

# STATISTICAL ASPECTS OF THE PARAMAGNETIC SYSTEMS IN THE PRESENCE OF A MINIMAL LENGTH

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This work attempts to investigate the influence of a minimal length scale on the statistical aspects of the paramagnetic system. The angular momentum operator and the magnetostatic field in 3-dimensional space described by the Kempf algebra is studied in the special case of  $\alpha' = 2\alpha$  up to the first order over the deformation parameter  $\alpha$ . The modified thermodynamical characteristics of the paramagnetic system such as mean energy, entropy, magnetization are obtained. It is shown that the relative magnetization approximately depends on the deformation parameter and orbital angular momentum. The upper limit of the deformation parameter is estimated.

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## 1. Introduction

In the recent years, theoretical physicists have tried to find the unification between the general relativity and the Standard Model of particle physics [1]. Today, it is widely accepted that perturbative string theory and loop quantum gravity lead to a minimal length scale in nature [2]. The minimal length appears due to a modification of the usual Heisenberg uncertainty principle. The modified uncertainty principle is known as Generalized Uncertainty Principle (GUP) [3, 4]. This generalized uncertainty principle can be written as

$$\Delta X \Delta P \geq \frac{\hbar}{2} [1 + \alpha(\Delta P)^2], \quad (1)$$

where  $\alpha$  is a positive parameter [5, 6]. It is obvious that in Eq. (1),  $\Delta X$  is always larger than  $(\Delta X)_{\min} = \hbar\sqrt{\alpha}$ . Kempf and his collaborators showed that finite resolution of length can be obtained from the deformed Heisenberg algebra [7–9]. The Kempf algebra leading to the existence of a minimal

length in a  $D$ -dimensional space is characterized by the following deformed commutation relations:

$$\begin{aligned} [X^i, P^j] &= i\hbar \left[ \left(1 + \alpha \vec{P}^2\right) \delta^{ij} + \alpha' P^i P^j \right], \\ [X^i, X^j] &= i\hbar \frac{(2\alpha - \alpha') + (2\alpha + \alpha')\alpha \vec{P}^2}{1 + \alpha \vec{P}^2} (P^i X^j - P^j X^i), \\ [P^i, P^j] &= 0, \end{aligned} \quad (2)$$

where  $\alpha, \alpha'$  are two positive deformation parameters. In Eq. (2),  $X^i$  and  $P^i$  are position and momentum operators in the GUP framework and it is easy to find that a minimal length scale equals  $\hbar\sqrt{(D\alpha + \alpha')}$ . There are many papers about the effects of minimal length on various problems such as harmonic oscillator, hydrogen atom, gravitational quantum well, Lamb's shift and particles scattering [10–14]. In the recent years, reformulations of quantum mechanics, quantum field theory and statistical mechanics in the presence of a minimal measurable length have been studied [15–20]. It seems that the thermodynamical characteristics of the physical systems such as the mean energy, partition function and entropy in the deformed space will be changed. In this work, we investigate the statistical properties of the paramagnetic system in the presence of a minimal length. This paper is organized as follows: In Sec. 2, the angular momentum operator and magnetostatic field in three spatial dimensions described by the Kempf algebra is introduced. In Sec. 3, thermodynamical characteristics of the paramagnetic system in the presence of a minimal length scale are obtained up to the order of  $\alpha$ . Our calculations are found for arbitrary total angular momentum  $J$ . We obtain also the modified magnetization of the paramagnetic system at both high and low temperatures. Our conclusions are presented in Sec. 4. We use SI units throughout this paper.

## 2. Angular momentum operator and magnetostatic field in the presence of a minimal length based on the Kempf algebra

In this section, we obtain the modified angular momentum operator and the modified magnetostatic field. For this aim, it is essential to introduce the representation of modified position and momentum operators satisfying the Kempf algebra in Eq. (2). Stetsko and Tkachuk used the approximate representation fulfilling the Kempf algebra in the first order over the deformation parameters  $\alpha$  and  $\alpha'$  [21]

$$\begin{aligned} X^i &= x^i + \frac{2\alpha - \alpha'}{4} (\vec{p}^2 x^i + x^i \vec{p}^2), \\ P^i &= p^i \left( 1 + \frac{\alpha'}{2} \vec{p}^2 \right), \end{aligned} \quad (3)$$

where the operators  $x^i$  and  $p^i = i\hbar \frac{\partial}{\partial x_i}$  are position and momentum operators in the ordinary Heisenberg algebra. It is emphasized that in the special case of  $\alpha' = 2\alpha$ , the position operators commute in linear approximation over the deformation parameter  $\alpha$ , i.e.  $[X^i, X^j] = 0$ . Considering this linear approximation, we can write the Kempf algebra in Eq. (2) as follows:

$$\begin{aligned} [X^i, P^j] &= i\hbar \left[ (1 + \alpha \vec{P}^2) \delta^{ij} + 2\alpha P^i P^j \right], \\ [X^i, X^j] &= 0, \\ [P^i, P^j] &= 0. \end{aligned} \quad (4)$$

Brau showed that the following representations satisfy (4) in the first order in  $\alpha$  [22]

$$X^i = x^i, \quad (5)$$

$$P^i = p^i (1 + \alpha \vec{p}^2). \quad (6)$$

### 2.1. The modified angular momentum operator

We know that in quantum mechanics the angular momentum operator is defined as follows [23]:

$$\vec{L} = \vec{x} \times \vec{p}, \quad (7)$$

where  $\vec{x}$  and  $\vec{p}$  are ordinary position and momentum operators. For obtaining the modified angular momentum operator based on the Kempf algebra, we must replace the usual position and momentum operators with the deformed position and momentum operators according to Eqs. (5) and (6), and we have

$$\vec{\mathcal{L}} = \vec{X} \times \vec{P}. \quad (8)$$

If we substitute Eqs. (5) and (6) into Eq. (8), we obtain the following angular momentum in the deformed space

$$\vec{\mathcal{L}} = (1 + \alpha \vec{p}^2) \vec{L}, \quad (9)$$

where  $\vec{L}$  is the ordinary angular momentum operator.

### 2.2. The modified magnetostatic field

The Lagrangian density for a magnetostatic field with an external current density  $\vec{J}(\vec{x}) = (J^1(\vec{x}), J^2(\vec{x}), J^3(\vec{x}))$  in three spatial dimensions can be written as follows [24]:

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{ij}(\vec{x}) F^{ij}(\vec{x}) - J^i(\vec{x}) A^i(\vec{x}), \quad (10)$$

where  $F_{ij}(\vec{x}) = \partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x})$  and  $\vec{A}(\vec{x}) = (A^1(\vec{x}), A^2(\vec{x}), A^3(\vec{x}))$  are the electromagnetic field tensor and the vector potential respectively. If we replace the usual position and derivative operators with the deformed position and derivative operators according to Eqs. (5) and (6), that is

$$x^i \longrightarrow X^i = x^i, \quad (11)$$

$$\partial^i \longrightarrow D^i := (1 - \alpha \hbar^2 \nabla^2) \partial^i, \quad (12)$$

we will obtain the electromagnetic field tensor in the presence of a minimal length as follows [25]:

$$\mathcal{F}_{ij}(\vec{X}) = F_{ij}(\vec{x}) - \alpha \hbar^2 \nabla^2 F_{ij}(\vec{x}) = (1 + \alpha p^2) F_{ij}(\vec{x}). \quad (13)$$

The three-dimensional magnetic induction vector  $\vec{B}(\vec{x})$  is defined as follows [26]:

$$F_{ij} = -\epsilon_{ijk} B^k, \quad F^{ij} = \epsilon^{ijk} B_k. \quad (14)$$

If we insert Eq. (14) into Eq. (13), we obtain the following magnetostatic field in the presence of a minimal length

$$\vec{b}_{\text{ML}}(\vec{x}) = (1 - \alpha \hbar^2 \nabla^2) \vec{B}(\vec{x}) = (1 + \alpha \vec{p}^2) \vec{B}(\vec{x}). \quad (15)$$

In the above equation, the term  $(\alpha \vec{p}^2) \vec{B}(\vec{x})$  can be considered as a minimal length effect.

### 3. Paramagnetic system in the presence of a minimal length scale

An especially interesting application of statistical mechanics is the paramagnetic behavior of substances. It is well-known that the atoms of many substances have a permanent magnetic dipole moment  $\vec{\mu}$  [27]. If such a substance is subject to an external magnetostatic field  $\vec{B}$ , then the dipoles try to align in the direction of the field, so that the potential energy can be shown as follows [28]:

$$U = -\vec{\mu} \cdot \vec{B}. \quad (16)$$

We consider that a paramagnetic system of unit volume contains  $N$  atoms which we treat as distinguishable. We denote the total angular momentum of the  $n^{\text{th}}$  atom by  $\vec{J}_n$ , and its magnetic moment is given by

$$\vec{\mu}_n = -g\mu_B \left( \frac{\vec{J}_n}{\hbar} \right), \quad (17)$$

where  $g$  is the Lande factor, and  $\mu_B$  the Bohr magneton. The atoms are assumed to interact only with an external magnetostatic field  $\vec{B}$ . Hence, the Hamiltonian describing the system is

$$H = - \sum_{n=1}^N \vec{\mu}_n \cdot \vec{B}. \quad (18)$$

It should be noted that the  $n^{\text{th}}$  atom must be in a simultaneous eigenstate of  $J_n^2$  and  $J_{nz}$ , which we denote  $|J, m_n\rangle$  or simply  $|m_n\rangle$ , because  $J$  is the same for all atoms. Hence, the energy eigenvalue of the quantum state  $|m_n\rangle$  is obtained by acting on  $|m_n\rangle$  with the Hamiltonian operator of Eq. (18). Since  $\mathbf{B} = B\hat{z}$ , we have

$$H = -B \sum_{n=1}^N \mu_{nz} = g\mu_B B \left( \frac{J_{1z}}{\hbar} + \frac{J_{2z}}{\hbar} + \cdots + \frac{J_{Nz}}{\hbar} \right). \quad (19)$$

Since  $J_{1z}|m_1\rangle = m_1\hbar|m_1\rangle$ , we find that the energy eigenvalues are given by

$$E_J = g\mu_B B(m_J), \quad (20)$$

where  $m_J$  is the component of  $\mathbf{J}$  in  $Z$ -direction and we have

$$m_J = -J, -(J-1), \dots, 0, 1, \dots, (J-1), J. \quad (21)$$

The magnetic moment of atoms is caused by electrons moving around the nucleus; *i.e.*, it is a quantum mechanical quantity. Therefore, we want to perform the same considerations once again for quantum mechanical dipole. In quantum mechanics,  $\vec{\mu}$  is an operator which is defined by [27]

$$\vec{\mu} = -g_J\mu_B\vec{J} = -g_J\mu_B(\vec{L} + \vec{S}), \quad (22)$$

where  $\vec{L}$  is the angular momentum operator and  $\vec{S}$  is the spin operator. Then,  $\vec{\mu}$  is no longer proportional to the total angular momentum  $\vec{J} = \vec{L} + \vec{S}$ . It should be emphasized that the Lande factor  $g_J$ , can be written as follows:

$$g_J = \left( \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)} \right). \quad (23)$$

### 3.1. Modified thermodynamical parameters

Now, we obtain the total energy, partition function, entropy and the magnetization of the paramagnetic system in the presence of a minimal length scale. For this purpose, let us write the modified Hamiltonian system as follows:

$$H_{\text{ML}} = - \sum_{n=1}^N (\vec{\mu}_n)_{\text{ML}} \cdot (\vec{b})_{\text{ML}} , \quad (24)$$

where  $(\vec{\mu}_n)_{\text{ML}} = -(g_J)_{\text{ML}} \mu_B (\vec{L}_n + \vec{S}_n)$ , and  $\vec{b}_{\text{ML}}$  is the modified magnetostatic field. Here, we consider the uniform magnetostatic field. From Eqs. (9) and (15), the modified Hamiltonian in Eq. (24) can be obtained as

$$\begin{aligned} H_{\text{ML}} &= (g_J)_{\text{ML}} \mu_B \sum_{n=1}^N \left[ (1 + \alpha \vec{p}^2) \vec{L}_n + \vec{S}_n \right] \cdot (1 + \alpha \vec{p}^2) \vec{B} \\ &= (g_J)_{\text{ML}} \mu_B \sum_{n=1}^N \left[ (1 + \alpha \vec{p}^2)^2 \vec{L} \cdot \vec{B} + (1 + \alpha \vec{p}^2) \vec{S} \cdot \vec{B} \right] . \end{aligned} \quad (25)$$

Assuming again a magnetostatic field  $\vec{B}$  in  $Z$ -direction, the energy eigenvalues in the presence of a minimal length is

$$\begin{aligned} \varepsilon_{\text{ML}} &= (g_J)_{\text{ML}} \mu_B B \sum_{n=1}^N [(1 + 2\alpha f(l)) L_z + (1 + \alpha f(l)) S_z] \\ &= (g_J)_{\text{ML}} \mu_B B \sum_{n=1}^N [(1 + \alpha f(l)) m_J + (\alpha f(l)) m_L] , \end{aligned} \quad (26)$$

where  $m_J, m_L$  are the quantum numbers and they vary from  $-j$  to  $+j$  and  $-l$  to  $l$ , respectively. In Eq. (26),  $f(l)$  is equal to  $\frac{\hbar^2 l(l+1)}{r_0^2}$  when we use the following definition for the operator  $\vec{p}^2$

$$\vec{p}^2 = -\hbar^2 \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\vec{L}^2}{\hbar^2 r^2} \right) . \quad (27)$$

It should be noted that in Eq. (26) the terms of the order of  $\alpha^2$  and higher are neglected. If we consider the simple case of atoms with spin  $\frac{1}{2}$ , we find that the orbital angular momentums are zero. According to Eq. (26), the modified energy levels are

$$\varepsilon_{\text{ML}} = g_S \mu_B B m_S . \quad (28)$$

It is interesting to note that the modified energy levels in Eq. (28) is equal to the usual energy levels. Therefore in this case, the modified thermodynamical parameters are not changed.

### 3.2. Atoms with arbitrary $J$ in the presence of a minimal length

Now, we want to calculate the modified thermodynamical characteristics of the paramagnetic systems when  $J$  is arbitrary. The partition function is obtained by summing the Boltzmann factor  $\exp(-\beta E_r)$  over all the quantum states  $|r\rangle$ , where  $\beta = \frac{1}{KT}$  and  $K$  is the Boltzmann factor and  $T$  denotes the temperature. Therefore, the partition function in the presence of a minimal length can be written as follows:

$$(Z)_{\text{ML}} = \sum_{m_J=-j}^j \exp(-\beta \epsilon_{\text{ML}}). \quad (29)$$

If we insert Eq. (26) into Eq. (29), we will obtain

$$\begin{aligned} (Z)_{\text{ML}}(T, B, N) &= [(Z)_{\text{ML}}(T, B, 1)]^N = \left[ \sum_{m_J=-j}^j \exp(y m_J) \sum_{m_L=-l}^l \exp(x m_L) \right]^N \\ &= \left[ \frac{\exp\left\{\left(j + \frac{1}{2}\right)\right\} - \exp\left\{-\left(j + \frac{1}{2}\right)\right\}}{\exp\left(\frac{y}{2}\right) - \exp\left(-\frac{y}{2}\right)} \frac{\exp\left\{\left(l + \frac{1}{2}\right)\right\} - \exp\left\{-\left(l + \frac{1}{2}\right)\right\}}{\exp\left(\frac{x}{2}\right) - \exp\left(-\frac{x}{2}\right)} \right]^N, \end{aligned} \quad (30)$$

where we have introduced the characteristic parameter  $y = \beta(g_J)_{\text{ML}}\mu_B(1 + \alpha f(l))$ , and  $x = \beta(g_J)_{\text{ML}}\mu_B\alpha f(l)$ . Using the hyperbolic sine, Eq. (30) can be rewritten as

$$(Z)_{\text{ML}}(T, B, N) = \left( \frac{\sinh\left(j + \frac{1}{2}\right) y \sinh\left(l + \frac{1}{2}\right) x}{\sinh\left(\frac{y}{2}\right) \sinh\left(\frac{x}{2}\right)} \right)^N. \quad (31)$$

The modified mean energy of the paramagnetic system can be found using the following equation

$$U_{\text{ML}} = -\frac{\partial}{\partial \beta} \ln(Z_{\text{ML}}). \quad (32)$$

Substituting Eq. (31) into Eq. (32), we have

$$U_{\text{ML}} = -N \left[ \frac{\partial \ln Z_J}{\partial y} \frac{\partial y}{\partial \beta} + \frac{\partial \ln Z_L}{\partial x} \frac{\partial x}{\partial \beta} \right], \quad (33)$$

where  $Z_J = \frac{\sinh(j+\frac{1}{2})y}{\sinh(\frac{y}{2})}$ , and  $Z_L = \frac{\sinh(l+\frac{1}{2})x}{\sinh(\frac{x}{2})}$ . After simplification, the modified mean energy in Eq. (33) leads to

$$U_{\text{ML}} = -N [(g_J)_{\text{ML}}\mu_B B j(1 + \alpha f(l)) B_J(y) + (g_J)_{\text{ML}}\mu_B B l(\alpha f(l)) B_L(x)], \quad (34)$$

where  $B_J(y)$  and  $B_L(x)$  are the Brillouin functions with the index  $J$  and  $L$  and they are defined as follows:

$$B_J(y) = \frac{1}{j} \left[ \left( j + \frac{1}{2} \right) \coth \left( j + \frac{1}{2} \right) y - \frac{1}{2} \coth \left( \frac{y}{2} \right) \right],$$

$$B_L(x) = \frac{1}{l} \left[ \left( l + \frac{1}{2} \right) \coth \left( l + \frac{1}{2} \right) x - \frac{1}{2} \coth \left( \frac{x}{2} \right) \right].$$

The first term  $U_0 = -N g_J \mu_B B j B_J(y)$  is the usual mean energy and the second term is its correction due to the considered minimal length effect. The mean energy shift can be obtained as

$$\frac{\Delta U}{U_0} = \alpha f(l) \left[ 1 + \frac{l B_L(x)}{j B_J(y)} \right]. \quad (35)$$

The modified entropy of the paramagnetic system from Eqs. (31) and (34) can be found as follows:

$$\begin{aligned} S_{\text{ML}} &= \frac{U_{\text{ML}}}{T} + K \ln(Z_{\text{ML}}) = -\frac{N}{T} [(g_J)_{\text{ML}} \mu_B B j (1 + \alpha f(l)) B_J(y) \\ &\quad + (g_J)_{\text{ML}} \mu_B B l (\alpha f(l)) B_L(x)] \\ &\quad + N K \ln \left( \sinh \left( j + \frac{1}{2} \right) y \sinh \left( l + \frac{1}{2} \right) x \right) \\ &\quad - N K \ln \left( \sinh \left( \frac{y}{2} \right) \sinh \left( \frac{x}{2} \right) \right). \end{aligned} \quad (36)$$

The magnetization in the presence of a minimal length is now obtained by the following partial differentiation

$$M_{\text{ML}} = \frac{1}{\beta} \frac{\partial \ln(Z_{\text{ML}})}{\partial B}. \quad (37)$$

If we substitute Eq. (31) into Eq. (37), we obtain the following result

$$\begin{aligned} M_{\text{ML}} &= N K T \frac{\partial}{\partial B} [\ln(Z_J) + \ln(Z_L)] \\ &= N \mu_B (g_J)_{\text{ML}} [j B_J(y) + \alpha f(l) (j B_J(y) + l B_L(x))]. \end{aligned} \quad (38)$$

Now, let us find the modified magnetization of the paramagnetic system first at high temperatures and second at low temperatures. For high temperatures,  $y \rightarrow 0$  and  $x \rightarrow 0$ ,  $B_J(y)$  and  $B_L(x)$  become linear functions as follows:

$$\begin{aligned} B_J(y) &\approx \frac{y}{3} (j + 1), \\ B_L(x) &\approx \frac{x}{3} (l + 1). \end{aligned} \quad (39)$$



Using Eqs. (38) and (39), the modified magnetization at high temperatures up to the first order of  $\alpha$  is given by

$$M_{\text{ML}} = N(g_J)_{\text{ML}}^2 \mu_{\text{B}}^2 j(j+1) B \frac{1 + \alpha f(l)}{3KT}. \quad (40)$$

Also, the modified magnetic susceptibility can be written as

$$\chi = N(g_J)_{\text{ML}}^2 \mu_{\text{B}}^2 j(j+1) \frac{1 + \alpha f(l)}{3KT}. \quad (41)$$

Here, the first term  $M_0 = N g_J^2 \mu_{\text{B}}^2 \frac{j(j+1)}{3KT}$  corresponds to the ordinary magnetization of the paramagnetic system. The second term is the minimal length effect. The relative modification of magnetization would be obtained as follows:

$$\frac{\Delta M}{M_0} \approx 2\alpha f(l). \quad (42)$$

For low temperatures,  $y \rightarrow \infty$  and  $x \rightarrow \infty$ ,  $B_J(y)$  and  $B_L(x)$  are equal to 1. Then, the modified magnetization at low temperatures is

$$M_{\text{ML}} = N(g_J)_{\text{ML}} \mu_{\text{B}} [j + \alpha f(l)(j+l)]. \quad (43)$$

It should be noted that for  $\alpha \rightarrow 0$ , the modified thermodynamical parameters in Eqs. (31), (34), (36) and (38) become the usual thermodynamical parameters of the paramagnetic system. Now, let us estimate the deformation parameter  $\alpha$  by using the modified magnetic susceptibility in Eq. (41). According to Eq. (27), we can find Eq. (41) in the following form

$$\chi = N(g_J)_{\text{ML}}^2 \mu_{\text{B}}^2 j(j+1) \frac{r_0^2 + \alpha \hbar^2 l(l+1)}{3KT r_0^2}, \quad (44)$$

where  $r_0$  is the Bohr radius and  $l$  is the orbital angular momentum quantum number. If we consider high temperature about  $T = 1000K$  and the experimental value of magnetic susceptibility (magnetic susceptibility of water is  $9.04 \times 10^{-6}$ ) and also  $l = 1$ , we can estimate the following upper bound for deformation parameter  $\alpha$

$$10^{-5} \approx \alpha \times 10^{-55}$$

or

$$\alpha \leq 10^{50}. \quad (45)$$

It should be noted that the upper bound in Eq. (45) is near to upper bound that is set by Landau levels [5] and also the upper bound on the deformation parameter in Eq. (45) is close to the value of upper bounds in Ref. [32].

#### 4. Conclusions

Today we know that every theory of quantum gravity predicts the existence of a minimal length scale which leads to a GUP. An immediate consequence of GUP is a generalization of position and momentum. Hence, the GUP influences the basis of quantum mechanics and deforms Hamiltonian of any system. In the recent years, many attempts have been made to compute the corrections of statistical mechanics in the presence of a minimal length [29–31]. In this study, we have investigated the statistical characteristics of the paramagnetic system in the framework of GUP. The modified angular momentum operator and the modified magnetostatic field in three spatial dimensions were obtained up to the first order over the deformation parameter  $\alpha$ . The modified thermodynamical quantities of the paramagnetic system such as the mean energy, entropy, magnetization have been achieved by using the modified energy eigenvalues. Our calculations were obtained for arbitrary angular momentum  $J$  and total angular momentum  $J = \frac{1}{2}$ . Also, we have found the magnetization of the paramagnetic system at both high- and low-temperatures limits. We have shown that at high temperatures, the relative magnetization depends on the deformation parameter and orbital angular momentum. The upper limit on the deformation parameter has been found. It is necessary to note that in the limit of  $\alpha \rightarrow 0$ , all of modified thermodynamical parameters become the usual thermodynamical parameters. The upper limit on the deformation parameter has been estimated using the experimental value of magnetic susceptibility.

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