STRESS-ENERGY TENSOR OF THE QUANTIZED MASSIVE FIELDS IN SPATIALLY-FLAT D-DIMENSIONAL FRIEDMANN–ROBERTSON–WALKER SPACETIMES

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We construct and investigate the general stress-energy tensor, T_a^b, of the quantized massive field in the D-dimensional spatially-flat Friedmann–Robertson–Walker spacetime within the framework of the adiabatic approximation. The behavior of T_a^b for $4 \leq D \leq 12$ is examined for the exponential (in the conformal time) and power-law cosmological models with the special emphasis put on the conformal and minimal curvature coupling. It is shown that time component of the stress-energy tensor is proportional to the spatial component and that the proportionality constant can be calculated without the detailed knowledge of the tensor. In the exponential expansion in even dimensions, the energy density of the quantized field identically vanishes for the conformally coupled fields and is positive for the minimal coupling. The analogous formulas for the odd-dimensional spacetimes do not exhibit this simple behavior and the energy density is positive for physical values of the coupling. The relations of the adiabatic method to the Schwinger–DeWitt approach is briefly discussed.

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1. Introduction

Despite our everyday experience, gravitation is still the most elusive of all fundamental interactions. It, of course, does not mean that at the classical level we have no deep and satisfactory understanding of gravitation. Quite the contrary, numerous tests that have been carried out to date, clearly show that General Relativity satisfactorily describes gravitational phenomena, and that the description of the spacetime as the differentiable manifold endowed with the metric tensor and additional structures is valid unless the Planck regime is approached [1]. However, our understanding of the gravitational interaction is severely limited as there is no quantum theory of gravity. Instead, we have various approaches and each of them has its own merits and drawbacks. Consequently, an approach has been proposed in which the spacetime is treated classically whereas the matter fields are quantized. Especially interesting in this regard is the influence of the quantized fields upon the spacetime geometry in a process, which has been figuratively called "the back reaction on the metric". This simple idea evolved into the mature theory, with its own techniques, methods, fundamental results and folklore [2–4]. Perhaps the most important result of the quantum field theory in curved background is the prediction that black holes evaporate [5, 6].

The physical content of the quantum field theory in curved background is encoded in its most important observable: the (renormalized) stress-energy tensor calculated in suitably chosen state. Such a tensor serves as the source term (the right-hand side) of the semiclassical Einstein field equations, allowing, in principle, to study the evolution of the system unless the quantum gravity effects become dominant. Within the semiclassical approach, one can address quite a number of important and physically interesting problems, such as, for example, the physics of the quantum-corrected black holes and their interiors, influence of the quantized fields on the extreme or close to extreme black holes or the problem of isotropization of cosmological models, to name a few. On the other hand, needless to say that most of the researches are restricted to $D = 4$. For higher-dimensional results, see *e.g.*, Refs. [7–12] and references cited therein.

It is evident that the semiclassical Einstein field equations cannot be trusted at the singularity and its closest vicinity as the full analysis of this problem requires quantum gravity or even more fundamental theory. On the other hand, the semiclassical analysis can teach us a lot about the influence of the quantized fields on the black hole interior and about the tendency of the changes of the background geometry as the singularity is approached. Of course, from the point of view of the present work, the most important are the results obtained in cosmology (see *e.g.*, Refs [13–27] and the references cited therein) and in the black hole interiors [28–33].

Recent findings in the Kaluza–Klein-type theories, string theory and its low-energy limit, various large-dimensional scenarios, such as, for example, the braneworld scenario suggest that our physical world has more than 4 dimensions. This opens new interesting possibilities in the cosmological context.

There are two main approaches that may be singled out: *(i)* the numerical calculations of the mean value of the stress-energy tensor in some physically motivated state/states in a fixed background, and *(ii)* construction of the analytical approximations to the stress-energy tensor of the quantized field that depends functionally on the metric tensor or at least on a wide class of metrics. This distinction is not sharp and there are some hybrid approaches which we collectively put into *(i)*. The second approach is particularly appealing and we intend to construct the general form of the approximate stress-energy tensor of the quantized fields. Having such a tensor at one's disposal, it is possible to solve the semiclassical Einstein field equations in a self-consistent way. Unfortunately, regardless of the particular method adapted, the calculations of the stress-energy tensor of the quantized fields in a curved background are very complicated, long, error-prone and time consuming. It is simply because the bilinear objects needed in its construction are the operator-valued distributions and the whole problem is infected with unavoidable infinities. The structure of the infinite terms depends on the dimension of the differentiable manifold and the type of the quantum fields. (See *e.g.*, Ref. [4, 34] and references therein.) Moreover, the computational complexity rapidly grows with dimension, making the problem practically intractable in larger dimensions.

One of the most versatile approaches in the quantum field theory in curved background is the adiabatic regularization [15–20, 24–27] (see also [35–37]). The adiabatic calculations are based on the higher-order WKB approximation to the mode functions and their derivatives and subsequent integration (summation) of the functions thus constructed. It is particularly well-suited to calculations of the energy density and pressures in the higher-dimensional spatially-flat or open FRW model as well as in the anisotropic cosmologies. (In the $k = 1$ case, one has summation of the mode functions instead of integration, which is an obstacle in constructing the final compact expression.)

Of course, the adiabatic approach is not the only one available for constructing the stress-energy tensor. Equally powerful is the Schwinger–DeWitt method, which is based on the coefficients of the heat-kernel expansion [12, 25–27, 32, 33, 38–43]. It can be used in any spacetime provided the Compton length associated with the mass of the field, λ_{C}, is much smaller that the characteristic radius of the curvature, L, *i.e.*, $\lambda_{\mathrm{C}}/L \ll 1$.

It has been recently demonstrated that the Schwinger–DeWitt and adiabatic approaches give precisely the same results in the Friedmann–Robertson–Walker spacetime [25]. The Schwinger–DeWitt approach is even more general, but because of the geometric terms that have to be constructed, its applicability in quantum field theory in curved background is practically limited to $D = 4$. It should be noted that the equality of results obtained within the Schwinger–DeWitt and adiabatic frameworks must not be taken for granted. Indeed, it is expected that the discussed methods give the Green functions with the same structure of singularities in the coincidence limit, but this does not mean that the functions are the same.

Our aim is to construct and study the regularized stress-energy tensor of the quantized massive scalar field in the spatially-flat Friedmann–Robertson–Walker spacetime. Here, we shall restrict ourselves to $4 \leq D \leq 12$, with the special emphasis put on the power-law and de Sitter cosmologies. Of course, the multidimensional cosmological models will be influenced by the quantized fields via the semiclassical Einstein field equation and the knowledge of the general form of the source term gives a unique opportunity to analyze and compare the evolution of the models for various dimensions. This will be the subject of the subsequent papers.

The paper is organized as follows. The detailed calculations of the components of the stress-energy tensor expressed in terms o the WKB-mode functions are presented in Sec. 2. In Sec. 3, the general formulas are used in construction of the T_a^b of the quantized massive field in the power-law cosmological models and for the exponentially expanding (in conformal time) scale factor. The last section contains a brief discussion of the Schwinger–DeWitt method. Throughout the paper, the natural units are chosen (except for the short discussion of the range of validity of the approximation) and we follow the Misner, Thorne and Wheeler conventions [44].

2. General equations

Let us consider the neutral massive scalar field, satisfying the covariant Klein–Gordon equation in D-dimensional spatially flat Friedmann–Robertson–Walker spacetime

$$\Box\phi - \left(m^2 + \xi R\right)\phi = 0\,, \tag{1}$$

where m is the mass of the field, R is the curvature scalar and ξ is the (arbitrary) curvature coupling constant. Although there are no *a priori* restrictions on the curvature coupling parameter, the two particular values of ξ are of principal interest: the conformal and minimal coupling, for which $\xi = (D-2)/(4D-4)$ and $\xi = 0$, respectively. Other, more exotic values of ξ, are considered to be of lesser importance. The spatially-flat line element can be written in the form

$$\mathrm{d}s^2 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)\,, \tag{2}$$

where $i,j = 1,\ldots,D-1$ and η is the conformal time.

Our first task is to construct the solutions of the scalar field equation. We start the calculations by putting

$$\phi(x) = a^{-(D-2)/2}\mu(x) \tag{3}$$

into Eq. (1) and subsequently decomposing the function $\mu(x)$ as

$$\mu(x) = (2\pi)^{-(D-1)/2} \int \mathrm{d}^{D-1}k \left(\mu_k(\eta) e^{ik_a x^a} a_{\boldsymbol{k}} + \mu^*{}_k(\eta) e^{-ik_a x^a} a^\dagger_{\boldsymbol{k}} \right) , \quad (4)$$

where $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$ are the annihilation and creation operators. The functions μ_k and μ^*_k are normalized in such a way that the Wronskian condition

$$\mu_k \dot{\mu}^*_k - \dot{\mu}_k \mu^*_k = i \quad (5)$$

is satisfied, where a dot denotes differentiation with respect to the conformal time. This ensures that the canonical commutation relations of the field operators and the conjugate momenta give the standard relations for the operators $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$

$$\begin{aligned} \left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}\right] &= \left[a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}\right] = 0 \,, \\ \left[a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{k}'}\right] &= \delta\left(\boldsymbol{k} - \boldsymbol{k}'\right) . \end{aligned} \quad (6)$$

The ground state of the field is defined as

$$a_{\boldsymbol{k}}|0\rangle = 0 \,, \quad (7)$$

i.e., we choose the functions $\mu_k(\eta)$ in Eq. (4) to be positive-frequency solutions of the equation

$$\ddot{\mu}_k + \left(k^2 + m^2 a^2\right) \mu_k + (\xi - \xi_{\mathrm{c}}) \left(2(D-1)\frac{\ddot{a}}{a} + (D-4)(D-1)\frac{\dot{a}^2}{a^2} \right) \mu_k = 0 \,, \quad (8)$$

where $k = \sqrt{k_1^2 + \cdots k_{D-1}^2}$ and $\xi_{\mathrm{c}} = (D-2)/(4D-4)$.

Thus far our analysis has been exact. Unfortunately, (8) is rather complicated and, in general, it cannot be solved in terms of the known functions. It is natural that one should either treat the problem numerically (that is beyond the scope of the present paper) or look for reasonable approximations. Our method of choice is the WKB approximation, which allows us to construct the adiabatic solutions iteratively. This approach defines the adiabatic vacuum $|0\rangle_A$. The adiabatic approach can be used in the Friedmann–Robertson–Walker spacetime if the chain of conditions $\dot{a}/a, \ddot{a}/a, \dots \ll \sqrt{k^2 + m^2 a^2}$ for $0 \le k < \infty$ is satisfied [3].

The WKB mode functions $\Omega_k(\eta)$ are defined as

$$\mu_k = \frac{1}{\sqrt{2\Omega_k}} e^{-i\int \Omega_k \mathrm{d}\eta} \quad (9)$$

and their form guarantees that the Wronskian condition (5) is automatically satisfied. The resulting equation that is satisfied by Ω_k is given by

$$\begin{aligned}\Omega_k^2 &= k^2 + m^2a^2 - \frac{\ddot{\Omega}_k}{2\Omega_k} + \frac{3\dot{\Omega}_k^2}{4\Omega_k^2} \\ &\quad + \left(\xi - \frac{D-2}{4(D-1)}\right)\left(2(D-1)\frac{\ddot{a}}{a} + (D-4)(D-1)\frac{\dot{a}^2}{a^2}\right) . \end{aligned} \tag{10}$$

The WKB solution can be constructed iteratively assuming that the function Ω_k can be expanded

$$\Omega_k = \omega_0 + \omega_2 + \omega_4 + \cdots \tag{11}$$

with the zeroth-order solution taken to be $\omega_0 = \sqrt{k^2 + m^2a^2}$. The role of the small parameter is played by number of differentiations with respect to the conformal time. To simplify the calculations and to keep track of the order of terms in complicated expansions, one can introduce the (dimensionless) parameter ε by means of the formulas

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \to \varepsilon \frac{\mathrm{d}}{\mathrm{d}\eta} \quad \text{and} \quad \Omega_k = \sum_{i=0} \varepsilon^{2i}\omega_{2i} \,, \tag{12}$$

and collect the resulting terms with the like powers of ε. The parameter ε should be set to 1 at the final stage of calculations. For example, ω_2 is given by

$$\omega_2 = \frac{\alpha\left[(D-4)\dot{a}^2 + 2a\ddot{a}\right]}{8a^2\left(k^2+m^2a^2\right)^{1/2}} - \frac{m^2\left(\dot{a}^2 + a\dot{a}\right)}{4\left(k^2+m^2a^2\right)^{3/2}} + \frac{5m^4a^2\dot{a}^2}{8\left(k^2+m^2a^2\right)^{5/2}} \,, \tag{13}$$

where

$$\alpha = 4\xi(D-1) - (D-2) \,. \tag{14}$$

The higher-order functions ω_{2i} $(i \geq 2)$ can be constructed term-by-term by solving algebraic equations of ascending complexity. The results presented in this paper require the 14$^{\text{th}}$-order WKB approximation to the function Ω_k, *i.e.*,

$$\Omega_k = \sum_{i=0}^{7} \omega_{2i} \,. \tag{15}$$

Note that the knowledge of Ω_k is sufficient to construct the vacuum polarization [27]. Indeed, since

$$|\mu_k|^2 = \frac{1}{2\Omega_k} \,, \tag{16}$$

it is clear that the problem reduces to expanding the function Ω_k^{-1} in terms of the number of derivatives (parameter ε) to some definite order and integration of the thus constructed result over k with the appropriate measure.

Although interesting in its own right, the function Ω_k is only a building block of a more important quantity — the stress-energy tensor of the quantized field. The general form of T_{ik} of the (classical) massive scalar field is given by

$$T_{ik} = \nabla_i\phi\nabla_k\phi - \tfrac{1}{2}g_{ik}\left(\nabla_a\phi\nabla^a\phi + m^2\phi^2\right) + \xi G_{ab}\phi^2 + \xi g_{ik}\Box\phi^2 - \xi\nabla_i\nabla_k\phi^2\,, \tag{17}$$

where G_{ab} is a D-dimensional Einstein tensor. Now, let us analyze the T_{00} component, which for the line element (2) is given by

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}k^2\phi^2 + \frac{1}{2}m^2a^2\phi^2 + \frac{1}{2}\xi(D-2)(D-2)\frac{\dot{a}^2}{a^2} + 2\xi(D-1)\frac{\dot{a}}{a}\phi\dot{\phi}\,. \tag{18}$$

It should be noted that it suffices to find only the T_{00} component as the spatial components can easily be calculated from the conservation equation. Indeed, from the conservation equation $\nabla_a T^a_b = 0$, one has

$$T^1_1 = ... = T^{D-1}_{D-1} = T^0_0 - \frac{1}{D-1}\frac{a}{\dot{a}}\dot{T}^0_0\,. \tag{19}$$

Here, we shall proceed in a different manner and construct the trace of the tensor

$$T^i_i = \tfrac{1}{2}\alpha\phi_{.a}\phi^{.a} - \tfrac{1}{2}\beta m^2\phi^2 + \tfrac{1}{2}\alpha\xi R\phi^2\,, \tag{20}$$

where

$$\beta = D - 4\xi(D-1) \tag{21}$$

and

$$R = 2(D-1)\left(\frac{\ddot{a}}{a^3} - \frac{\dot{a}^2}{a^4}\right) + (D-1)(D-2)\left(\frac{\dot{a}^2}{a^4}\right)\,. \tag{22}$$

Other methods will be used as an useful check of the calculations.

Substituting (3) into (18) and (20), and subsequently making use of the relations

$$\mu_k\mu_k^* = \frac{1}{2\Omega_k}\,, \tag{23}$$

$$\mu_k\dot{\mu}_k^* + \dot{\mu}_k\mu_k^* = -\frac{\dot{\Omega}_k}{2\Omega_k^2} \tag{24}$$

and

$$\dot{\mu}_k\dot{\mu}_k^* = \frac{1}{2}\Omega_k + \frac{\dot{\Omega}_k^2}{\Omega_k^3}\,, \tag{25}$$

one obtains for the energy density ($\rho = -T_0^0$) the following compact formula:

$$\rho = \frac{1}{N(D)}\int \mathrm{d}k\, k^{D-2}\left\{\frac{\Omega_k}{2} + \frac{1}{2\Omega_k}\left[k^2 + m^2a^2 - \xi\left(2 - 3D + D^2\right)h^2\right] - (D-1)\xi h\frac{\dot{\Omega}_k}{\Omega_k^2} + \frac{1}{2\Omega_k}\left[\frac{\dot{\Omega}_k}{2\Omega_k} + \frac{1}{2}(D-2)h\right]^2\right\}, \tag{26}$$

where

$$N(D) = (4\pi)^{(D-1)/2}\Gamma\left(\frac{D-1}{2}\right)a^D\,, \tag{27}$$

$h = \dot{a}/a$ and $\Gamma(x)$ is the Euler Gamma function. Similarly, for the trace of the tensor, one has

$$T_i^i = \frac{1}{N(D)}\int \mathrm{d}k\, k^{D-2}\left\{\frac{1}{2\Omega_k}\left[k^2\alpha - \frac{1}{4}(D-2)^2\alpha h^2 + \left(R\alpha\xi - \frac{1}{2}\beta m^2\right)a^2\right] - \frac{1}{2}\alpha\Omega_k - \frac{1}{2\Omega_k}\left[\frac{1}{2}(D-2)h\alpha\frac{\dot{\Omega}_k}{\Omega_k} + \frac{1}{4}\alpha\frac{\dot{\Omega}_k^2}{\Omega_k^2}\right]\right\}. \tag{28}$$

Now, making use of the standard formulas describing the division of the power series

$$\frac{\sum\limits_{q=0}^{\infty} A_q\varepsilon^q}{\sum\limits_{q=0}^{\infty} B_q\varepsilon^q} = \frac{1}{B_0}\sum_{q=0}^{\infty} C_q\varepsilon^q\,, \tag{29}$$

where

$$C_q = -\frac{1}{B_0}\sum_{p=1}^{q} C_{q-p}B_p - A_q \tag{30}$$

and raising to powers

$$\left(\sum_{q=0}^{\infty} A_q\varepsilon^q\right)^n = \sum_{q=0}^{\infty} B_q\varepsilon^q\,, \tag{31}$$

where

$$B_0 = A_0^n\,, \qquad B_q = \frac{1}{qA_0}\sum_{p=1}^{q}(pn - q - p)A_pB_{q-p} \quad \text{for} \quad m \geq 1\,, \tag{32}$$

one obtains the adiabatic expansion for the energy density ρ and the trace of the stress-energy tensor. Each term in this rather complicated expansions contains an integral over k of $k^p/\left(k^2+m^2a^2\right)^{q/2}$ which is finite for $q > p+1$. Consequently, it can be shown that for a given dimension D, the terms up to the adiabatic order $2\lfloor D/2\rfloor$ diverge, where $\lfloor x\rfloor$ denotes the floor function, *i.e.*, it gives the largest integer less than or equal to x. This statement needs clarification: Of course, not all the terms of a given adiabatic order lead to the divergent integrals. The regularization prescription, however, requires to subtract all the terms of a given adiabatic order if at least one of them is divergent. Finally, making use of the formula

$$\int\limits_0^\infty \mathrm{d}k \frac{k^p}{(k^2+m^2a^2)^{q/2}} = \frac{1}{2}(ma)^{1+p-q} B\left(\frac{1+p}{2}, \frac{q-p-1}{2}\right) = \frac{1}{2}(ma)^{1+p-q} \frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{q-p-1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)}, \tag{33}$$

where B is the beta function, one can calculate the stress-energy tensor to the required adiabatic order. Moreover, using precisely the same algorithm, the next-to-leading and higher-order terms of the stress-energy tensor can be constructed. Here, however, we shall concentrate on the leading order contribution to the stress-energy tensor defined by the WKB series, which is, of course, the most important.

3. The stress-energy tensor

Using the adiabatic method, we have calculated the stress-energy tensor of the quantized massive scalar field in the spatially-flat Friedmann–Robertson–Walker spacetime for $4 \leq D \leq 12$. Because of the complexity of the calculations, we used various methods and computational strategies, and each component of T_a^b has been calculated at least 3 times. The results are stored in Mathematica syntax and can easily be converted into any other format. Since the general results valid for any scale factor are not especially illuminating, to avoid unnecessary proliferation of complex formulas, we do not present them here, instead, we shall discuss some special cases of physical interest. The general results can be obtained on request from the first author.

Before we start to discuss the stress-energy tensor, let us briefly investigate the classical Einstein field equations. For the spatially-flat line element, one has

$$\frac{1}{2}(D-2)(D-1)\frac{\dot{a}^2}{a^4} = \frac{\rho_{\mathrm{c}}}{M_{\mathrm{Pl}}^{D-2}} \tag{34}$$

and

$$-\frac{1}{2}(D-2)\left[(D-5)\frac{\dot{a}^2}{a^4}+2\frac{\ddot{a}}{a^3}\right]=\frac{p_{\rm c}}{M_{\rm Pl}^{D-2}}\,, \tag{35}$$

where $M_{\rm Pl}$ is the D-dimensional (reduced) Planck mass and the pressure, $p_{\rm c}$, is related to the energy density, $\rho_{\rm c}$, through the equation of state $p_{\rm c}=\kappa\rho_{\rm c}$. From (34), one has for a classical background

$$H^2\sim\frac{\rho_{\rm c}}{M_{\rm Pl}^{D-2}}\,, \tag{36}$$

where we have introduced the (proper time) Hubble parameter $H=\dot{a}/a^2$.

3.1. The power-law cosmologies

Now, let us limit ourselves to the class of the power-law cosmologies with the scale factor

$$a(\eta)=\left(\frac{\eta_0}{\eta}\right)^n\,. \tag{37}$$

For the power-law cosmologies, the scale factor expressed in terms of the proper time assumes simple form[1]

$$a(t)=\left(\frac{t}{t_0}\right)^\beta\,,\qquad \beta=\frac{n}{n-1}\,. \tag{38}$$

Since the conformal time is related to the proper time through $\mathrm{d}t=a(\eta)\mathrm{d}\eta$, it is relatively easy to construct the stress-energy tensor in the coordinates $(t,x^1,\ldots,x^{D-1})$ once the energy density (26) is known. Although our discussion will be carried out in the former representation, reinterpretation of the results within the context of the proper time scale factor is unproblematic.

From the Einstein field equations (for $D\geq 4$ and $n\neq 0$), one has

$$n=-\frac{2}{D-3+(D-1)\kappa}\,. \tag{39}$$

For the scale factor (37), the stress-energy tensor of the quantized massive fields considerably simplifies. However, before we go further, let us analyze how far one can go without the exact form of the stress-energy tensor.

[1] The notation has been slightly abused: in this section, α and β have a different meaning than α and β defined in the previous section.

First, observe that on the general ground, the energy density of the massive quantum field has the following form [27]

$$\rho = \begin{cases} \dfrac{H^{D+2}}{m^2} f_D(n,\xi) & \text{for even } D\,, \\ \dfrac{H^{D+1}}{m} f_D(n,\xi) & \text{for odd } D\,, \end{cases} \tag{40}$$

where, for a given dimension, $f_D(n,\xi)$ is function of the exponent n and the curvature coupling, ξ. There is a simple explanation of this fact: the adiabatic expansion is, in fact, an expansion in the number of derivatives and this together with (37) gives (40). Now, making use of the conservation equation, one has

$$\alpha = -\frac{T_x^x}{T_0^0} = -\frac{\eta}{n(D-1)}\frac{\mathrm{d}}{\mathrm{d}\eta}\ln T_0^0 - 1\,, \tag{41}$$

where $T_x^x \equiv p$ is the spatial component of the stress-energy tensor and $T_0^0 \equiv -\rho$. Finally, substitution of (40) into (41) gives the desired result

$$\alpha = \begin{cases} \dfrac{3n-(D+2)}{(D-1)n} & \text{for even } D\,, \\ \dfrac{2n-(D+1)}{(D-1)n} & \text{for odd } D\,. \end{cases} \tag{42}$$

Since α is constant for a given n and D, the pressure and the energy density of the quantized field are related by a simple equation state. The energy density and pressure have opposite signs for $0 < n < (D+2)/3$ and $0 < n(D+1)/2$ for D even and odd, respectively.

The parameter α can be easily related to the parameter κ of the equation of state describing the classical source. Making use of Eqs. (39) and (42), one has

$$\alpha = \begin{cases} \frac{1}{2}\left[D + \kappa(D+2)\right] & \text{for even } D\,, \\ \frac{1}{2}\left[\kappa - 1 + D(\kappa+1)\right] & \text{for odd } D\,. \end{cases} \tag{43}$$

From (43), one sees that the parameter α for even D is the same as the parameter for $D+1$. A few important special cases are collectively listed in Table I. It should be noted that for $\kappa = -1$, one has still $\alpha = -1$. Similarly, for a phantom universe governed by the equation of state with $-(1+\delta)$ and $(\delta > 0)$, the parameter α is even more exotic.

TABLE I

The dimensional dependence of the parameter α for a few exemplary forms of the classical matter. The equation of state governed by α is calculated for the radiation-dominated, dust, de Sitter and phantom ($\delta > 0$) universe.

κ	$1/(D-1)$	0	-1	$-(1+\delta)$
α (even D)	$\frac{D^2+2}{2(D-1)}$	$\frac{1}{2}D$	-1	$-1-\frac{1}{2}(D+2)\delta$
α (odd D)	$\frac{D^2-D+2}{2(D-1)}$	$\frac{1}{2}(D-1)$	-1	$-1-\frac{1}{2}(D+1)\delta$

It is instructive to compare the energy density of the quantized field and the background energy density. From (36) and (40), one has

$$\frac{\rho}{\rho_{\rm c}} \sim \begin{cases} \dfrac{H^D}{m^2 M_{\rm Pl}^{D-2}} & \text{for even } D\,, \\ \dfrac{H^{D-1}}{m M_{\rm Pl}^{D-2}} & \text{for odd } D\,. \end{cases} \tag{44}$$

Since the adiabaticity condition can be written as $m \gg H$, one concludes that $\rho_{\rm c} \gg \rho$ unless $H \gg M_{\rm Pl}$.

Similar estimates follow from the condition $\lambda_{\rm C}/L \ll 1$. Indeed, taking $L \sim K^{-1/4}$, where K is the Kretschmann scalar

$$K = R_{abcd}R^{abcd} = 2(D-1)\left[D\frac{\dot{a}^4}{a^8} - 8\frac{\dot{a}^2\ddot{a}}{a^7} + 2\frac{\ddot{a}^2}{a^6}\right]\,, \tag{45}$$

in the power-law cosmological model, one has $K \sim H^4$ and hence $H \ll m$.

Now, let us return to the stress-energy tensor calculated within the framework of the adiabatic method. The expressions describing the energy density of the minimally and conformally coupled quantized field for a power-law cosmology are relegated to Appendix. Here, we shall concentrate on a few interesting physical cases. First, observe that for the radiation-dominated cosmology with the equation of state of the form of $p = \rho/(D-1)$, the equation of state of the quantized field is "very stiff" (with the "stiffness" increasing with D) and for a given dimension, the energy density for $\xi = 0$ and $\xi = \xi_{\rm c}$ is of the same sign. More detailed results are presented for $4 \le D \le 8$ in Table II.

For the matter-dominated universe ($\kappa = 0$), the equation of state of the quantized fields is still of the "stiff-type" and the sign of the energy density depends on dimension (see Table III). For example, for $D = 4$ and $D = 5$, the energy density is always negative.

TABLE II

The dimensional dependence of the characteristics of the minimally ($\xi = 0$) and conformally coupled ($\xi = \xi_c$) quantized field in the spatially flat radiation-dominated ($\kappa = 1/(D-1)$) universe. The pressure has the same sign as the energy density ($\alpha > 0$).

D	κ	n	α	β	$\xi = 0$	$\xi = \xi_c$
4	1/3	−1	3	1/2	$\rho > 0$	$\rho > 0$
5	1/4	−2/3	11/4	2/5	$\rho > 0$	$\rho > 0$
6	1/5	−1/2	19/5	1/3	$\rho < 0$	$\rho < 0$
7	1/6	−2/5	11/3	2/7	$\rho < 0$	$\rho < 0$
8	1/7	−1/3	33/7	1/4	$\rho > 0$	$\rho > 0$

TABLE III

The dimensional dependence of the characteristics of the minimally ($\xi = 0$) and conformally coupled ($\xi = \xi_c$) quantized field in the spatially flat matter-dominated ($\kappa = 0$) universe. The pressure has the same sign as the energy density ($\alpha > 0$).

D	n	α	β	$\xi = 0$	$\xi = \xi_c$
4	−2	2	2/3	$\rho < 0$	$\rho < 0$
5	−1	2	1/2	$\rho < 0$	$\rho < 0$
6	−2/3	3	2/5	$\rho > 0$	$\rho < 0$
7	−1/2	3	1/3	$\rho > 0$	$\rho < 0$
8	−2/5	4	2/7	$\rho < 0$	$\rho > 0$

For $\kappa = -1$, one has $n = 1$, $\alpha = -1$ and $\beta = \infty$. The energy density is always negative for the minimal coupling and positive for the conformal one. The background configuration in this case corresponds to the cosmological constant, which, in turn, is modified by the quantized field obeying the simple relation $p = -\rho$.

3.2. The exponential expansion

In this section, we shall briefly analyze the cosmological models with the scale factor

$$a(\eta) = \exp(\eta/\eta_0)\,. \tag{46}$$

This model represents the linear expansion in the proper time. Now, the classical background is a solution of the Einstein field equations with the source term satisfying the equation of state with

$$\kappa = -\frac{D-3}{D-1}\,. \tag{47}$$

On the other hand, the equation of state of the quantum matter is characterized by

$$\alpha = \begin{cases} \dfrac{3}{D-1} & \text{for even } D\,, \\ \dfrac{2}{D-1} & \text{for odd } D\,. \end{cases} \tag{48}$$

For the even dimensions, the stress-energy tensor simplifies considerably and can be written

$$\rho = \frac{H^{D+2}}{m^2} F_D(\xi)\,, \tag{49}$$

where

$$F_4 = -\frac{(66\xi - 13)(1-6\xi)^2}{192\pi^2}\,, \tag{50}$$

$$F_6 = -\frac{5(33\xi - 7)(1-5\xi)^3}{36\pi^3}\,, \tag{51}$$

$$F_8 = -\frac{63(222\xi - 49)(3-14\xi)^4}{20480\pi^4}\,, \tag{52}$$

$$F_{10} = -\frac{4(84\xi - 19)(2-9\xi)^5}{15\pi^5}\,, \tag{53}$$

$$F_{12} = -\frac{34375(474\xi - 109)(5-22\xi)^6}{24772608\pi^6}\,. \tag{54}$$

Inspection of the above equations shows that each F-function contains a factor $(\xi-\xi_{\rm c})^{D/2}$, where $\xi_{\rm c} = (D-2)/(4D-4)$, and consequently ρ vanishes for a conformally coupled fields. It is also zero for a more exotic values of the parameters ξ. On the other hand, the energy density is positive for minimally coupled fields.

The analogous formulas for the odd-dimensional spacetimes do not exhibit this simple behavior and will be not presented here. However, the components of the stress-energy tensor for $\xi = 0$ and $\xi = \xi_{\rm c}$ can easily be obtained from the functions listed in Appendix. Indeed, taking a limit $n \to \infty$, one obtains

$$\rho = \frac{H^{D+1}}{m} F_D(\xi)\,, \tag{55}$$

where $F_D = \lim_{n\to\infty} f(n,\xi)$. Equally well one can take $n \to -\infty$ limit. The functions F_D at $\xi = 0$ and $\xi = \xi_{\rm c}$ are positive. Finally, observe that due to $\alpha > 0$, the pressure-component of the stress-energy tensor of the quantized massive field has to be always positive.

4. Final remarks

In this paper (which can be thought of as a sequel to [27] and a natural generalization of the results presented in Refs. [24–26]), we have calculated the stress-energy tensor of the quantized massive scalar field in the spatially-flat D-dimensional Friedmann–Robertson–Walker spacetime. Thus far, however, the main emphasis has been on the adiabatic approximation, which has proven to be an excellent tool in this context. There is, however, equally powerful method, which relies heavily on the purely geometric objects constructed form the (differential) curvature invariants. In the Schwinger–DeWitt method, the stress-energy tensor is constructed form the one-loop effective action in a standard way

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{ab}} W^{(1)}_{\text{eff}}\,, \tag{56}$$

where

$$W^{(1)}_{\text{eff}} = \frac{1}{2(4\pi)^{D/2}} \int \mathrm{d}^D x \sqrt{g} \frac{[a_k]}{(m^2)^{k-D/2}} \Gamma\left(k - \frac{D}{2}\right) \tag{57}$$

and $k = \lfloor \frac{D}{2} \rfloor + 1$. Here, $[a_k]$ is the coincidence limit of the k^{th} Hadamard–DeWitt coefficient constructed from the Riemann tensor, its covariant derivatives up to $(2k-2)$-order and contractions. Unfortunately, the complexity of the stress-energy tensor rapidly grows with k and the dimension of the spacetime. Nevertheless, we have calculated the components of the stress-energy tensor for $4 \leq D \leq 8$ that, in turn, required knowledge of $[a_3]$ (for $D = 4, 5$), $[a_4]$ (for $D = 6, 7$) and $[a_5]$ (for $D = 8$), and their functional derivatives with respect to the metric. In all cases, we obtained the same results as those obtained within the framework of the adiabatic approximation, as expected.

5. Appendix

Here, we shall concentrate on the physical values of the parameter ξ, namely the minimal coupling ($\xi = 0$) and the conformal coupling ($\xi = (D-2)/(4D-4)$). In the first case, the energy density is described by

$$\rho = \frac{H^{D+2}}{m^2} f_D(n, \xi = 0)\,, \tag{58}$$

where

$$f_4 = \frac{1365n^4 + 4716n^3 - 4029n^2 - 4742n + 3060}{20160\pi^2 n^4}\,, \tag{59}$$

$$f_5 = \frac{2398n^4 + 1827n^3 - 4362n^2 - 1045n + 1350}{10080\pi^2 n^4}\,, \tag{60}$$

$$
\begin{aligned}
f_6 &= \frac{1}{120960\pi^3 n^6}\Big(117600n^6 + 223536n^5 - 297004n^4 - 325976n^3 \\
&\quad +257210n^2 + 76140n - 45675\Big) , && (61)\\
f_7 &= \frac{1}{645120\pi^3 n^6}\Big(1470552n^6 + 889680n^5 - 2986768n^4 - 898156n^3 \\
&\quad +1580375n^2 + 145944n - 175140\Big) , && (62)\\
f_8 &= \frac{1}{\pi^4 n^8}\Big(12.2093n^8 + 17.7149n^7 - 32.5261n^6 - 32.3395n^5 \\
&\quad +31.6037n^4 + 14.8324n^3 - 10.7232n^2 - 1.421n + 0.81733\Big) , && (63)\\
f_9 &= \frac{1}{\pi^4 n^8}\Big(24.3812n^8 + 12.9255n^7 - 56.1621n^6 - 18.0962n^5 \\
&\quad +40.3373n^4 + 6.30852n^3 - 9.62435n^2 - 0.463751n + 0.5335\Big) , && (64)\\
f_{10} &= \frac{1}{\pi^5 n^{10}}\Big(162.133n^{10} + 200.998n^9 - 471.558n^8 - 436.697n^7 \\
&\quad +520.298n^6 + 289.714n^5 - 245.264n^4 - 63.1615n^3 \\
&\quad +42.8619n^2 + 3.21193n - 1.79838\Big) , && (65)\\
f_{11} &= \frac{1}{\pi^5 n^{10}}\Big(296.804n^{10} + 144.584n^9 - 771.987n^8 - 255.695n^7 \\
&\quad +692.1n^6 + 136.602n^5 - 251.312n^4 - 24.0048n^3 \\
&\quad +33.6381n^2 + 0.991118n - 1.1099\Big) , && (66)
\end{aligned}
$$

and

$$
\begin{aligned}
f_{12} &= \frac{1}{\pi^6 n^{12}}\Big(2363.29n^{12} + 2641.16n^{11} - 7539.76n^{10} - 6637.74n^9 \\
&\quad +9420.92n^8 + 5677.44n^7 - 5564.29n^6 - 1948.25n^5 + 1508.29n^4 \\
&\quad +245.392n^3 - 159.039n^2 - 7.65612n + 4.20787\Big) . && (67)
\end{aligned}
$$

On the other hand, for the conformally coupled quantized field, one has

$$\rho = \frac{H^{D+1}}{m} f_D(n, \xi = \xi_{\rm c}) , \tag{68}$$

where

$$
\begin{aligned}
f_4 &= \frac{24n^3 - 15n^2 - 41n + 30}{5040\pi^2 n^4} , && (69)\\
f_5 &= \frac{457n^4 + 1512n^3 - 3588n^2 - 1408n + 2880}{2580480\pi^2 n^4} , && (70)
\end{aligned}
$$

$$f_6 = \frac{96n^5 - 4n^4 - 596n^3 + 470n^2 + 360n - 315}{120960\pi^3 n^6}, \tag{71}$$

$$f_7 = \frac{(n-2)\left(3287n^5 + 24094n^4 - 5900n^3 - 77656n^2 + 13248n + 40320\right)}{165150720\pi^3 n^6}, \tag{72}$$

$$f_8 = \frac{(n-2)\left(288n^6 + 788n^5 - 2196n^4 - 2108n^3 + 4338n^2 + 585n - 1620\right)}{1900800\pi^4 n^8}, \tag{73}$$

$$f_9 = \frac{1}{\pi^4 n^8}\left(2.79769 \times 10^{-6} n^8 + 0.0000205636n^7 - 0.0000768524n^6 - 0.000175441n^5 + 0.00053688n^4 + 0.000204701n^3 - 0.00089077n^2 + 0.0000507305n + 0.000319602\right), \tag{74}$$

$$f_{10} = \frac{1}{\pi^5 n^{10}}\left(0.0000326261n^9 + 0.0000515903n^8 - 0.000716464n^7 + 0.000202078n^6 + 0.0035634n^5 - 0.0032242n^4 - 0.0043084n^3 + 0.00506189n^2 + 0.000865606n - 0.0015024\right), \tag{75}$$

$$f_{11} = \frac{1}{\pi^5 n^{10}}\left(4.67381 \times 10^{-7} n^{10} + 4.36237 \times 10^{-6} n^9 - 0.00001906n^8 - 0.0000659n^7 + 0.000234n^6 + 0.0001871n^5 - 0.0008377n^4 - 0.00001398n^3 + 0.0009307n^2 - 0.000132667n - 0.000281\right) \tag{76}$$

and

$$f_{12} = \frac{1}{\pi^6 n^{12}}\left(7.825067 \times 10^{-6} n^{11} + 0.0000191n^{10} - 0.000255n^9 - 0.0000297n^8 + 0.002199n^7 - 0.001744n^6 - 0.0057854n^5 + 0.0068403n^4 + 0.004394n^3 - 0.00683n^2 - 0.00051n + 0.00167411\right). \tag{77}$$

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