NEW SOLUTION TO THE TWO-DIMENSIONAL LENZ–ISING–ONSAGER PROBLEM

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(Received April 15, 2016)

In this work, a new analytical solution to the two-dimensional Lenz–Ising–Onsager (2D LIO) problem in zeroth external magnetic field is presented. The developed approach is based on using twice the transfer-matrix method and generalized Jordan–Wigner transform. This allows to reduce the initial problem to the problem of the many-particle (fermions) interaction.

DOI:10.5506/APhysPolB.47.2079

1. Introducion

In the year 1920, Lenz [1] presented, now very well-known, a simplified model of the magnetic material as a system of N interacting with each other magnetic dipoles (spins) which are in the vertices of crystal lattice. Five years later, in 1925, Ising [2] presented an exact solution to the one-dimensional Lenz model with the nearest-neighbors interaction and arbitrary external magnetic field. It was shown in particular, that the phase transition from paramagnetic to ferromagnetic state at non-zero temperature is absent in the frame of this model. In the year 1944, Onsager in his path-breaking work [3] presented the exact solution for two-dimensional problem in zeroth external magnetic field ($H = 0$). His solution indicated the existence of paramagnetic-to-ferromagnetic state transition at non-zero temperature. Since then, many other methods of obtaining the exact solution have been proposed (see, for example, Refs. [4, 5]). It is known that till now, there is no any exact solution for this model in non-zero external magnetic field ($H \neq 0$) and this is despite the tremendous efforts undertaken by the physicists and mathematicians during the last seven decades.

All of the up-to-now proposed methods for obtaining Onsager's solutions encounter great difficulties when one tries to generalize them for the case of non-zero external magnetic field. To obtain such solution for 2D Lenz–

Ising–Onsager (2D LIO) model in arbitrary external magnetic field is very important in regard of the scaling hypothesis [7]. One of the methods for obtaining Onsager's solutions proposed almost 18 years ago [5] reduces the problem to the summation over all Hamiltonian graphs (loops) on the simple rectangular $(N \times M)$ lattice. However, also in this case, taking into account an external magnetic field encounters great difficulties.

In this work, we present new analytical approach to obtaining the Onsager's solution, which could be useful for obtaining an exact solution for 2D LIO model in arbitrary external magnetic field. Despite the fact that the proposed approach looks like more complicated in comparison, for instance, with that proposed in Ref. [4], it seems that our ideas can be useful in searching for the solution mentioned above.

2. Hamiltonian of the 2D LIO model

Consider the well-known two-dimensional model of magnetic material in an external magnetic field (H), described by the following Hamiltonian (2D LIO model)

$$\mathcal{H}(\sigma) = -J_1 \sum_{nm} \sigma_{nm}\sigma_{n+1,m} - J_2 \sum_{nm} \sigma_{nm}\sigma_{n,m+1} - H \sum_{nm} \sigma_{nm} \,, \tag{1}$$

defined on the rectangular lattice with $(N \times M)$ vertices and with the nearest-neighbors interaction $(J_{1,2} > 0)$. Each 'spin' of the system $(\sigma_{nm} = \pm 1)$ interacts with the external magnetic field directed 'up'. The statistical sum of the system is

$$Z(T,H) = \sum_{\{\sigma\}} \exp(-\beta\mathcal{H}\{\sigma\}) \,, \qquad \beta = \frac{1}{k_{\mathrm{B}}T} \,, \tag{2}$$

where the sum runs over all possible configurations of spins $\{\sigma_{nm} = \pm 1\}$, T is the temperature and k_{B} is Boltzmann constant.

In Ref. [4] by means of transfer-matrix method, the following expression for the statistical sum was obtained

$$Z(\beta,H) = \mathrm{Tr}\left(\hat{T}\right)^N \,, \qquad \hat{T} = T_1 T_2 T_h \,, \tag{3}$$

where the matrices $T_{1,2,h}$ are defined by the formulae

$$\begin{aligned} T_1 &= (2\sinh 2K_1)^{M/2} \exp\left(K_1^* \sum_m \tau_m^x\right) , \qquad T_2 = \exp\left(K_2 \sum_m \tau_m^z \tau_{m+1}^z\right) , \\ T_h &= \exp\left(h \sum_m \tau_m^z\right) , \qquad K_{1,2} \equiv \beta J_{1,2} \,, \qquad h \equiv \beta H \,, \qquad K_{1,2} > 0 \,, \end{aligned} \tag{4}$$

and $(\tau_m^{x,z})$ are $(2^M \times 2^M)$-Pauli matrices, while the constants K_1 and K_1^* obey the following relations

$$\tanh K_1^* = e^{-2K_1}\,, \qquad \tanh K_1 = e^{-2K_1^*}\,, \qquad \sinh 2K_1 \sinh 2K_1^* = 1\,. \quad (5)$$

The difficulties with taking into account an external magnetic field h in this approach [4] are well-known. They are related to the fact that in the interacting fermions $(\alpha_m^\dagger, \alpha_m)$ representation, T_h-matrix has the operator form

$$T_h = \exp\left[h \sum_{m=1}^{M} \chi_m \left(c_m^\dagger + c_m\right)\right]\,, \quad (6)$$

which includes the phase factors χ_m

$$\chi_m \equiv \exp\left(\imath\pi \sum_{j=1}^{m-1} c_j^\dagger c_j\right) = (-1)^{\sum_{j=1}^{m-1} c_j^\dagger c_j} = \prod_{j=1}^{m-1} (-1)^{c_j^\dagger c_j} = \prod_{j=1}^{m-1} \left(1-2c_j^\dagger c_j\right)\,. \quad (7)$$

These factors practically cannot be eliminated and above all, the last one makes the diagonalization of $\hat{T}$-operator (3) impossible. Besides, the operator of the total number of fermions ($\hat{M} = \sum_{m=1}^M c_m^\dagger c_m$) does not commute with the T_h-operator (6). That the operators $\hat{M} = \sum_{m=1}^M c_m^\dagger c_m$ and T_h do not commute in the framework of this formalism means that there is the lack of conservation of states with the odd and even numbers of fermions. That is, the external field does mix them. This provides the indirect proof of the lack of phase transition at non-zero temperature and $\Re h \neq 0$ (Lee–Yang Theorem, [6]).

In order to take into account the external field h, in Ref. [5], a new approach to the solution of 2D LIO problem was proposed. This one is based on the considering the three-dimensional problem 3D LIO in external field with the Hamiltonian of the form

$$\begin{aligned}\mathcal{H} \;=\; -\sum_{(n,m,k)=1}^{NMK} &\Big(J_1\sigma_{nmk}\sigma_{n+1,mk} + J_2\sigma_{nmk}\sigma_{n,m+1,k} \\ &+J_3\sigma_{nmk}\sigma_{nm,k+1} + H\sigma_{nmk}\Big)\,, \end{aligned} \quad (8)$$

whose statistical sum is

$$\begin{aligned} Z_3(h) \;=\; \sum_{\sigma_{111}=\pm1} \cdots \sum_{\sigma_{NMK}=\pm1} e^{-\beta\mathcal{H}} = \sum_{\{\sigma_{nmk}=\pm1\}} \exp\Bigg[\sum_{nmk}\Big(K_1\sigma_{nmk}\sigma_{n+1,mk} \\ +K_2\sigma_{nmk}\sigma_{n,m+1,k} + K_3\sigma_{nmk}\sigma_{nm,k+1} + h\sigma_{nmk}\Big)\Bigg]\,. \end{aligned} \quad (9)$$

Here, the following notations are used

$$\beta = 1/k_{\rm B}T\,, \qquad K_{1,2,3} \equiv \beta J_{1,2,3}\,, \qquad h \equiv \beta H\,, \qquad K_{1,2,3} > 0\,.$$

Now, by means of the transfer-matrix method, one can write down the next expression for the statistical sum (9)

$$Z_3(h) = \mathrm{Tr}(T)^K\,, \qquad T = T_3 T_2 T_1 T_h\,, \tag{10}$$

$$T_1 = \exp\left(K_1 \sum_{nm} \tau^z_{nm}\tau^z_{n+1,m}\right)\,, \qquad T_2 = \exp\left(K_2 \sum_{nm} \tau^z_{nm}\tau^z_{n,m+1}\right)\,, \tag{11}$$

$$T_3 = (2\sinh 2K_3)^{NM/2} \exp\left(K_3^* \sum_{nm} \tau^x_{nm}\right)\,, \qquad T_h = \exp\left(h \sum_{nm} \tau^z_{nm}\right)\,, \tag{12}$$

where $(\tau^{x,z}_{nm})$ are $(2^{NM} \times 2^{NM})$-Pauli matrices, while the constants K_3 and K_3^* obey the following relations

$$\tanh K_3^* = e^{-2K_3}\,, \qquad \tanh K_3 = e^{-2K_3^*}\,, \qquad \sinh 2K_3 \sinh 2K_3^* = 1\,. \tag{13}$$

Further on, the author's of Ref. [5] main idea was to use the generalized Jordan–Wigner transforms [8] and write down the matrices $T_{1,2,3,h}$ (11)–(12) in the interacting fermions representation and only then pass to zero interacting constant K_3 ($K_3 \to 0$, $K = 1$). This one enables to eliminate the phase factors in the T_h-operator (12) and for the statistical sum (10), one gets the next expression

$$Z_2(h) = 2^{NM} \langle 0| T_h T_2 T_1 |0\rangle\,, \tag{14}$$

where $\langle 0| (...) |0\rangle$ is the vacuum matrix element in the finite-dimensional Fock space ($\alpha_{nm}|0\rangle = 0$, $\beta_{nm}|0\rangle = 0$), while the operators $T_{1,2}$ and T_h are given by the following formulae

$$T_1 = \exp\left[K_1 \sum_{n,m=1}^{N,M} \left(\beta^\dagger_{nm} - \beta_{nm}\right)\left(\beta^\dagger_{n+1,m} + \beta_{n+1,m}\right)\right]\,, \tag{15}$$

$$T_2 = \exp\left[K_2 \sum_{n,m=1}^{N,M} \left(\alpha^\dagger_{nm} - \alpha_{nm}\right)\left(\alpha^\dagger_{n,m+1} + \alpha_{n,m+1}\right)\right]\,, \tag{16}$$

$$T_h = (\cosh h)^{NM} \exp\left\{\alpha^2 \sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{M-m} \alpha_{n,m+k}\alpha_{nm}\right\}\,. \tag{17}$$

Here, $\alpha = \tanh h$, while α_{nm} and β_{nm} are Fermi's creation and annihilation operators, which interrelate with each other by means of unitary transforms of the form [5, 8]

$$\begin{aligned}
\alpha^{\dagger}_{nm} &= \exp\left(i\pi \sum_{k=1}^{n-1} \sum_{l=1}^{M} \tau^{+}_{kl}\tau^{-}_{kl} + i\pi \sum_{l=1}^{m-1} \tau^{+}_{nl}\tau^{-}_{nl} \right) \tau^{+}_{nm} \,, \\
\beta^{\dagger}_{nm} &= \exp\left(i\pi \sum_{k=1}^{N} \sum_{l=1}^{m-1} \tau^{+}_{kl}\tau^{-}_{kl} + i\pi \sum_{k=1}^{n-1} \tau^{+}_{km}\tau^{-}_{km} \right) \tau^{+}_{nm} \,, \\
\alpha^{\dagger}_{nm} &= \exp(i\pi\varphi_{nm})\beta^{\dagger}_{nm} \,, \\
\varphi_{nm} &= \left[\sum_{k=n+1}^{N} \sum_{p=1}^{m-1} + \sum_{k=1}^{n-1} \sum_{p=m+1}^{M} \right] \alpha^{\dagger}_{kp}\alpha_{kp} = [\cdots]\beta^{\dagger}_{kp}\beta_{kp} \,.
\end{aligned} \tag{18}$$

At last, knowing the statistical sum (14), one can write down the next expression for the free-energy of the system with respect to the single spin in the thermodynamic limit

$$-\beta f_2(h) = \lim_{N,M\to\infty} \frac{1}{NM} \ln Z_2(h) \,. \tag{19}$$

Let us pay attention to the fact that statistical sum (14) obviously is even function with respect to external field h, and hence the free-energy (19) is also even with respect to h.

As it was shown by the author of [5], expression (14) for statistical sum can be reduced to the summation over all possible Hamiltonian loops on the rectangular lattice $(N \times M)$. In the case of zeroth external field $(h = 0)$, such summation can be done in closed form, and as a result, in the thermodynamic limit, one gets Onsager's solution. Unfortunately, in the case of $(h \neq 0)$, this approach encounters great difficulties, the last one brings us to search for other methods of solving the discussed problem.

3. T_1-operator transformation

It is obvious that the problem would be solved if one could somehow eliminate the phase factors $\theta_{nm} = \exp(i\pi\varphi_{nm})$ expression (15) for the T_1-operator

$$T_1 = \exp\left[K_1 \sum_{n,m=1}^{N,M} \theta_{nm} \left(\alpha^{\dagger}_{nm} - \alpha_{nm} \right) \theta_{n+1,m} \left(\alpha^{\dagger}_{n+1,m} + \alpha_{n+1,m} \right) \right] . \tag{20}$$

In this expression, we pass from β Fermi-operators to α-operators in accordance with relations (18). All the attempts to eliminate directly the phase factors θ_{nm} in ket-vector $T_1 |0\rangle$ encounter great difficulties. Below, we present one of the possible intermediate strategies of the elimination the aforementioned factors, which is based on the sequence of transformations of T_1-operator. First of all, one can write down the bra-vector $\langle 0| T_h T_2$ in the form [5]

$$\begin{aligned}\langle 0| T_h T_2 &= (\cosh h)^{NM} \left(\prod_{0<q,p<\pi} A_2(p,h) \right)^2 \\ &\quad \times \langle 0| \exp\left[\sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{M-m} b(k) \alpha_{n,m+k} \alpha_{nm} \right], \end{aligned} \tag{21}$$

where $b(k)$, $A_2(p,h)$ i $B_2(p,h)$ are given by the following formulae

$$\begin{aligned} b(k) &= \frac{1}{M} \sum_{0<p<\pi} 2B_2(p,h) \sin(kp)\,, \\ A_2(p,h) &= \cosh 2K_2 - \sinh 2K_2 \cos p + \alpha(h,p) \sinh 2K_2 \sin p\,, \\ B_2(p,h) &= \frac{\alpha(h,p)[\cosh 2K_2 + \sinh 2K_2 \cos p] + \sinh 2K_2 \sin p}{A_2(p,h)}\,, \\ \alpha(h,p) &= \tanh^2 h \frac{1+\cos p}{\sin p}\,. \end{aligned} \tag{22}$$

In order to eliminate the phase factors θ_{nm} in the ket-vector $T_1 |0\rangle$, we present the T_1-operator (15) in terms of Pauli operators $\tau^{\pm}_{nm}$

$$T_1 = \exp\left[K_1 \sum_{nm} \left(\tau^+_{nm} + \tau^-_{nm}\right) \left(\tau^+_{n+1,m} + \tau^-_{n+1,m}\right) \right], \tag{23}$$

where $\tau^{\pm}_{nm}$-operators obey the well-known relations

$$\begin{aligned} &\left\{\tau^+_{nm}, \tau^-_{nm}\right\}_+ = I\,, \qquad \left(\tau^+_{nm}\right)^2 = \left(\tau^-_{nm}\right)^2 = 0\,, \qquad \left[\tau^{\pm}_{nm}, \tau^{\pm}_{n'm'}\right]_- = 0\,, \\ &(nm) \neq \left(n'm'\right)\,, \qquad \left(\tau^+_{nm} + \tau^-_{nm}\right)^2 = I\,. \end{aligned} \tag{24}$$

Now, taking into account these formulae, one can present T_1 (23) as

$$T_1 = e^{-NMK_1} \exp\left[\frac{K_1}{2} \sum_{nm} \left[\tau^+_{nm} + \tau^-_{nm} + \tau^+_{n+1,m} + \tau^-_{n+1,m} \right]^2 \right]. \tag{25}$$

Using the expression

$$\exp\left(\boldsymbol{A}^2\right) = \frac{1}{\pi^{1/2}} \int\limits_{-\infty}^{\infty} \exp\left(-\xi^2 + 2\boldsymbol{A}\xi\right) \mathrm{d}\xi \,, \tag{26}$$

which is also valid for the finite-dimensional matrices (operators), one can introduce the next quantities ($K_1 > 0$)

$$A_{nm} = \sqrt{\frac{K_1}{2}} \left[\tau^+_{nm} + \tau^-_{nm} + \tau^+_{n+1,m} + \tau^-_{n+1,m}\right] .$$

Then, since the operators A_{nm} commute, using formula (26) for the operator T_1 (25), one gets the following expression

$$\begin{aligned} T_1 &= e^{-NMK_1} \int\limits_{-\infty}^{+\infty} \cdots \int\limits_{-\infty}^{+\infty} \prod_{nm} \left(\frac{\mathrm{d}\xi_{nm}}{\sqrt{\pi}} e^{-\xi^2_{nm}}\right) \\ &\quad \times \prod_{m=1}^{M} \prod_{n=1}^{N} e^{\sqrt{2K_1}(\xi_{n-1,m}+\xi_{nm})\left(\tau^+_{nm}+\tau^-_{nm}\right)} \,, \end{aligned} \tag{27}$$

where for the operators $\tau^{\pm}_{nm}$, we assume the cyclic boundary conditions ($\tau^{\pm}_{N+1,m} = \tau^{\pm}_{1m}$) as well as for the integration variables ξ_{nm}, for which it should be also assumed $\xi_{1-1,m} \to \xi_{N,m}$. Now, passing from Pauli operators $\tau^{\pm}_{nm}$ to Fermi-operators ($\alpha_{nm}, \alpha^{\dagger}_{nm}$), by means of generalized Jordan–Wigner transforms [8], one gets the equality

$$\begin{aligned} &\prod_{m=1}^{M} \prod_{n=1}^{N} e^{\sqrt{2K_1}(\xi_{n-1,m}+\xi_{nm})\left(\tau^+_{nm}+\tau^-_{nm}\right)} \\ &= \prod_{m=1}^{M} \prod_{n=1}^{N} e^{\sqrt{2K_1}(\xi_{n-1,m}+\xi_{nm})\chi_{nm}\left(\alpha^{\dagger}_{nm}+\alpha_{nm}\right)} \,, \end{aligned} \tag{28}$$

where the phase factors χ_{nm} are given by the formulae

$$\chi_{nm} = \exp\left[i\pi\left(\sum_{k=1}^{n-1}\sum_{p=1}^{M} \alpha^{\dagger}_{kp}\alpha_{kp} + \sum_{p=1}^{m-1} \alpha^{\dagger}_{np}\alpha_{np}\right)\right] , \qquad n(m) = 1, 2, \ldots \tag{29}$$

Since the operators $\chi_{nm}(\alpha^{\dagger}_{nm} + \alpha_{nm})$ commute for different$(nm) \neq (n'm')$, operators (28) can be arranged in arbitrary order, for instance,

$$(n\!=\!1; m\!=\!1, 2, \ldots, M)(n\!=\!2; m\!=\!1, 2, \ldots, M) \cdots (n\!=\!sN; m\!=\!1, 2, \ldots, M) \,. \tag{30}$$

Operator T_1 (27) in accordance with formulae (28) is the function of Fermi-operators $(\alpha_{nm}, \alpha^\dagger_{nm})$ and acts on the ket-vector $|0\rangle$ (14). The last one allows the phase factors χ_{nm} to be 'let through' the ket-vector $|0\rangle$, since the following equality holds

$$\chi_{nm}\,|0\rangle = 1\,|0\rangle\ . \tag{31}$$

Now, in accordance with the structure of χ_{nm}-operators (29) and the established order (30) of operators (28), all the phase factors χ_{nm} 'pass from left side to the right side' through the operators which are 'on the right', and as a result, in accordance with (31), they 'cancel each other out' in the $|0\rangle$-state. Then, one can write down the next expression for the ket-vector $T_1\,|0\rangle$

$$\begin{aligned} T_1\,|0\rangle \;=\; & e^{-NMK_1}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\prod_{nm}\left(\frac{\mathrm{d}\xi_{nm}}{\sqrt{\pi}}e^{-\xi^2_{nm}}\right) \\ & \times\prod_{n=1}^{N}\left[\prod_{m=1}^{M}\left[\cosh\left(\sqrt{2K_1}\,(\xi_{n-1,m}+\xi_{nm})\right)\right.\right. \\ & \left.\left.+\alpha^\dagger_{nm}\sinh\left(\sqrt{2K_1}\,(\xi_{n-1,m}+\xi_{nm})\right)\right]\right]|0\rangle\ , \end{aligned} \tag{32}$$

where the equality

$$e^{\alpha\, t} = \cosh(t) + \alpha\,\sinh(t)\,, \qquad \alpha^2 = 1 \tag{33}$$

was used. In formula (32) all the operators α_{nm} were also ignored, because the equality $\alpha_{nm}\,|0\rangle = 0$ is valid. Expression (32) after some simple transformations and integration over ξ_{nm} becomes

$$\begin{aligned} T_1\,|0\rangle = & e^{-NMK_1}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\prod_{nm}\left(\frac{\mathrm{d}\xi_{nm}}{\sqrt{\pi}}e^{-\xi^2_{nm}}\right) \\ & \times\prod_{n=1}^{N}\left[\prod_{m=1}^{M}\left[\frac{e^{\sqrt{2K_1}(\xi_{n-1,m}+\xi_{nm})}}{2}\,e^{\alpha^\dagger_{nm}}+\frac{e^{-\sqrt{2K_1}(\xi_{n-1,m}+\xi_{nm})}}{2}\,e^{-\alpha^\dagger_{nm}}\right]\right]|0\rangle \\ = & e^{-NMK_1}\sum_{\{\sigma_{nm}=\pm1\}}\int\limits_{-\infty}^{+\infty}\cdots\int\limits_{-\infty}^{+\infty}\prod_{nm}\left[\frac{\mathrm{d}\xi_{nm}}{\sqrt{\pi}}e^{-\xi^2_{nm}+\sqrt{2K_1}(\sigma_{nm}+\sigma_{n-1,m})\xi_{nm}}\right] \\ & \times\left[\prod_{m=1}^{M}\frac{e^{\sigma_{1m}\alpha^\dagger_{1m}}}{2}\prod_{m=1}^{M}\frac{e^{\sigma_{2m}\alpha^\dagger_{2m}}}{2}\cdots\prod_{m=1}^{M}\frac{e^{\sigma_{Nm}\alpha^\dagger_{Nm}}}{2}\right]|0\rangle \end{aligned}$$

$$= \frac{1}{2^{NM}} \sum_{\{\sigma_{nm}=\pm 1\}} \left[e^{K_1 \sum_{nm} \sigma_{nm}\sigma_{n+1,m}} \prod_{m=1}^{M} e^{\sigma_{1m}\alpha_{1m}^{\dagger}} \prod_{m=1}^{M} e^{\sigma_{2m}\alpha_{2m}^{\dagger}} \right.$$
$$\left. \cdots \prod_{m=1}^{M} e^{\sigma_{Nm}\alpha_{Nm}^{\dagger}} \right] |0\rangle \ . \qquad (34)$$

One can easily check that expression (34) is correct, since at $(K_1 = 0)$ for the ket-vector, one gets $T_1(K_1 = 0)\,|0\rangle = 1 \cdot |0\rangle$, in accordance with (15). Then, as it can be easily demonstrated [5], formulae (14), (19) together with (16), (17) lead exactly to Ising solution [2] for the system's free-energy with respect to the single spin.

The way for elimination the phase factors θ_{nm} in expression (20) for $T_1\,|0\rangle$ presented here is not the only possible (which will be demonstrated on the occasion of discussion the 3D LIO problem), but leads to very useful form $T_1\,|0\rangle$ in terms of statistical sum calculation. The last formula (34) is the key expression for the proposed method of obtaining Onsager solution.

4. Statistical sum

Collecting the obtained results (21), (34), one can get the next expression for the statistical sum (14)

$$\begin{aligned} Z_2(h) \;=\; & (\cosh h)^{NM} \left(\prod_{0<q,p<\pi} A_2(p,h) \right)^2 \sum_{\{\sigma_{nm}=\pm 1\}} \left[e^{K_1 \sum_{nm} \sigma_{nm}\sigma_{n+1,m}} \right] \\ & \times \langle 0|\exp\left(\sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{M-m} b(k)\alpha_{n,m+k}\alpha_{nm} \right) \\ & \times \prod_{m=1}^{M} e^{\sigma_{1m}\alpha_{1m}^{\dagger}} \prod_{m=1}^{M} e^{\sigma_{2m}\alpha_{2m}^{\dagger}} \cdots \prod_{m=1}^{M} e^{\sigma_{Nm}\alpha_{Nm}^{\dagger}} |0\rangle \ . \end{aligned} \qquad (35)$$

Further on, since the operators $\alpha_{n,m+k}\alpha_{nm}$ and $\alpha_{nm}^{\dagger}$ commute at different $n' \neq n$, for the vacuum matrix element $\langle 0|\,(...)\,|0\rangle$ in (35), one can write

$$\langle 0|\,(\cdots)\,|0\rangle = \prod_{n=1}^{N} \langle 0|\, e^{\sum_{m=1}^{M}\sum_{k=1}^{M-m} b(k)\alpha_{n,m+k}\alpha_{nm}} \prod_{m=1}^{M} e^{\sigma_{nm}\alpha_{nm}^{\dagger}}\,|0\rangle \ . \qquad (36)$$

It is not difficult to note that one can apply the transfer-matrix method to expressions (35), (36) and then use the well-known Baker–Hausdorff formula $(\alpha, \beta = \mathrm{const})$

$$\begin{aligned}
\exp(\alpha\, x)\, \exp(\beta\, y) &= \exp\left(\alpha\, x + \beta\, y + \frac{\alpha\, \beta}{2}[x, y]\right) , \\
[[x, y], x] &= [[x, y], y] = 0
\end{aligned} \tag{37}$$

in order to transform the operator product $(\prod_{m=1}^{M} e^{\tau_m^z \alpha_m^\dagger})$, where τ_m^z are the $(2^M \times 2^M)$-Pauli matrices. It is also easy to note that one gets the same result if one proceeds in the reversed order, that is, at first, applies Baker–Hausdorff formula (37) to the operator product $(\prod_{m=1}^{M} e^{\sigma_{nm}\alpha_{nm}^\dagger})$ and then uses the transfer-matrix method.

Now, applying the transfer-matrix method (see, for instance, [4, 5]) to expression (35) together with (36) and the Baker–Hausdorff formula (37), one gets the following expression for the statistical sum

$$Z_2(h) = (\cosh h)^{NM} \left(\prod_{0<q,p<\pi} A_2(p, h) \right)^2 \mathrm{Tr}(T_1^* T_2^*)^N , \tag{38}$$

where the matrices T_1^* and T_2^* are given by

$$T_1^* = (2 \sinh 2K_1)^{M/2} \prod_{m=1}^{M} e^{K_1^* \tau_m^x} , \tag{39}$$

$$T_2^* = \langle 0| \prod_{m=1}^{M} \prod_{k=1}^{M-m} e^{b(k)\alpha_{m+k}\alpha_m} \prod_{m=1}^{M} \prod_{k=1}^{M-m} e^{\tau_m^z \tau_{m+k}^z \alpha_m^\dagger \alpha_{m+k}^\dagger} |0\rangle , \tag{40}$$

and where (τ_m^x, τ_m^z) are the $(2^M \times 2^M)$-Pauli matrices, while the constants K_1 and K_1^* are interrelated by

$$e^{-2K_1} = \tanh K_1^* , \qquad e^{-2K_1^*} = \tanh K_1 , \qquad \sinh 2K_1^* \sinh 2K_1 = 1 . \tag{41}$$

The quantities $b(k)$ in (40) are given by formulae (22)

$$\begin{aligned}
b(k) &= \tanh^k \tilde{K}_2 + \tanh^2 \tilde{h} \frac{1 - \tanh^k \tilde{K}_2}{\left(1 - \tanh \tilde{K}_2\right)^2} , \\
\tanh 2\tilde{K}_2 &= \frac{\sinh 2K_2 \left(1 - \tanh^2 h\right)}{\cosh 2K_2 + \tanh^2 h\, \sinh 2K_2} , \\
\tanh^2 \tilde{h} &= \tilde{\beta}\, \tanh^2 h\, \frac{e^{2K_2}}{\cosh^2 \tilde{K}_2} , \\
\tilde{\beta} &= \left[1 + 2 \tanh^2 h\, \sinh 2K_2\, e^{2K_2}\right]^{-1/2} .
\end{aligned}$$

At last, one can write down the expression for statistical sum (38) in the form which is symmetric with respect to T_1^*-operator

$$Z_2(h) = (\cosh h)^{NM} \left(\prod_{0<q,p<\pi} A_2(p,h) \right)^2 \mathrm{Tr}(T)^N \,, \qquad T \equiv T_1^{*1/2} T_2^* T_1^{*1/2} \,, \tag{42}$$

where the transfer-matrix T is the $(2^M \times 2^M)$-matrix in regard to corresponding basis.

5. Ising and Onsager solutions

It follows from formulae (39) and (40) that in order to diagonalize T-matrix, one needs to calculate the vacuum matrix element in terms of Fermi-operators $(\alpha_m, \alpha_m^\dagger)$. It is not difficult to observe that the following relations are valid

$$\begin{aligned} \langle 0|\, \alpha_{m+k}\alpha_m \alpha^\dagger_{m'}\alpha^\dagger_{m'+k'}\, |0\rangle &= 0 \,, \qquad \left(m \neq m' \;,\; k \neq k'\right) \,, \\ \langle 0|\, \alpha_{m+k}\alpha_m \alpha^\dagger_{m'}\alpha^\dagger_{m'+k'}\, |0\rangle &= 1 \,, \qquad \left(m = m' \;,\; k = k'\right) \,. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\left[\alpha_{m+k}\alpha_m, \alpha^\dagger_{m'}\alpha^\dagger_{m'+k'}\right]_- \neq 0 \,, \\ &\left(m' = m + k \;,\; m = m' + k' \;,\; m + k = m' + k'\right) \,, \end{aligned}$$

and this makes the calculations of vacuum matrix element (40) very difficult. We shall present one of the possible ways to eliminate this difficulty in one of the next papers. Below, following the proposed formalism, we consider the well-known Ising and Onsager solutions.

5.1. Ising solution

It is not difficult to prove that at $K_1 = 0$, $N = 1$ expressions (38)–(41) lead to the correct result for free-energy of one-dimensional Ising model in the external field. Indeed, in this case, the contribution to expression (38) due to the trace $\mathrm{Tr}(T_1^* T_2^*)$ is equal to 2^M (see for details [5]) and for the statistical sum (38), we have

$$Z_1(K_1 = 0, K_2, h) = (2\cosh h)^M \prod_{0<p<\pi} A_2(p,h) \,,$$

where $A_2(p,h)$ is given by (22). The above expression for statistical sum $Z_1(K_1 = 0, K_2, h)$ leads exactly to the Ising solution [2] for the free-energy of the system with respect for the single spin

$$
\begin{aligned}
-\beta f(K_2,h) &= \lim_{M\to\infty}\frac{1}{M}\ln Z_1(K_1=0,K_2,h)\\
&= \ln\left[e^{K_2}\cosh h+\left(e^{2K_2}\sinh^2 h+e^{-2K_2}\right)^{1/2}\right].
\end{aligned}
$$

Some different situation arises in the case of $K_2=0,\ M=1$. Now, in expression (40) for the T_2^*-operator, one cannot directly assume $M=1$. In this case, as it can be shown, the operator $[\tilde{T}_2\equiv\prod_{0<p<\pi}A_2(p,h)T_2^*]$, where T_2^* is defined by (40), at $K_2=0,\ M=1$ can be expressed as

$$
T_2\equiv\tilde{T}_2(K_2=0,M=1)=\langle 0|\, e^{\tanh(h)\alpha}e^{\tau^z\alpha^\dagger}\,|0\rangle\ . \tag{43}
$$

Indeed, at $(K_2\neq 0)$, the operator $[\tilde{T}_2\equiv\prod_{0<p<\pi}A_2(p,h)T_2^*]$ has the form (see Ref. [5] for details)

$$
\begin{aligned}
\tilde{T}_2 &= \langle 0|\prod_{0<p<\pi}A_2(p,h)\prod_{m=1}^{M}\prod_{k=1}^{M-m}e^{b(k)\alpha_{m+k}\alpha_m}\prod_{m=1}^{M}\prod_{k=1}^{M-m}e^{\tau_m^z\tau_{m+k}^z\alpha_m^\dagger\alpha_{m+k}^\dagger}\,|0\rangle\\
&= \langle 0|\prod_{0<p<\pi}A_2(p,h)\prod_{m=1}^{M}\prod_{k=1}^{M-m}e^{b(k)\alpha_{m+k}\alpha_m}\prod_{m=1}^{M}e^{\tau_m^z\alpha_m^\dagger}\,|0\rangle\\
&= \langle 0|\prod_{m=1}^{M}e^{\tanh(h)\alpha_m}\prod_{m=1}^{M}e^{K_2\left(\alpha_m^\dagger-\alpha_m\right)\left(\alpha_{m+1}^\dagger+\alpha_{m+1}\right)}\prod_{m=1}^{M}e^{\tau_m^z\alpha_m^\dagger}\,|0\rangle
\end{aligned} \tag{44}
$$

and assuming $K_2=0,\ M=1$ in (44), for the operator $[T_2\equiv\tilde{T}_2(K_2=0,\ M=1)]$, one gets expression (43). At last, for the statistical sum (38) at $K_2=0,\ M=1$, one has

$$
Z_1(K_1,h)=(\cosh h)^N\,\mathrm{Tr}(T_1T_2)^N\,, \tag{45}
$$

where the matrices T_1 and T_2 are of the form

$$
T_1=(2\sinh 2K_1)^{1/2}e^{K_1^*\tau^x}\,,\qquad T_2=\langle 0|\,e^{\tanh(h)\alpha}e^{\tau^z\alpha^\dagger}\,|0\rangle=1+\tanh(h)\tau^z\,. \tag{46}
$$

Matrix T_1T_2 is of (2×2), and its eigenvalues are equal

$$
\lambda^{\pm}=e^{K_1}\pm\sqrt{e^{2K_1}\tanh^2 h+e^{-2K_1}\left(1-\tanh^2 h\right)}\,. \tag{47}
$$

Here, we used relations (34). In the thermodynamic limit, expressions (45) and (47) lead exactly to the Ising solution [2] for the free-energy with respect to the single spin

$$\begin{aligned} -\beta f(K_1, h) &= \lim_{N\to\infty} \frac{1}{N} \ln Z_1(K_1, h) \\ &= \ln\left[e^{K_1}\cosh h + \left(e^{2K_1}\sinh^2 h + e^{-2K_1}\right)^{1/2}\right]. \end{aligned}$$

5.2. Onsager solution

In the case of zeroth external field ($h = 0$), the obtained expression for T_2^*-operator (40) leads to the Onsager solution [3]. Indeed, first of all, it is not difficult to check directly that for the ket-vector $(\cdots)|0\rangle$ in expression (40), the following formula holds

$$\begin{aligned} &\prod_{m=1}^{M}\prod_{k=1}^{M-m} e^{\tau_m^z \tau_{m+k}^z \alpha_m^\dagger \alpha_{m+k}^\dagger}|0\rangle \\ &= \prod_{m=1}^{M}\left[1 + \tau_m^z\tau_{m+1}^z\left(\alpha_m^\dagger - \alpha_m\right)\left(\alpha_{m+1}^\dagger + \alpha_{m+1}\right)\right]|0\rangle\ . \end{aligned}$$

On the other hand, for bra-vector $\langle 0|(\cdots)$ in (40) at ($h = 0$, $b(k) = \tanh^k K_2$), one can get the following equality (see [5] for details)

$$\begin{aligned} &\langle 0|\prod_{m=1}^{M}\prod_{k=1}^{M-m} e^{b(k)\alpha_{m+k}\alpha_m} \\ &= \frac{1}{\cosh^M K_2}\langle 0|\prod_{m=1}^{M}\left[\cosh K_2 + \left(\alpha_m^\dagger - \alpha_m\right)\left(\alpha_{m+1}^\dagger + \alpha_{m+1}\right)\sinh K_2\right]\ . \end{aligned}$$

Since the operators $(\alpha_m^\dagger - \alpha_m)(\alpha_{m+1}^\dagger + \alpha_{m+1})$ commute at different $m \neq m'$ and the equality $[(\alpha_m^\dagger - \alpha_m)(\alpha_{m+1}^\dagger + \alpha_{m+1})]^2 = I$ holds, one can get for operator (40) the following formula ($\alpha_m|0\rangle = 0$, $\langle 0|\alpha_m^\dagger = 0$)

$$\begin{aligned} T_2^* &= \frac{1}{\cosh^M K_2}\langle 0|\prod_{m=1}^{M}\left[\cosh K_2 + \left(\alpha_m^\dagger - \alpha_m\right)\left(\alpha_{m+1}^\dagger + \alpha_{m+1}\right)\sinh K_2\right] \\ &\quad\times\left[1 + \tau_m^z\tau_{m+1}^z\left(\alpha_m^\dagger - \alpha_m\right)\left(\alpha_{m+1}^\dagger + \alpha_{m+1}\right)\right]|0\rangle \\ &= \frac{1}{\cosh^M K_2}\prod_{m=1}^{M}\langle 0|\left[\cosh K_2 + \left(\alpha_m^\dagger - \alpha_m\right)\left(\alpha_{m+1}^\dagger + \alpha_{m+1}\right)\sinh K_2\right] \\ &\quad\times\left[1 + \tau_m^z\tau_{m+1}^z\left(\alpha_m^\dagger - \alpha_m\right)\left(\alpha_{m+1}^\dagger + \alpha_{m+1}\right)\right]|0\rangle \end{aligned}$$

$$= \frac{1}{\cosh^M K_2} \prod_{m=1}^{M} \left[\cosh K_2 + \tau_m^z \tau_{m+1}^z \sinh K_2\right]$$
$$= \frac{1}{\cosh^M K_2} e^{K_2 \sum_{m=1}^{M} \tau_m^z \tau_{m+1}^z} .$$

Now, taking into account that the factor $(1/\cosh^M K_2)$ and $[\prod_{0<p<\pi} A_2(p,h)]$ cancel exactly each other out, for the statistical sum (42) at $h=0$, one gets

$$Z_2(h=0) = \mathrm{Tr}(T)^N , \qquad T \equiv T_1^{1/2} T_2 T_1^{1/2} , \tag{48}$$

where matrices T_1 and T_2 are given by

$$\begin{aligned} T_1 &= (2\sinh 2K_1)^{M/2} e^{K_1^* \sum_{m=1}^{M} \tau_m^x} , \qquad T_2 = e^{K_2 \sum_{m=1}^{M} \tau_m^z \tau_{m+1}^z} , \\ e^{-2K_1} &= \tanh K_1^* , \qquad e^{-2K_1^*} = \tanh K_1 , \qquad \sinh 2K_1^* \sinh 2K_1 = 1 . \end{aligned} \tag{49}$$

Finally, expressions (48) and (49) in the thermodynamic limit lead exactly to Onsager solution for the free-energy of the spin system in respect to a single spin (see [4] for details)

$$\begin{aligned} -\beta f_2(T) &= \ln 2 + \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \ln[\cosh 2K_1 \cosh 2K_2 \\ &\qquad - \cos q \sinh 2K_1 - \cos p \sinh 2K_2] \mathrm{d}q \mathrm{d}p . \end{aligned} \tag{50}$$

The well-known Ising and Onsager solutions obtained above verify completely the correctness of our consideration. Formalism presented in the work based on the transfer-matrix method twice-repeated and the generalized Jordan–Wigner transforms [5, 8] enables not only to reproduce correctly the well-known results for 1D and 2D Ising model, but also allows to hope for finding the exact analytical solution of 2D LIO in arbitrary external magnetic field.

I am grateful to I. Tralle for help in preparation the final form of this paper. This work was done due to the support from the Center for Innovation and Transfer of Natural Sciences and Engineering Knowledge at the University of Rzeszów.

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