POSITIVITY AND UNITARITY CONSTRAINTS ON DIPOLE GLUON DISTRIBUTIONS*

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Dedicated to Andrzej Bialas in honour of his 80th birthday

In the high-energy domain, gluon transverse-momentum dependent distributions in nuclei obey constraints coming from positivity and unitarity of the colorless QCD dipole distributions through Fourier–Bessel transformations. Using mathematical properties of Fourier-positive functions, we investigate the nature of these constraints which apply to dipole model building and formulation.

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1. Introduction

The QCD dipole formalism [1] has proven to be quite successful as a tool for describing "low-x physics". This is the domain of high-energy scattering of particles on nuclei at high energy and moderate but high enough transverse momentum exchange allowing the QCD coupling constant to be small. One relevant example is the description and model building for the *transverse-momentum dependent* (TMD) gluon distributions in nuclei. They appear in the formulation of physical observables such as forward jet production [2] and forward dijet correlations [3] off nuclei by scattering of protons. In the QCD dipole formalism, *i.e.* in the large $N_{\rm c}$ and leading-log approximation of perturbative QCD, they are related to the *size-dependent* distribution of colorless gluon–gluon (gg) dipoles in the target nucleus.

Within some simplifying assumptions [2], this relation takes the form of a pair of Fourier–Bessel transforms, namely

$$\begin{aligned} \mathcal{G}(Y,k) &= \int_0^\infty r\mathrm{d}r J_0(kr)\mathcal{S}(Y,r)\,, \\ \mathcal{S}(Y,r) &= \int_0^\infty k\mathrm{d}k J_0(kr)\mathcal{G}(Y,k)\,, \end{aligned} \tag{1.1}$$

where $1-\mathcal{S}(Y,r)=\mathcal{T}(Y,r)$ is the gg distribution in the target as a function of their size r at total center-of-mass energy e^Y. Its Fourier–Bessel partner $\mathcal{G}(Y,k)$ enters into the expression of the TMD gluon distribution

$$xG(x,k) = \frac{k^2 N_\mathrm{c}}{2\pi^2\alpha_\mathrm{S}}\; A_\perp\; \mathcal{G}(Y,k)\,, \tag{1.2}$$

where $x=e^{-Y}$, α_S is the QCD coupling constant, and $A_\perp$ the target transverse area[1].

Interestingly enough, the dipole formalism is submitted to positivity and unitarity conditions which gives rise to nontrivial constraints on the pair of Fourier transforms (1.1). Let us specify them as follows:

(i) Positivity constraint on $\mathcal{T}(Y,r)$. The dipole gluon distribution $xG(x,k)$ is expected to be positive. In fact, it is required to be so, since it is proportional [2, 3] to physical observables. Hence, from (1.2), $\mathcal{G}(Y,k)$ is positive and through (1.1), it induces nontrivial mathematical constraints on the gg dipole distribution in the target $\mathcal{T}(Y,r)$. Its Fourier–Bessel transform should be positive.

(ii) Unitarity constraint on $\mathcal{G}(Y,k)$. The dipole distribution $\mathcal{T}(Y,r)$ is also expressed as the dipole-target total cross section at total incident energy $E=e^Y$, up to a normalization. As such, it obeys the S-matrix unitarity condition $\mathcal{S}=1-\mathcal{T}(Y,r)\geq 0$. Hence, through relations (1.1), unitarity implies that $\mathcal{G}(Y,k)$ has to have a positive Fourier–Bessel transform.

In both cases, one has functions whose Fourier transforms are positive (here, 2-dimensional radial *i.e.* Fourier–Bessel ones). The aim of the present contribution is to show some interesting physical consequences of this mathematical property. We call it $\mathcal{F}$-positivity.

[1] This is the so-called *dipole gluon* distribution which is to be distinguished from the Weizsäcker–Williams gluon distribution. The Weizsäcker–Williams distribution cannot be expressed as the Fourier transform of a QCD dipole distribution [3].

2. Fourier-positivity

$\mathcal{F}$-positivity is the mathematical property of real-valued functions whose Fourier transforms are positive [4]. Contrary to expectation, there is no explicit parametrization of the set of $\mathcal{F}$-positive functions. They instead can be characterized by an infinite set of necessary conditions, which constitute the Böchner theorem [5]. For a given $\mathcal{F}$-positive real function $\psi(\vec{v})$ whose d-dimensional Fourier transform $\varphi(\vec{w})$ is positive, the Böchner theorem states that the function ψ is *positive definite*, that is

$$\sum_{i,j=1}^{n} u_i \ \psi\left(\vec{v_i} - \vec{v_j}\right) u_j > 0 \,, \qquad \forall u_i \,, \quad \forall \vec{v}_i \,, \quad \forall n \,. \tag{2.1}$$

In the case of the Fourier–Bessel transforms (1.1), the conditions (2.1) apply to any set of two-dimensional vectors $\vec{v}_i$. Hence, for any $n \in \mathbb{N}$ and for any set of numbers $\{u_i, i = 1, \ldots, n\}$, the $n \times n$ matrix $\mathbb{M}$ with elements $\psi(|\vec{v_i} - \vec{v_j}|)$ is positive definite. This is equivalent to the property that the lowest eigenvalue of $\mathbb{M}$ remains positive for all $\vec{v}_i, u_i$, and all values of n. In the case of (1.1), two-dimensional transverse position coordinates $\vec{r}$ (with $r = |\vec{r}\,|$) or transverse momentum space $\vec{k}$ (with $k = |\vec{k}|$) may be involved, depending on the required physical constraint we shall consider later on.

Applying the whole set of consitions (2.1) is not realistic for practical purposes. For this sake, we have developed in the recent years [6–8] specific tools for practical tests of $\mathcal{F}$-positivity. They are formulated, in various forms, in terms of an optimized finite subset of necessary conditions issued from 1- and 2-dimensional versions of relations (2.1).

One of the conditions coming from the Böchner theorem which appears to be relevant for our problems is the following. Let us consider the 3×3 matrix $\mathbb{M}_3$ with matrix elements

$$\{\mathbb{M}_3\}_{i,j} \equiv \ \{\psi\left(|\vec{v_i} - \vec{v_j}|\right)\} \,, \qquad \vec{v_i} = \{0, 0\} \,, \{0, v\} \,, \{v \sin\theta, v \cos\theta\} \,, \tag{2.2}$$

which leads to the $\mathcal{F}$-positivity conditions for the matrix

$$\mathbb{M}_3 = \begin{pmatrix} \psi(0) & \psi(v) & \psi\left(2v \sin\frac{\theta}{2}\right) \\ \psi(v) & \psi(0) & \psi(v) \\ \psi\left(2v \sin\frac{\theta}{2}\right) & \psi(v) & \psi(0) \end{pmatrix} . \tag{2.3}$$

Positive-definiteness implies positivity of the matrix determinant and of its minors along its diagonal. This leads, up to a rescaling of v, to the inequalities

$$\psi(0) > \psi(v) > 2\frac{\psi^2\left(v/\left[2\sin\frac{\theta}{2}\right]\right)}{\psi(0)} - \psi(0) \,, \qquad \forall v > 0 \,, \qquad \forall \theta \in [0, \pi] \,. \tag{2.4}$$

Note that a larger number of points, provided they include the points of (2.2), still leads to condition (2.4), together with others, forming a hierarchy of necessary condition [8] for $\mathcal{F}$-positivity.

An important addendum to the $\mathcal{F}$-positivity tests for a 2-dimensional radial function $\psi(v)$ has been noticed in [7]. They can be extended to the action of the radial Laplacian on $\psi(v)$, namely

$$\begin{aligned} \triangle_2\left[\psi\right](v) &\equiv -\frac{1}{v}\frac{\mathrm{d}}{\mathrm{d}v}\left(v\frac{\mathrm{d}\psi(v)}{\mathrm{d}v}\right) \\ &= \int_0^\infty w^3 \mathrm{d}w J_0(vw)\,\varphi(w) > 0\,, \end{aligned} \tag{2.5}$$

where $\phi(w) > 0$ is the positive Fourier–Bessel transform of $\psi(v)$. Hence, $\mathcal{F}$-positivity tests then apply not only to ψ, but also to $\triangle_2[\psi]$, and its iterations, provided the integrals, such as in (2.5) for the first one, remain convergent.

3. Positivity constraints on the dipole distribution

Through the second equation (1.1), the positivity of the gluon distribution $\mathcal{G}(Y,k)$ induces $\mathcal{F}$-positivity constraints on the dipole amplitude $\mathcal{T}(Y,r) = 1 - \mathcal{S}(Y,r)$.

In order to conveniently formulate these $\mathcal{F}$-positivity constraints, let us turn to the first equation (1.1). By double integration by part on the right-hand side, one obtains

$$\begin{aligned} \mathcal{G}(Y,k) &= \int_0^\infty r\mathrm{d}r J_0(kr)(1-\mathcal{T}(Y,r)) \\ &= \int r\mathrm{d}r J_1(kr)\frac{\partial}{\partial r}\mathcal{T}(Y,r) \\ &= \int r\mathrm{d}r J_0(kr)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\mathcal{T}(Y,r)\right) \\ &= \int r\mathrm{d}r J_0(kr)\triangle_2\left[\mathcal{S}\right](Y,r) > 0\,. \end{aligned} \tag{3.1}$$

Performing the integrations, we used the known derivative relations between Bessel functions, successively, $rJ_0(r) = \frac{\partial}{\partial r}J_1(r)$ and $J_1(r) = -\frac{\partial}{\partial r}J_0(r)$.

Now, the key point, as discussed in our recent work [9], is the behavior of $\mathcal{T}(Y,r)$ when the dipole size $r \to 0$. The standard leading order QCD behavior near the origin is given by the property of "color transparency",

$\mathcal{T}(Y,r) \propto r^2$. Following [9], higher order QCD corrections of the dipole amplitude near the origin leads to the modified behavior

$$\mathcal{T}(Y,r) \propto r^{2+\epsilon} \quad \text{when} \quad r \to 0\,, \tag{3.2}$$

where $0 < |\epsilon| \ll 1$ parameterizes the slight deviations from color transparency. On the one hand, they are expected to come from the running of the coupling constant. On the other hand, they are generated by the resummation of the perturbative QCD expansion in the low-x (high Y) domain of dipole models.

Interestingly enough, the $\mathcal{F}$-positivity inequalities (2.4) impose the simple but nontrivial condition $\epsilon \le 0$ in (3.2). This condition has consequences [9] on various dipole models when the running QCD coupling constant $\alpha_{\rm S}(r)$ at short dipole separation is taken into account.

Let us consider, for instance, the saturation model consistent with the leading order DGLAP evolution [10]. The dipole amplitude reads

$$\mathcal{T}(Y,r) = 1 - \mathcal{S}(Y,r) = 1 - \exp\left(-\frac{\pi^2 r^2}{3\sigma_0}\alpha\left(\mu^2\right) xg\left(x,\mu^2\right)\right), \qquad x \equiv e^{-Y}\,. \tag{3.3}$$

Here, $xg(x,\mu^2)$ is the gluon distribution function in the proton considered at momentum fraction x with r-dependent momentum scale $\mu^2 = C/r^2 + \mu_0^2$, with $\alpha(\mu^2) \propto \log \Lambda^2_{\rm QCD}/\mu^2$ and σ_0, C are phenomenological constants fitted to deep-inelastic data. The interest of this model is that it combines the saturation effect $\mathcal{S} \to 1$ at large dipole size r with a behavior at small r compatible with the DGLAP evolution equation.

In Eq. (3.3), the color transparency behavior $\mathcal{T}(Y,r) \propto r^2$, $r \to 0$, is naturally obtained for fixed $\alpha_{\rm S}$ and constant $xg(x,\mu^2)$ at first order of the QCD perturbative expansion. This is equivalent to the original Golec-Biernat and Wüsthoff model [11] which reads

$$\mathcal{T}(Y,r) = 1 - \exp\left(-\frac{r^2}{4}Q_{\rm S}^2(Y)\right), \tag{3.4}$$

where $Q_{\rm S}(Y)$ is the "saturation scale". In fact, the model verifies the $\mathcal{F}$-positivity constraints, as it is obvious from (1.1) by Gaussian integration.

Adding higher orders in the coupling constant modifies that behavior and its consequences. The running of $\alpha_{\rm S}(\mu^2) \sim 1/\log(1/r^2)$ leads to an effective value $\epsilon_{\rm run} < 0$ in Eq. (3.2). On the other hand, the summation of the double leading logarithms of the QCD perturbative expansion at small x leads to

$$xg\left(x,\mu^2\right) \approx \sum_n \frac{\left[\,Y\int_{\mu_0^2}^{\mu^2}\alpha\left(k^2\right){\rm d}k^2/k^2\right]^n}{(n!)^2} \sim e^{{\rm cst.}\sqrt{Y\ \log\log\frac{1}{r^2}}}\,. \tag{3.5}$$

Hence, in this case, the overall modification of the color transparency behavior due to the increase of the gluon parton distribution function at small r leads to a positive contribution $0 < \epsilon_{\rm resum} \ll 1$ to the behavior (3.2). The overall effect of both higher order contributions leads to

$$\frac{\mathcal{T}(Y,r)}{r^2} \approx \frac{e^{\mathrm{cst.}\sqrt{Y \log\log \frac{1}{r^2}}}}{\log \frac{1}{r^2}} \to 0 \quad \text{when} \quad r \to 0 \,. \tag{3.6}$$

Hence, one finds that the compensation of the running effect by the resummation one in model (3.3) is incomplete, leading to $\epsilon = \epsilon_{\rm run} + \epsilon_{\rm resum} < 0$ in Eq. (3.2). The effect may be small in absolute value, but is non-zero.

$\mathcal{F}$-positivity of the model (3.3) is thus violated. Hence, the corresponding $\mathcal{G}(Y,k)$ is not everywhere positive. This can be verified by explicit Fourier transform. The phenomenological and theoretical relevance of such and similar behavior in various dipole models has been discussed in Ref. [9]. It may lead to a reformulation of dipole models in the presence of higher order QCD corrections. We shall discuss this conclusion further on in Section 5.

4. Unitarity constrains on the gluon distribution

The unitarity condition on the dipole distribution $\mathcal{S}(Y,r) = 1 - \mathcal{T}(Y,r) \geq 0$, induces $\mathcal{F}$-positivity of the gluon TMD distribution $\mathcal{G}(Y,k)$ as shown by the relations Eq. (1.1). Indeed, let us consider the second line of Eq. (1.1). The constraint reads

$$\mathcal{S}(Y,r) = \int k\mathrm{d}k J_0(kr) \mathcal{G}(Y,r) \geq 0 \,, \tag{4.1}$$

which involves the $\mathcal{F}$-positivity constraints (2.1) for $\mathcal{G}(Y,r)$. The necessary condition (2.4) of Section 2 reads

$$\mathcal{G}(Y,k=0) > \mathcal{G}(Y,k) > 2\frac{\mathcal{G}^2\left(Y, k/\left[2\sin\frac{\theta}{2}\right]\right)}{\mathcal{G}(Y,0)} - \mathcal{G}(Y,k=0)\,. \tag{4.2}$$

The limiting quantity $\mathcal{G}(Y,k=0)$ is difficult to estimate directly from the gluon transverse momentum spectrum. However, from the first line of (1.1), one finds the expression

$$\mathcal{G}(Y,0) = \int_0^\infty r\mathrm{d}r \mathcal{S}(Y,r) \,, \tag{4.3}$$

which can be estimated from the associated dipole model. In the original GBW model (3.4), one finds $\mathcal{G}(Y,0) = 2\, Q_{\rm S}^{-2}(Y)$, which in a characteristic length squared scale of saturation models. More generally, a model verifying "geometric scaling" [12], $\mathcal{S}(Y,r) = \mathcal{S}(rQ_{\mathrm{S}}(Y))$ gives rise to a similar dependence on the saturation scale, namely

$$\mathcal{G}(Y,0) = Q_{\mathrm{S}}^{-2}(Y) \int_0^\infty \rho \mathrm{d}\rho \; \mathcal{S}(\rho) = \lambda^{-2} \; Q_{\mathrm{S}}^{-2}(Y) \,, \tag{4.4}$$

where $\lambda = \mathcal{O}(1)$ is specified by the corresponding dipole model. All in all, for generic saturation models (having only at most small violations of geometric scaling), one expects $\mathcal{G}(Y,0) \sim \mathcal{O}(Q_{\mathrm{S}}^{-2}(Y))$.

Combining Eqs. (4.2) and (4.4), one obtains

$$1 > \tilde{g}\,(Y,k) > 2\tilde{g}^2 \left(Y, \frac{k}{2\sin\frac{\theta}{2}} \right) - 1 \,, \tag{4.5}$$

where we introduced the short-hand notation

$$\tilde{g}\,(Y,k) \equiv \frac{\mathcal{G}\,(Y,k)}{\mathcal{G}\,(Y,0)} = \lambda^2 Q_{\mathrm{S}}^2(Y)\; \mathcal{G}\,(Y,k) \;. \tag{4.6}$$

Hence, the relation (4.5) induces bounds on the magnitude of the TMD gluon distribution in the whole momentum range. In particular, from the leftmost inequality in (4.5), it appears that

$$\tilde{g}\,(Y,k) \equiv \lambda^2 Q_{\mathrm{S}}^2(Y) \mathcal{G}\,(Y,k) < 1 \,. \tag{4.7}$$

Thus, $\mathcal{G}$ cannot rise significantly above the inverse squared of the saturation scale. The quantitative *upper* bound on $\tilde{g}$ is a function of the constant λ and thus of the dipole model in use.

It is also interesting to take into account the second inequality of Eq. (4.5). For a given value k of the transverse momentum and varying the angle $\theta \in [0,\pi]$, the corresponding range is $[k/2,\infty]$. Quoting $\tilde{g}_{\mathrm{max}}(Y,k) \leq 1$, the maximum value of $\tilde{g}$ in this range, we obtain now a *lower* bound on $\tilde{g}$. Combining the upper and lower bounds, one gets

$$1 > \tilde{g}_{\mathrm{max}}(Y,k) > \tilde{g}\,(Y,\kappa) > 2\tilde{g}_{\mathrm{max}}(Y,k)^2 - 1 \,, \qquad \kappa \in [k/2,\infty] \,. \tag{4.8}$$

However, in order for the lower bound in Eq. (4.8) to be operating, one has the condition

$$\tilde{g}_{\mathrm{max}}(Y,k) > \sqrt{2}/2 \,, \tag{4.9}$$

which limits the range of validity of the lower bound in k.

One typical example is when $\tilde{g}$ is a monotonically decreasing function of k then with $\tilde{g}_{\mathrm{max}}(Y,k) = \tilde{g}(Y,k/2)$. Then, (4.9) translates into

$$1 > \tilde{g}\,(Y,k) > 2\tilde{g}^2(Y,k/2) - 1 \tag{4.10}$$

with the condition $\tilde{g}(Y,k/2) > \sqrt{2}/2$. In all cases, Eq. (4.10) works in the lower k range.

5. Discussion

There are interesting phenomenological and theoretical consequences of the mathematical constraints due to positivity and unitarity derived in the previous sections. In particular, the positivity constraints of Section 3 appear to cast a doubt on some formulations of QCD dipole models when the running of the QCD coupling constant in coordinate space $\alpha_S(r)$ is taken into account. Asymptotic freedom for short separation of the gluons in the QCD dipole apparently leads to a contradiction with $\mathcal{F}$-positivity.

In order to show the relevance of positivity and unitarity constraints on an example, we show in figure 1 the results for the McLerran–Venugopalan (MV) model with running coupling [13], using phenomenologically realistic rapidity and model parameters. For concreteness, we thus choose a specific version used in a recent summation model [14].

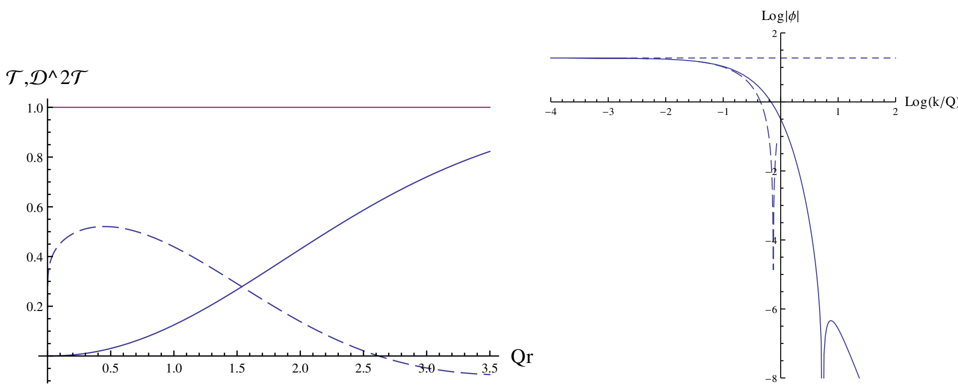

Fig. 1. Positivity and unitarity tests of the McLerran–Venugopalan model with running coupling. Left: the dipole amplitude: solid line, $\mathcal{T}(Y,r)$, dashed line, $\mathcal{D}^2\mathcal{T}(Y,r)$, see (5.2), as a function of rQ. Right: its Fourier-transform $\varphi(Y,k/Q) \equiv \mathcal{G}(Y,k)$ in a $\log|\varphi|, \log(k/Q)$ plot, and the upper and lower bounds from Eq. (4.10) due to unitarity: solid line, the absolute value of $\varphi(Y,k/Q) \equiv \mathcal{G}(Y,k)$, showing the positivity violation at a value of k/Q after the dip signalling the zero. The upper and lower bounds due to unitarity are shown with discontinulous lines: short-dashed line, the upper bound $\varphi(Y,0)$, long-dashed line, the absolute value of the lower bound. Note that the lower bound becomes negative (and thus not operating) beyond the dip signaling the zero.

The corresponding dipole amplitude reads

$$\mathcal{T}(Y,r) = \left\{1 - \exp\left[-\left(\frac{1}{4}(rQ(Y))^2\alpha_S(rC)\left[1+\log\left(\frac{\bar{\alpha}_{\rm sat}}{\bar{\alpha}_S(rC)}\right)\right]\right)^p\right]\right\}^{1/p}, \tag{5.1}$$

where $\alpha_S(rC) \propto \log^{-1}(1/(rC)$ is the running QCD coupling in transverse coordinate space and $C, \bar{\alpha}_{\rm sat}, p$ are phenomenological constants. Following Eq. (3.1), $\mathcal{F}$-positivity constrains the double derivative of the dipole amplitude (5.1) defined as follows

$$\mathcal{D}^2\mathcal{T}(Y,r) \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\mathcal{T}(Y,r)\right) . \tag{5.2}$$

(i) *Positivity constraints.* The positivity violation and its source appear clearly on the curves drawn in Fig. 1. The figure on the left-hand side shows both the amplitude $\mathcal{T}$ and its double derivative $\mathcal{D}^2\mathcal{T}$, Eq. (5.2). This last function has a positive derivative near the origin (*i.e.* $\epsilon > 0$ in Eq. (3.2)) which violates the $\mathcal{F}$-positivity constraint (2.4). On the right-hand side, the positivity violation is made manifest by explicit computation of the Fourier–Bessel transform. The positivity violation can be traced back to the inverse logarithmic coupling of the running constant which is not compensated by the log–log term which appears into brackets in (5.1). Hence, the formulation of QCD dipole models with a r-dependent coupling constant, $\alpha_S(r)$, leads to a violation of the expected positivity of the TMD gluon distribution. This violation appears on the *ultra-violet* side ($k/Q > 1$) of the TMD gluon distribution spectrum.

(ii) *Unitarity constraints.* Figure 1, left shows explicitely that the unitarity constraints (4.10) are satisfied by the amplitude of the McLerran–Venugopalan model with running coupling, as can be easily checked on Eq. (5.1). The constraints are shown by discontinuous lines in Fig. 1, right. They are clearly operating on the *infra-red* ($k/Q < 1$) behavior of the gluon TMD distributions. This comes from the upper bound being the limiting value $\mathcal{G}$ at $k = 0$, and from the limiting condition (4.9) on the lower bound to be valid.

As a concluding remark, both positivity and unitarity constraints provide quite general constraints on the QCD dipole models, which are of common and useful use in high energy phenomenology. We see that even the application of the Böchner theorem on the low rank 3×3 matrix case, leads to nontrivial consequences. These are remarkably distributed between the ultra-violet (for positivity) and infra-red (for unitarity) sectors of the TMD gluon spectrum.

Our study was limited to leading-log orders of dipole models and to the lower rank matrix tests of the Böchner theorem. As an outlook, it is clear that our observations ask for a developed study of $\mathcal{F}$-positivity constraints beyond leading orders and smaller matrix rank.

The studies presented here mainly come from a thorough and long term collaboration with Bertrand Giraud (IPhT, Saclay) and I warmly thank him for this fruitful common work.

The present contribution has been elaborated and written in honor of Andrzej Bialas with whom I have collaborated during so many years and whom I consider both as a master and a close friend. Many domains of my research, including the dipole formalism (remembering exciting after-dinner sessions in our neighboring homes in the mid of 90's!), found their starting point and developments in our lively discussions and our studies in common. For the dipole models, see among other common publications [15].

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