A COMMENTARY ON SINGLE-PHOTON WAVE FUNCTION ADVOCATED BY BIAŁYNICKI-BIRULA

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We present in this paper how the single-photon wave function for transversal photons (with the direct sum of ordinary unitary representations of helicity 1 and -1 acting on it) is subsumed within the formalism of Gupta–Bleuler for the quantized free electromagnetic field. The rigorous Gupta–Bleuler quantization of the free electromagnetic field is based on our generalization (published formerly) of the Mackey theory of induced representations which includes representations preserving the indefinite Krein-inner-product given by the Gupta–Bleuler operator. In particular, it follows that the results of Białynicki-Birula on the single-photon wave function may be reconciled with the causal perturbative approach to QED.

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This short account is a commentary on the single-photon wave function as advocated by Białynicki-Birula [1] and references therein. His works on the subject enjoy a considerable attention and popularity. This is because on the one hand, the single-photon wave function is a concept which is accompanied with controversial opinions. Some authors, *e.g.* [2], even claim that position wave function for photon does not exist. But on the other hand, the subject being of fundamental importance is still not systematically explored.

We agree *e.g.* with [3] and [1] and the authors cited there that the single-photon wave function is already implicitly present in quantum field theory: generally, a free quantum field is constructed by the application of the symmetrized/antisymmetrized tensoring and direct sum operations (the so-called second quantized functor) to a specific representation of the double covering of the Poincaré group acting in a space, which may be identified with the space of single-particle wave functions, and which depends on the specific quantum field. At the level of the free electromagnetic field, one can start at the Hilbert space of transversal single-photon states acted on by the direct sum of the unitary zero mass helicity 1 and -1 representations, respectively (in the language of the classical by now Wigner–Mackey–Gelfand–Bargmann classification of irreducible unitary representations of the Poincaré group). In more physical terms, the representation has been described *e.g.* in [1] together with its relation to the Riemann–Silbertstein vector wave function. It is true that the (free) quantum electromagnetic field has its own peculiarities making some differences in comparison to massive and non-gauge fields which still serve as a source of misunderstandings and still are not well-understood.

The first peculiarity of a zero mass quantum (free) field, even non-gauge field (as we assume for a while in order to simplify situation), is that now, the representation of the Poincaré group to which we apply Segal's functor of second quantization, although being unitary in ordinary sense, is specified within the Wigner–Mackey classification scheme by the orbit in the momentum space which is the light cone (without the apex), contrary to the massive case, where the orbit is the smooth sheet of the two-sheeted hyperboloid. The apex being a singular point of the cone (in the sense of the ordinary differential structure of the cone as embedded into the $\mathbb{R}^4$-manifold) causes serious difficulties of infrared character. This is because the quantum field is, in fact, an operator-valued distribution (as motivated by the famous Bohr–Rosenfeld analysis [4] of the measurement of the quantum electromagnetic field) which needs a test function space. It is customary to use the standard Schwartz space of rapidly decreasing functions as the universal test space even for zero mass fields, and this is not the correct test space for zero mass field. Recall that the construction mentioned above of a free quantum field achieved by the second quantization functor Γ applied to a representation specified by a fixed orbit allows to construct creation and annihilation families of *operators* in the Fock space. In order to construct the field as operator valued distribution, we have to proceed slightly further along the construction given by Streater and Wightman in their well-known monograph [5, Ch. 3]. In the construction of Wightman, we consider the restrictions of Fourier transforms (*i.e.* functions in the momentum space) of the test functions to the orbit in question. The construction works if the restriction is a continuous map from the test function space in $\mathbb{R}^4$ to the test function space in $\mathbb{R}^3$ which is really the case in the massive case as the orbit is a smooth manifold in that case. Unfortunately, it seems that it has escaped due attention of physicists that the correct test function space in the momentum representation for the zero mass field is the closed subspace $\mathcal{S}_0$ of the Schwartz space $\mathcal{S}$ of those functions which vanish at zero together with all their derivatives and the test function space $\mathcal{S}_{00}$ in the position representation is given by the inverse Fourier image of the space $\mathcal{S}_0$. This, in turn, causes additional difficulties concerned with exploring and correct use of the principles of locality character, because, in particular, the space $\mathcal{S}_{00}$ does not contain any function of compact support (except the trivial zero function) which immediately follows from the generalized Paley–Wiener theorem. Fields have to be carefully extended outside the test space $\mathcal{S}_0$ on functions whose derivatives *up to only a finite order* vanish at zero (similarly as the creation and annihilation families of operators, which make sense outside the test space) in order to account for the locality-type principles correctly in the zero mass case.

The situation for the electromagnetic field is still more delicate as the field is accompanied by the gauge freedom and the ordinary unitarity is untenable and has to be replaced with a weaker condition of preservation of the indefinite Krein-inner-product — which is the second main peculiarity of the electromagnetic field, shared with the other zero mass gauge fields of the Standard Model. This requires, however, the theory of non-unitary representations of the Poincaré group which preserve indefinite inner product defined by the Gupta–Bleuler operator, which should allow us to work effectively with tensor products of such representations, Frobenius reciprocity theorem, imprimitivity system theorem, *etc.* Such a theory had not existed until 2015, compare [6] where it appears for the first time.

Therefore, the construction of the field by the second quantization functor Γ applied to a single-particle representation should be extended on representations which are not unitary but only Krein-isometric.

The Gupta–Bleuler quantization (or generally the BRST method) is the only known quantization of gauge fields compatible with the causal perturbative approach of Stuckelberg–Bogoliubov (compare [7]), this is a well-established fact, compare *e.g.* [8] or [9] and references therein. The point is that causal approach allows us to avoid completely the UV divergences (so that we can dispense completely with UV renormalization), but requires the local transformation law for A_μ. Importance of this method for understanding of UV divergences is well-known. The problem of mathematically rigorous construction of the (free) quantum electromagnetic potential is therefore an important problem, and it was pointed out by many specialists that the indefinite character of the metric makes the problem difficult, compare *e.g.* [10]. Using our previous results [6], we give the solution to this problem which allows the solution within causal perturbative approach. From the point of view of causal method, the work of Białynicki-Birula on single-photon wave function, although important, is not entirely complete, giving the construction of the single-particle subspace of the Fock space of the quantum Riemann–Silberstein vector. First, because we need A_μ with local transformation and not the electric or magnetic fields themselves to construct (perturbatively) interacting fields. And second: although we can, of course, appeal to the Coulomb gauge, the additional gauge term in the transformation formula for A_μ, saving relativistic invariance is non-local together with non-local Coulomb interaction (which is, of course, well-known), the infinite UV divergences and the UV renormalization are unavoidable in this approach, [8] or [9]. We construct here the quantum A_μ with local transformation (within the Gupta–Bleuler scheme) which allows us the application of the causal method and which allows to avoid completely UV divergences, then we show that the physical subspace of the single-particle states is exactly the same as the single-particle state space in the Fock space of the quantum Riemann–Silberstein vector of Białynicki-Birula paper [1]. The lack of a rigorous construction of the quantum A_μ with local transformation (which appears for the first time in this paper) was heavily felt, and was raised by many prominent specialists, compare *e.g.* [10]. For example, for the construction of the quantum field A_μ, an extension of spectral theorem for class of operators in the Krein space was needed as well as an extension of unitary representations to Krein-isometric representations [6]. This extension problem was open and only a partial solution has been given by a well-known and prominent mathematician Neumark, but in a form which is not sufficient for the needs of QFT, as he analysed Krein-unitary representations in the so-called Pontriagin spaces, with the strong simplification that the subspace on which the inner product is negative has finite dimension, compare *e.g.* [11].

It should be stressed that the mathematically rigorous construction of the free gauge fields (*e.g.* quantum free electromagnetic potential) is not merely a matter of pedantry. In the case of QED, the ultraviolet problem is fully solved by the extension of the Bogoliubov–Epstein–Glaser method [7] to QED, compare [9]. The infrared divergences are controlled by the adiabatic switching of the interaction. However, the infrared problem is only partially solved for QED in this way. One aspect is that charged particles cannot be eigenstates of the mass operator. The other aspect are the divergences which appear in the adiabatic limit. Here, we comment shortly the second aspect (although it seems that these two aspects are interconnected). These divergences are logarithmic in QED and cancel out in the cross section, at least at lower order terms of the perturbative series [9]. Blanchard and Seneor [12] extended only partially on QED the result of Epstein–Glaser of the existence of the adiabatic limit for scalar massive field and proved the existence of the adiabatic limit for Wightman and Green functions for QED (for non-Abelian gauge fields, the situation is still less explored). In the Epstein–Glaser proof (for the scalar massive field), spectral condition is crucial, and essentially means that the orbit of the representation determining the single-particle space is separated from zero and the only behaviour of the test function which plays a role goes through the restriction to the orbit of its Fourier transform (the test functions are just the Schwartz rapidly decreasing functions $\mathcal{S}$). Because the orbit for the free electromagnetic field is not separated from zero being just the light cone, then the Epstein–Glaser proof does not work in QED with the test function space $\mathcal{S}$. But in the treatment of QED (and the other non-Abelian gauge fields), we have not been so much pedantic in the construction of the free field, because we have many relatively simple methods for making the correct guess as to the shape of distribution functions giving the pairings of free fields plying the immediate role in computation of the cross section. Here enters our rigorous construction, because our construction of the zero mass gauge fields revealed at least one point which must have been missed at the heuristic level of the construction of the free field. Namely, the test function space has to be changed for the zero mass gauge fields, and in the momentum picture, it is just $\mathcal{S}_0$. This, in particular, means that we have a God-given infrared cut-off assured by the very existence of the zero mass gauge field as a well-defined operator valued distribution. In particular, the method of Epstein–Glaser for the proof of the existence of the adiabatic limit should be revisited, because the fact that the light-cone orbit is not separated from zero is compensated for by the infrared cut-off of the elements of $\mathcal{S}_0$. Another infrared problem which can be solved by the use of our rigorous construction is the strict proof of the Bogoliubov–Shirkov quantization hypothesis for free fields, as stated in their monograph. This problem lies among the problems which were unsolved and are concerned with the existence of integrals of local conserved currents corresponding to conserved symmetries [13]. In the case of zero mass gauge fields, any endeavour of proving the existence of these integrals and their eventual equality to the generator of the corresponding one-parameter subgroup have permanently been accompanied by infrared divergences. Our rigorous method allows to solve these problems without encountering any divergences.

After this general introduction, let us concentrate on the main theme of our commentary and give some details of the single-photon Krein-isometric representation and the closed subspace of transversal photon states. Let us start with a brief description of the Krein-isometric single-photon representation in the momentum picture, which we call *Łopuszański representation*. We give at once the form of the representation which has the multiplier independent of the momentum, so that the Fourier transform of the momentum functions, *i.e.* position wave functions, have local transformation formula. Namely, the representation acts in a Krein space[1] $(\mathcal{H}', \mathfrak{J}')$, *i.e.* an ordinary Hilbert space $\mathcal{H}'$ endowed with the fundamental symmetry $\mathfrak{J}'^2 = I$, $\mathfrak{J}'^* = \mathfrak{J}'$,

[1] We use the notation of [14].

and the Krein-isometric representation preserves the Krein-inner-product $(\cdot, \mathfrak{J}' \cdot)$, but for detailed definition compare Sect. 2 of [6] as the peculiarities like unboundedness (with respect to the ordinary Hilbert space product) cannot be excluded from the outset here in contrast to the ordinary unitary representations and indeed, our representation is unbounded. The Hilbert space $\mathcal{H}'$ consists of all measurable four-component functions $\widetilde{\varphi}$ on the light cone $\mathscr{O}_{\bar{p}}$ in momentum space, which we may naturally regard as the functions of the spatial momentum components $\boldsymbol{p} \in \mathbb{R}^3$ with $p^0(\boldsymbol{p}) = r(\boldsymbol{p}) = \sqrt{\boldsymbol{p} \cdot \boldsymbol{p}}$, and which have finite Hilbert space norm $\sqrt{(\cdot, \cdot)}$. The Hilbert space inner product $(\cdot, \cdot)$ in $\mathcal{H}'$ is equal

$$\big(\widetilde{\varphi}, \widetilde{\varphi}'\big) = \big(\widetilde{\varphi}, B\widetilde{\varphi}'\big)_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \,,$$

where the self-adjoint positive operator B, regarded as operator, *e.g.* in $L^2(\mathbb{R}^3, \mathbb{C}^4)$, is equal to the operator of point-wise multiplication by the matrix operator

$$\frac{1}{2r} B(p) \,, \qquad p \in \mathscr{O}_{\bar{p}}$$

which is strictly positive and self-adjoint in $\mathbb{C}^4$ with

$$B(p) = \begin{pmatrix} \frac{r^{-2}+r^2}{2} & \frac{r^{-2}-r^2}{2r} p^1 & \frac{r^{-2}-r^2}{2r} p^2 & \frac{r^{-2}-r^2}{2r} p^3 \\ \frac{r^{-2}-r^2}{2r} p^1 & \frac{r^{-2}+r^2-2}{2r^2} p^1 p^1 + 1 & \frac{r^{-2}+r^2-2}{2r^2} p^1 p^2 & \frac{r^{-2}+r^2-2}{2r^2} p^1 p^3 \\ \frac{r^{-2}-r^2}{2r} p^2 & \frac{r^{-2}+r^2-2}{2r^2} p^2 p^1 & \frac{r^{-2}+r^2-2}{2r^2} p^2 p^2 + 1 & \frac{r^{-2}+r^2-2}{2r^2} p^2 p^3 \\ \frac{r^{-2}-r^2}{2r} p^3 & \frac{r^{-2}+r^2-2}{2r^2} p^3 p^1 & \frac{r^{-2}+r^2-2}{2r^2} p^3 p^2 & \frac{r^{-2}+r^2-2}{2r^2} p^3 p^3 + 1 \end{pmatrix},$$

again strictly positive self-adjoint on $\mathbb{C}^4$. For each $p \in \mathscr{O}_{\bar{p}}$,

$$w_{{}_1}{}^{+}(p) = \begin{pmatrix} 0 \\ \frac{p^2}{\sqrt{(p^1)^2+(p^2)^2}} \\ \frac{-p^1}{\sqrt{(p^1)^2+(p^2)^2}} \\ 0 \end{pmatrix}, \qquad w_{{}_1}{}^{-}(p) = \begin{pmatrix} 0 \\ \frac{p^1 p^3}{\sqrt{(p^1)^2+(p^2)^2}\, r} \\ \frac{p^2 p^3}{\sqrt{(p^1)^2+(p^2)^2}\, r} \\ -\frac{\sqrt{(p^1)^2+(p^2)^2}}{r} \end{pmatrix},$$

$$w_{{}_{r^{-2}}}(p) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \frac{p^1}{r} \\ \frac{1}{\sqrt{2}} \frac{p^2}{r} \\ \frac{1}{\sqrt{2}} \frac{p^3}{r} \end{pmatrix}, \qquad w_{{}_{r^2}}(p) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \frac{p^1}{r} \\ -\frac{1}{\sqrt{2}} \frac{p^2}{r} \\ -\frac{1}{\sqrt{2}} \frac{p^3}{r} \end{pmatrix}$$

are the eigenvectors of the matrix $B(p)$ which are orthonormal in $\mathbb{C}^4$, where $w_{{}_1}{}^{+}(p), w_{{}_1}{}^{-}(p)$ correspond to the eigenvalue equal $+1$, and $w_{{}_{r^{-2}}}(p)$, $w_{{}_{r^2}}(p)$ correspond to the eigenvalues r^{-2}, r^2, respectively.

The fundamental symmetry $\mathfrak{J}'$ is equal to the operator of point-wise multiplication by the matrix

$$\mathfrak{J}'_p = \mathfrak{J}_{\bar{p}} B(p) \,, \qquad p \in \mathscr{O}_{\bar{p}}$$

with $\mathfrak{J}_{\bar{p}}$ equal to the following constant matrix

$$\mathfrak{J}_{\bar{p}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

being a fundamental symmetry in $\mathbb{C}^4$.

If for each $\alpha \in SL(2,\mathbb{C})$, we denote by $\alpha \mapsto \Lambda(\alpha)$ the natural antihomomorphism of $SL(2,\mathbb{C})$ into the Lorentz group, and by $U(\alpha)$ the representors of $\alpha \in SL(2,\mathbb{C})$ and by $T(a)$, $a \in \mathbb{R}^4$ the representors of translations, then we have

$$\begin{aligned} U(\alpha)\widetilde{\varphi}(p) &= \Lambda\left(\alpha^{-1}\right)\widetilde{\varphi}(\Lambda(\alpha)p) \,, \\ T(a)\widetilde{\varphi}(p) &= e^{ia\,p}\widetilde{\varphi}(p) \,, \qquad \widetilde{\varphi} \in \mathcal{H}' \,. \end{aligned}$$

The inverse Fourier transforms φ

$$\varphi(x) = (2\pi)^{-3/2} \int\limits_{\mathscr{O}_{\bar{p}}} \widetilde{\varphi}(p) e^{-ip\,x} \, \mathrm{d}\mu|_{\mathscr{O}_{\bar{p}}}(p) \,, \qquad \widetilde{\varphi} \in \mathcal{H}'$$

compose the single-photon Krein space $(\mathcal{H}'', \mathfrak{J}'')$ in the position picture with the representation giving the local four-vector transformation law in the position picture. In the last formula, $\mathrm{d}\mu|_{\mathscr{O}_{\bar{p}}}(p)$ stands for the invariant measure $2^{-1}r^{-1}\mathrm{d}^3\boldsymbol{p}$ on the cone $\mathscr{O}_{\bar{p}}$.

Together with the Łopuszański representation (T, U), we consider the conjugate representation $([T]^{*-1}, [U]^{*-1}) = (\mathfrak{J}'T\mathfrak{J}', \mathfrak{J}'U\mathfrak{J}') = (T, \mathfrak{J}'U\mathfrak{J}')$, which likewise preserves the same Krein-inner-product $(\cdot, \mathfrak{J}'\cdot)$.

We apply to this conjugate representation the functor of second quantization obtaining the families $a(\widetilde{\varphi}), a(\widetilde{\varphi})^+$ of creation and annihilation operators in the Fock space $\Gamma(\mathcal{H}') \cong \Gamma(\mathcal{H}'')$ with the Gupta–Bleuler operator $\eta = \Gamma(\mathfrak{J}')$. We claim that η fulfils the correct commutation relations which are to be expected for the Gupta–Bleuler operator.

We must be careful in preparing the fields as Wightman operator-valued distributions. This can be achieved by application of the Schwartz–Woronowicz kernel theorem [15] to the test function spaces $\mathcal{S}_0$ and $\mathcal{S}_{00}$ mentioned above and to the operator valued distribution (quantum vector potential)

$$A(\varphi) = A^{\mu}(\varphi_{\mu}) = a\left(\widetilde{\varphi}|_{\mathscr{O}_{\bar{p}}}\right) + \eta a\left(\widetilde{\varphi}|_{\mathscr{O}_{\bar{p}}}\right)^{+} \eta \,,$$

where $\varphi \in \mathcal{S}_{00}(\mathbb{R}^4)$, its Fourier transform $\widetilde{\varphi}$ belongs to $\mathcal{S}_0(\mathbb{R}^4)$ and where $\widetilde{\varphi} \mapsto \widetilde{\varphi}|_{\mathscr{O}_{\bar{p}}}$ is the restriction to the cone, which turns out to be, indeed, a continuous map of nuclear spaces $\mathcal{S}_0(\mathbb{R}^4) \to \mathcal{S}_0(\mathbb{R}^3)$.

It turns out that, indeed, the commutator

$$[A(\varphi), A(\varphi')]$$

defines the kernel distribution equal to the Pauli–Jordan function multiplied by the Minkowskian metric; in the proof, one can apply *e.g.* the kernel theorem as stated in [15] (the ordinary Schwartz kernel theorem is not sufficient for the construction of the Wick product); and it follows that $A(\varphi)$ is the Wightman field transforming locally as a four-vector field.

It should be stressed that, in general, the elements $\widetilde{\varphi}$ of the single-particle space of the Łopuszański representation (and its conjugation) in the momentum picture do not fulfil the condition $p^\mu \widetilde{\varphi}_\mu = 0$, so that their Fourier transforms φ do not preserve the Lorentz condition $\partial^\mu \varphi_\mu = 0$. This corresponds to the well-known fact that the Lorentz condition cannot be preserved as an operator equation. It can be preserved in the sense of the Krein-product average on a subspace of Lorentz states which arise from the closed subspace $\mathcal{H}_{\mathrm{tr}}$ of the so-called transversal states together with all their images under the action of the Łopuszański representation and its conjugation. We are now going to define the closed subspace $\mathcal{H}_{\mathrm{tr}}$.

The closed subspace $\mathcal{H}_{\mathrm{tr}} \subset \mathcal{H}'$ consists of all functions of the form of

$$\widetilde{\varphi} = {w_1}^+ f_+ + {w_1}^- f_-$$

with f_+, f_- ranging over all pairs of measurable scalar functions on the light cone $\mathscr{O}_{\bar{p}}$ square integrable with respect to the invariant measure $2^{-1} r^{-1} \mathrm{d}^3\boldsymbol{p}$ on the cone. It follows that Hilbert space $\mathcal{H}'$ inner product

$$(\widetilde{\varphi}, \widetilde{\varphi}) = \int\limits_{\mathscr{O}_{\bar{p}}} |f_+(p)|^2 \, 2^{-1} r^{-1} \mathrm{d}^3\boldsymbol{p} + \int\limits_{\mathscr{O}_{\bar{p}}} |f_-(p)|^2 \, 2^{-1} r^{-1} \mathrm{d}^3\boldsymbol{p}$$

of any element $\widetilde{\varphi} \in \mathcal{H}_{\mathrm{tr}}$ is equal to the Krein-inner-product $(\widetilde{\varphi}, \mathfrak{J}' \widetilde{\varphi})$, and thus the Krein-inner-product is strictly positive on $\mathcal{H}_{\mathrm{tr}}$.

We claim that the action of the Łopuszański representation and its conjugation generate modulo unphysical states of Krein-norm zero and Krein orthogonal to $\mathcal{H}_{\mathrm{tr}}$, exactly the same representation $(\mathbb{T}, \mathbb{U})$:

$$\mathbb{U}(\alpha) \begin{pmatrix} f_+ \\ f_- \end{pmatrix}(p) = \begin{pmatrix} \cos\Theta(\alpha, p) & \sin\Theta(\alpha, p) \\ -\sin\Theta(\alpha, p) & \cos\Theta(\alpha, p) \end{pmatrix} \begin{pmatrix} f_+(\Lambda(\alpha)p) \\ f_-(\Lambda(\alpha)p) \end{pmatrix},$$

$$\mathbb{T}(a) \begin{pmatrix} f_+ \\ f_- \end{pmatrix}(p) = e^{ia\cdot p} \begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix}$$

on $\mathcal{H}_{\mathrm{tr}}$, which is unitary for the strictly positive inner product on $\mathcal{H}_{\mathrm{tr}}$ induced by the Krein-inner-product $(\cdot, \mathfrak{J}'\cdot)$, for the proof compare [14]. Therefore, $(\mathbb{T}, \mathbb{U})$ is an ordinary unitary representation of the Poincaré group, which may be shown to be unitary equivalent to the direct sum $[m = 0, h = +1] \oplus [m = 0, h = -1]$ of zero mass helicity $+1$ and of helicity -1 representations [14]. For the concrete form of the phase Θ, we refer to [14]. The representation $(\mathbb{T}, \mathbb{U})$, after a simple unitary transform on $\mathcal{H}_{\mathrm{tr}}$, gives exactly the single-photon representation of [1], §4.3, formulas (4.22) and (4.23) with exactly the Hilbert space of §5.1 of [1], which can be identified with our $\mathcal{H}_{\mathrm{tr}}$, compare [14].

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