c_\theta/(1+\gamma_m)}$ which implies subdiffusion. That behaviour is possible due to a nonlocal character of the multiplicative term for the Stratonovich interpretation: the diminishing of the noise intensity with the distance is then taken into account in the evaluation of the stochastic integral, making the variance finite and slowing down the transport. This is not the case for the Itô interpretation when the slope of the distribution tail is the same as that for the driving noise: $p(x,t) \propto |x|^{-1-\alpha}$ for any γ_m. The comparison of the above expression with Eq. (12) discloses the essential difference between both interpretations which cannot be removed by the addition of a drift to the stochastic equation since then, the density would converge to a stationary state.

Next, let us assume $f(x)$ in the form of

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq L \\ L^{\gamma_m}|x|^{-\gamma_m} & \text{for } |x| > L \end{cases} \tag{13}$$

which can be interpreted as a consequence of the fact that the medium has a limited size. In the bulk ($|x| < L$), the dynamics is driven by the additive noise whereas at the boundary, a multiplicative noise in the Stratonovich interpretation is switched on; then, the external region represents a diffused boundary. Moreover, traps are distributed according to $g(x)$ in both regions and γ_m must be sufficiently large to ensure the finite variance. $f(x)$ in the form of (13) has been introduced rather phenomenologically but it is physically natural. First of all, when we think about disordered media, the medium structure is expected to be more complex near the surface that in the bulk — due to a possible presence of impurities, defects and fractures. The additive noise may not be sufficient to describe such a complicated system, and a generalisation to take into account a position dependence of the random stimulation may be necessary. Moreover, there exists an experimental evidence that the multiplicative noise should be introduced near the boundary [19].

Equation (11) for $f(x)$ in the form of (13) can be analytically solved for $\theta = 0$. Two cases are distinguished. For a relatively small time, the diffusion takes place mainly in the bulk and role of the boundary resolves itself to suppressing the long tails of the noise; then, we obtain the same result as for the Gaussian case: $\langle x^2\rangle(t) \propto t^{\beta}$. On the other hand, in the long time limit, the transport in the region of the diffused boundary is important and a direct calculation yields $\langle x^2\rangle(t) \propto t^{2\beta/\alpha(1+\gamma_m)}$. The motion is subdiffusive and slower than for the diffusion in the bulk. The general case $\theta \neq 0$ can be solved only numerically [20], since the transformation $x \to y$ becomes complicated for $f(x)$ in the form of (13). However, if $|x|/L$ is large, the relation between x and y assumes a scaling form, $|x| = (1+\gamma_m)^{1/(1+\gamma_m)} L^{\gamma_m/(1+\gamma_m)} |y|^{1/(1+\gamma_m)}$. Then, the multiplicative factor in the Langevin equation is a power law as well and the problem is reduced to the case $f(x) = |x|^{-\gamma_m}$, discussed above. The direct calculation produces the variance

$$\left\langle x^2 \right\rangle (t) \propto t^{\frac{2\beta}{\alpha(1+\gamma_m+\theta\beta)}}\,, \tag{14}$$

which means that the nonhomogeneity of the medium makes the diffusion slower ($\theta > 0$) or faster ($\theta < 0$), compared to the homogeneous case. Equation (14) constitutes a limiting form of the variance, valid for a very long-time evolution, and it is rather formal since corresponds to large distances which can hardly be interpreted as the surface region. We present this result for the sake of comparison with the numerical simulations which are presented in Fig. 1. They were obtained from a time evolution of Eq. (11) and the multiplicative noise $f(x)$ was evaluated from the definition of the stochastic integral in the Stratonovich interpretation. Slope of the variance (14) agrees with the numerical result if time is sufficiently large.

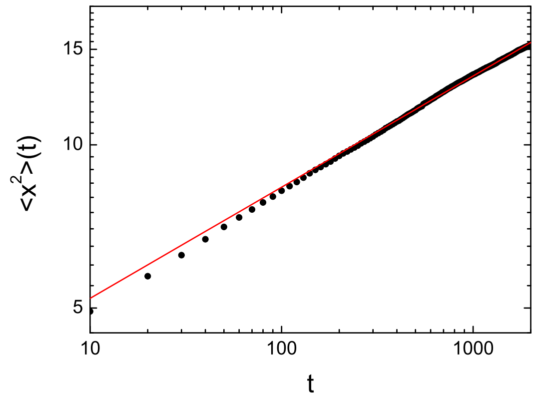

Fig. 1. Variance as a function of time evaluated from numerical simulations for $\gamma_m = 2$, $L = 0.1$, $\beta = 0.5$, $\theta = 0.5$ and $\alpha = 1.5$ (points). Slope of the straight line follows from Eq. (14).

For the case of the general, nonscaling dependence $y(x)$, the numerical analysis shows that the variance always has the form $\langle x^2 \rangle(t) \propto t^\mu$ and μ depends on β, θ and α, whereas it is independent of L and γ_m. Figure 2 illustrates that power-law dependence and demonstrates that, in contrast

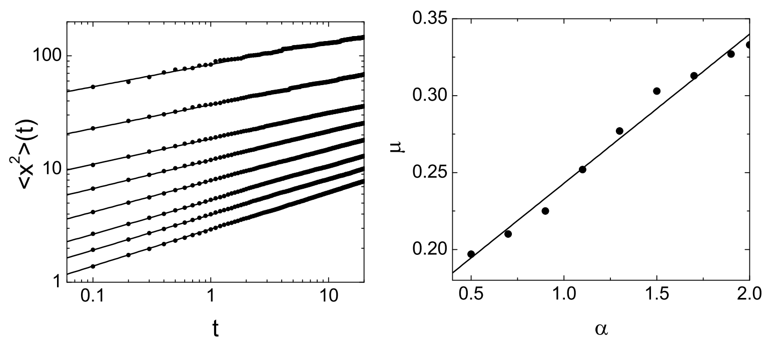

Fig. 2. Left part: Variance as a function of time for $\gamma_m = 2$, $L = 10$, $\beta = 0.5$, $\theta = 2$ and the following values of α: 0.5, 0.7, 0.9, 1.1, 1.3, 1.5, 1.7 and 1.9 (dots, from top to bottom). Straight lines represent fits to the data. Right part: Corresponding slope of the time dependence of the variance t^μ. Straight line $\mu(\alpha) = 0.146+0.097\alpha$ is a fit to the data.

to the homogeneous case, μ varies with α. This finding is presented in the right part of Fig. 2; $\mu(\alpha)$ exhibits a linear growth. As regards the dependence on the other parameters [14], μ rises with β and the growth is faster for smaller θ. Finally, μ diminishes with θ since for a large θ the trapping is stronger ($g(x)$ is larger).

4. Stationary states

Up to now, we discussed the diffusion case; the particle was subjected to the random stimulation, without any external potential. Now, we switch on a deterministic force in a form of a nonlinear oscillator

$$F_{\rm d} = -\lambda|x|^{\gamma}{\rm sign}(x)\,, \tag{15}$$

where $\lambda \geq 0$, and the system converges with time to a stationary state. Our ultimate goal is to determine a relaxation process to that state as a function of the physical time t, but first we must derive that state itself. Let us consider a subordinated process which follows from Eq. (3) and is given by the Langevin equation

$${\rm d}x(\tau) = -\lambda|x|^{\gamma}\nu(x){\rm sign}(x){\rm d}\tau + \nu(x)^{1/\alpha}\eta({\rm d}\tau)\,; \tag{16}$$

it corresponds to a Fokker–Planck equation

$$\frac{\partial p_0(x,\tau)}{\partial\tau} = \lambda\frac{\partial}{\partial x}\Big[|x|^{\gamma}\nu(x){\rm sign}(x)p_0(x,\tau)\Big] + \frac{\partial^{\alpha}[\nu(x)p_0(x,\tau)]}{\partial|x|^{\alpha}}\,. \tag{17}$$

We will restrict our considerations to the asymptotic regime of large $|x|$ which means small wave numbers k. The Fourier transform from Eq. (17) yields

$$0 = -\lambda k\frac{\partial}{\partial k}\mathcal{F}\Big[|x|^{\gamma-1}\nu(x)p_0^{(s)}(x)\Big] - |k|^{\alpha}\mathcal{F}\Big[\nu(x)p_0^{(s)}(x)\Big]\,, \tag{18}$$

where we introduced a stationary limit $p_0^{(s)}(x) = \lim_{\tau\to\infty}p_0(x,\tau)$. Since $\nu(x) = |x|^{-\theta\beta}$ ($\theta\beta > -\alpha$) and $p_0^{(s)}(x)$ is expected in the power-law form, the solution of the above equation requires the evaluation of the Fourier transform from a power law. This can be performed [21] by means of a Tauberian theorem [22] which constitutes an exact and unique relation between the density for $|x| \gg 1$ and its characteristic function for $|k| \ll 1$. Applying this theorem in Eq. (18) and neglecting terms higher than $|k|^{\alpha}$ produces a final solution

$$p_0^{(s)}(x) = \frac{\langle\nu(x)\rangle}{\lambda\pi}\Gamma(\alpha)\sin(\alpha\pi/2)\;|x|^{-\alpha-\gamma+\theta\beta}\,, \tag{19}$$

where the following conditions must be satisfied: $\gamma > 1-\alpha+\theta\beta$ and $\gamma > 1-\alpha$. The slope rises with the potential parameter γ but it depends on the effective potential: becomes stronger for a negative θ. Therefore, variance may be finite even for relatively weak potentials. The constant coefficient in Eq. (19) cannot be uniquely determined without additional assumptions since $\langle\nu(x)\rangle$ depends on the density at small $|x|$. For $\theta = 0$, $\langle\nu(x)\rangle = 1$ and we recover a well-known result [23].

The next step is to derive a time-dependent solution, $p_0(x,\tau)$. This can be exactly accomplished for the linear case, *i.e.* for $\gamma - \theta\beta = 1$, when the transformed Fokker–Planck equation is of the form of

$$\frac{\partial}{\partial\tau}\widetilde{p}_0(k,\tau) = -\lambda k\frac{\partial}{\partial k}\widetilde{p}_0(k,\tau) - |k|^\alpha\mathcal{F}\left[|x|^{-\theta\beta}p_0(x,\tau)\right] . \tag{20}$$

Presence of the factor $|k|^\alpha$ indicates a scaling form of the solution

$$p_0(x,\tau) = a(\tau)p_0(a(\tau)x) = a(\tau)^{-\alpha}|x|^{-\alpha-1} , \tag{21}$$

where $a(\tau)$ is an unknown function which can be determined by inserting Eq. (21) into Eq. (20). By taking the Fourier transform, one can demonstrate [21] that the equation is satisfied, provided we neglect the terms higher than $|k|^\alpha$, and

$$a(\tau) = A\left[1 - e^{-\lambda(\alpha+\theta\beta)\tau}\right]^{-1/(\alpha+\theta\beta)} ; \tag{22}$$

in the above expression

$$A = \left(\frac{\pi\lambda}{h_0\mathit{\Gamma}(\alpha)\sin(\alpha\pi/2)}\right)^{1/(\alpha+\theta\beta)} \tag{23}$$

and $h_0 = \int |x|^{-\theta\beta}p_0(x)\mathrm{d}x$.

Therefore, the problem is completely solved: both the position and time dependence of the density is uniquely determined. However, the constant h_0 has to remain unspecified unless we introduce additional assumptions about the density form at small $|x|$. A natural assumption in this context could be a stable distribution of $p(x,t)$ which satisfies Eq. (20) in the asymptotic limit if $\theta\beta < 1$ [24]. Then, the solution of the Fokker–Plank equation can be expressed in the form of the H-function [25]

$$p_0(x,\tau) = \frac{1}{\alpha}a(\tau)H_{2,2}^{1,1}\left[a(\tau)x\left|\begin{array}{l}(1-1/\alpha,1/\alpha),(1/2,1/2)\\ \\ (0,1),(1/2,1/2)\end{array}\right.\right] . \tag{24}$$

h_0, which follows from the evaluation of the Mellin transform and the comparison with Eq. (21), reads: $h_0 = \frac{2}{\pi\alpha}\mathit{\Gamma}(\theta\beta/\alpha)\mathit{\Gamma}(1-\theta\beta)\sin(\pi\theta\beta/2)$. For $\theta = 0$, $h_0 = 1$ and the stable distribution (24) is exact for all values of both arguments [26].

5. Relaxation to the stationary state

Having the τ-dependent solution calculated, we are in position to determine a relaxation to the stationary state as a function of the physical time. There are two typical relaxation patterns [1]. The ordinary Fokker–Planck equation is characterised by single modes decaying exponentially (Debye relaxation). On the other hand, if the problem is nonlocal in time and the Fokker–Planck equation contains a fractional derivative, a Mittag–Leffler pattern is observed; asymptotically, this corresponds to a power-law decay. In this section, we address the question how the relaxation pattern is affected by the medium nonuniformity. We restrict our considerations to the linear case and derive the density $p(x,t)$ from $p_0(x,\tau)$, Eq. (21), by applying Eq. (8). The Fourier–Laplace transform from the density reads

$$\widetilde{p}(k,u) = \frac{1}{u} - c|k|^{\alpha}u^{\beta-1}\int\limits_0^\infty a(\tau)^{-\alpha}e^{-\tau u^\beta}\mathrm{d}\tau\,, \tag{25}$$

where $c = \pi/\Gamma(\alpha+1)\sin(\alpha\pi/2)$. The exponentiation and integration term-by-term yields

$$\widetilde{p}(k,u) = \frac{1}{u} - cA^{-\alpha}|k|^{\alpha}\left[\frac{1}{u} + \sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\frac{u^{\beta-1}}{u^\beta+\lambda(\alpha+\theta\beta)j}\prod_{i=0}^{j-1}(q-i)\right], \tag{26}$$

where $q = \alpha/(\alpha+\theta\beta)$. The final expression follows from inverting of both transforms

$$\widetilde{p}(k,t) = 1 - cA^{-\alpha}[1-\chi(t)]|k|^{\alpha}\,, \tag{27}$$

and contains the relaxation function

$$\chi(t) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}E_\beta\Big[-\lambda(\alpha+\theta\beta)jt^\beta\Big]\prod_{i=0}^{j-1}(q-i)\,; \tag{28}$$

in the above formula, $E_\beta(x)$ stands for the Mittag–Leffler function. Since $\chi(t)$ does not depend on h_0, it is uniquely determined. The long-time limit of $\chi(t)$ results from the asymptotic behaviour of $E_\beta(x)$: $E_\beta(-t^\beta) \sim t^\beta/\Gamma(1-\beta)$. The straightforward calculation yields $\chi(t) = Rt^{-\beta}$ $(t \gg 1)$, where

$$R = \frac{[\lambda(\alpha+\theta\beta)]^{1/\beta}}{\Gamma(1-\beta)}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}j^{1/\beta}}{j!}\prod_{i=0}^{j-1}(q-i)\,. \tag{29}$$

Therefore, the algebraic decay pattern appears independent of θ but the actual relaxation time strongly varies with θ which is illustrated in Fig. 3.

The particle is attracted to the origin for negative θ which hampers the relaxation process and makes the relaxation time long. For positive θ, we observe the opposite effect. The decline of $R(\theta)$ is especially strong for small α and it becomes infinite when θ approaches the value $-\alpha\beta$.

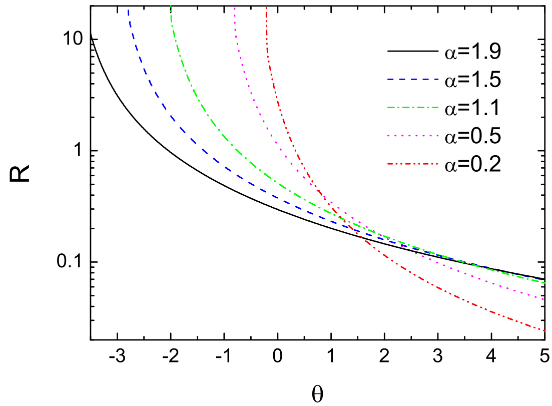

Fig. 3. Function $R(\theta)$ calculated from Eq. (29) for $\beta = 1/2$, $\lambda = 1$ and some values of α.

We note, finally, that the system of Langevin equations (3) for $F_{\mathrm{d}}(x)\nu(x) = 1-\lambda x$ corresponds to a generalised Fokker–Planck equation [21]

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\beta}\left[\lambda\frac{\partial}{\partial x}\left[xp(x,t')\right] + \frac{\partial^\alpha}{\partial|x|^\alpha}\left(|x|^{-\theta\beta}p(x,t)\right)\right]. \tag{30}$$

It contains two fractional derivatives: beside the Riesz–Weyl derivative, the Riemann–Liouville derivative

$${}_0D_t^{1-\beta}f(t) = \frac{1}{\Gamma(\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t \mathrm{d}t'\frac{f(t')}{(t-t')^{1-\beta}}\,. \tag{31}$$

6. Summary and conclusions

We have discussed stochastic properties of the particle subjected to a random force possessing long tails of the distribution and characterised by a long memory. This memory results from the presence of traps which are nonhomogeneously distributed: in the present study, algebraic with the parameter θ. The problem has been formalized by a subordination technique with a position-dependent random time distribution. In the absence of any potential, the variance exists and the diffusion rate can be described by its time dependence when one introduces a multiplicative noise near a boundary.

The variance rises with time as a power law and the diffusion index μ can be both smaller and larger than one indicating the anomalous transport. As expected, it depends on θ and the memory parameter β but the observed dependence $\mu(\alpha)$ — μ rises linearly with the stability index — is in contrast with the homogeneous case when $\mu = \beta$. Therefore, the diffusion index can vary with the stability index only if the trap distribution is nonhomogeneous.

In the presence of the external potential, the system converges with time to a stationary state. The case of the nonlinear oscillator was analysed. The asymptotics of the stationary solution is a power law and the potential makes the tail steeper than for the force-free case which may result in finite moments. However, also the medium heterogeneity influences the density slope and, if θ is negative, the variance may be finite even for relatively weak potentials. The pattern according to which the density converges to the stationary state has been discussed for the linear case; it is always algebraic for a long time, corresponding to the asymptotics of the Mittag–Leffler function, but the actual relaxation time strongly depends on θ. This dependence is very strong for small α and the relaxation time becomes infinite when $\theta \to -\alpha\beta$.

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