ANALYSIS OF THE NEW TECHNIQUE TO SOLUTION OF FRACTIONAL WAVE- AND HEAT-LIKE EQUATION

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We have applied the new approach of homotopic perturbation method (NHPM) for wave- and heat-like equation featuring time-fractional derivative. A combination of NHPM and multiple fractional power series form has been used the first time to present analytical solution. In order to illustrate the simplicity and ability of the suggested approach, some specific and clear examples have been given. All computations were done using Mathematica.

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1. Introduction

In this research work, it has been proposed that the new HPM based on the multiple fractional power series can be employed to solve the wave- and heat-like equation featuring time-fractional derivative of the following form:

$$\begin{gathered} D_t^{\mu} v(x,y,z,t) = P(x,y,z)v_{xx} + Q(x,y,z)v_{yy} + L(x,y,z)v_{zz} + M(x,y,z)\,, \\ 0<x<a\,, \qquad 0<y<b\,, \qquad 0<z<c\,, \qquad 0<\mu\leq 2\,, \qquad t>0\,, \end{gathered} \tag{1.1}$$

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where

$$\begin{aligned} v(0,y,z,t) &= f_1(y,z,t)\,, & v_x(0,y,z,t) &= f_2(y,z,t)\,,\\ v(0,y,z,t) &= g_1(y,z,t)\,, & v_x(0,y,z,t) &= g_2(y,z,t)\,,\\ v(0,y,z,t) &= h_1(y,z,t)\,, & v_x(0,y,z,t) &= h_2(y,z,t) \end{aligned}$$

are boundary statues with under primary statues

$$v(x,y,z,0) = \varphi(x,y,z)\,, \qquad v_t(x,y,z,0) = \psi(x,y,z)$$

in which μ is the order of fractional derivative. The function $v(x,y,z,t)$ describes temperature as a function of time and space, while v_{xx}, v_{yy} and v_{zz} are the second spatial derivatives (thermal conditions) of temperature as a function within x, y, and z. In which $P(x,y,z)$, $Q(x,y,z)$, $L(x,y,z)$ and $M(x,y,z)$ are any functions within x, y and z.

In the case when

— $0 < \mu \le 1$, equation (1.1) refers to the fractional heat-like equation featuring variable coefficients,

— $1 < \mu \le 2$, equation (1.1) refers to the fractional wave-like equation which patterns anomalous diffusive and sub-diffusive systems, alliance of diffusion and wave propagation phenomena, description of fractional random walk [1, 2].

There are some more books related to fractional calculus for interested readers [3–5]. It should be noted that there are no accurate analytical solutions for most fractional differential equations. Consequently, for such equations, we have to employ some direct and iterative methods. Researchers have used various methods to solve differential equations featuring fractional derivative (FDEs) and partial differential equations featuring fractional derivative (FPDEs) in recent years. Some familiar methods are as follows: decomposition and Adomian's decomposition method [6, 7], variational iteration method [8, 9], homotopic perturbation method [10, 11], homotopic analysis method [12, 13] and so on [2, 14–19].

The arrangement of this work is as follows: we have presented some basic idea of the optimum q-homotopic analysis method in Section 2. In Section 3, the convergence of the suggested method is explained. Following that, in Section 4, the application of NHPM to the wave- and heat-like equation featuring time-fractional derivative is illustrated, and some numerical examples are presented. Finally, in Section 5, some conclusions regarding the proposed method are drawn.

2. Description of the new approach

To describe the NHPM for nonlinear time fractional PDE, we use

$$D_t^{\mu} v(\zeta,\,\tau) + N[v(\zeta,\,\tau)] = h(\zeta,\,\tau) \tag{2.1}$$

along with

$$v^{(i)}(\zeta,\,\tau_0)\big|_{\tau=\tau_0} = g_i(\zeta)\,, \qquad i = 0, 1, \cdots, n-1\,, \tag{2.2}$$

in which $\zeta = (\zeta_1, ..., \zeta_n)$, $n-1 < \mu \leq n$, N are linear and nonlinear operators, ζ and τ signify the independent variables, $v(\zeta,\,\tau)$ is a indeterminate function, D^{μ} denotes the Caputo fractional and $h(\zeta,\,\tau)$ is inhomogeneous term.

To get the solution of (2.1), by using this approach, we create the below homotopic:

$$(1-q)\left(D_\tau^{\mu} V(\zeta,\,\tau) - v_0(\zeta,\,\tau)\right) + q\left(D_\tau^{\mu} V(\zeta,\,\tau) + N(V(\zeta,\,\tau)) - h(\zeta,\,\tau)\right) = 0\,, \tag{2.3}$$

or

$$D_\tau^{\mu} V(\zeta,\,\tau) = v_0(\zeta,\,\tau) - q\left(v_0(\zeta,\,\tau) + N(V(\zeta,\,\tau)) - h(\zeta,\,\tau)\right)\,. \tag{2.4}$$

Now, by operating $L^{-1} = I_\tau^{\mu}(.)$, we get

$$V(\zeta,\,\tau) = V(\zeta,\,\tau_0) + I_\tau^{\mu} v_0(\zeta,\,\tau) - q\, I_\tau^{\mu}\left(v_0(\zeta,\,\tau) + N(V(\zeta,\,\tau)) - h(\zeta,\,\tau)\right)\,, \tag{2.5}$$

where

$$V(\zeta,\,\tau_0) = \sum_{i=0}^{n-1} g_i\left(\zeta\right) \frac{\tau^i}{i!}\,. \tag{2.6}$$

Let us present the solution of Eq. (2.5) as

$$V(\zeta,\,\tau) = V_0(\zeta,\,\tau) + q\,V_1(\zeta,\,\tau) + q^2\,V_2(\zeta,\,\tau) + \cdots\,, \tag{2.7}$$

where $V_j(\zeta,\,\tau)$, $j = 0, 1, 2, 3, \ldots$, are functions which should be specified.

Definition 2.1. A power series expansion

$$\sum_{j=0}^{\infty} c_j\left(\tau - \tau_0\right)^{j\mu} = c_0 + c_1(\tau-\tau_0)^{\mu} + c_2(\tau-\tau_0)^{2\mu} + \cdots\,,$$
$$0 \leq n-1 < \mu \leq n\,, \qquad \tau_0 \leq \tau$$

is named fractional power series PS around $\tau = \tau_0$.

Definition 2.2. A power series expansion of the form of

$$\sum_{j=0}^{\infty} f_j(\zeta)\,(\tau-\tau_0)^{j\mu}\,, \qquad 0\le n-1<\mu\le n,\ \tau_0\le\tau$$

is named multiple fractional power series PS around $\tau=\tau_0$.

Assume that the primary estimate of the solution of (2.1) can be represented as

$$v_0(\zeta,\,\tau)=\sum_{j=0}^{\infty} c_j(\zeta)\,q_j(\tau)\,, \tag{2.8}$$

where $c_j(\zeta)$, $j=0,1,2,3,\ldots,$ are indeterminate coefficients, and $q_j(\tau)$, $j=0,1,2,3,\ldots$ are specific functions.

It is worth mentioning that if $h(\zeta,\,\tau)$, and $v_0(\zeta,\,\tau)$ are analytic about $\tau=0$, their next Taylor series can be written as

$$v_0(\zeta,\,\tau)=\sum_{j=0}^{\infty} c_j(\zeta)\,t^{j\,\mu}\,. \tag{2.9}$$

Now, with (2.5) featuring (2.7) and (2.8), and equating the coefficients of p, featuring the same power, we get

$$\begin{aligned}
q^0:\ V_0(\zeta,\tau) &= V(\zeta,\,\tau_0)+\sum_{n=0}^{\infty} c_n(\zeta)\,I_\tau^\mu\,(q_n(\tau))\,,\\
q^1:\ V_1(\zeta,\tau) &= -\sum_{n=0}^{\infty} c_n(\zeta)\,I_\tau^\mu\,(q_n(\tau))-I_\tau^\mu\,(N\,(V_0(\zeta,\,\tau)-h(\zeta,\,\tau)))\,,\\
q^2:\ V_2(\zeta,\tau) &= -I_\tau^\mu\,(N\,(V_0(\zeta,\,\tau),V_1(\zeta,\,\tau)))\,,\\
&\cdots\\
p^k:\ V_j(\zeta,\tau) &= -I_\tau^\mu\,(N\,(V_0(\zeta,\,\tau),V_1(\zeta,\,\tau)\ldots,V_{j-1}(\zeta,\,\tau)))\,.
\end{aligned} \tag{2.10}$$

By solving these equations so that $V_1(\zeta,\,\tau)=0$, then Eqs. (2.10) defer to $V_k(\zeta,\tau)=0$, $k=2,3,\cdots$. Accordingly, we will obtain the accurate solution

$$v(\zeta,\,\tau)=V_0(\zeta,\,\tau)=V(\zeta,\,\tau_0)+\sum_{k=0}^{\infty} c_k(\zeta)\,\left(I_\tau^\mu q_k(\tau)\right)\,. \tag{2.11}$$

3. Convergence analysis

A large number of problems can be treated by NHPM through applying the methodology that has been elaborated in previous sections.

Lemma 3.1. *Suppose H_ns are He's polynomials [20]. Then, for the nonlinear term N, the following relation is satisfied:*

$$N\left(\sum_{i=0}^{\infty} V_i\, q^i\right) = \sum_{i=0}^{\infty} H_i\, q^i.$$

Lemma 3.2. *From Lemma 3.1, relation (2.10) is equivalent to the following formula:*

$$\begin{aligned}
V_0(\zeta,\tau) &= V(\zeta,\tau_0) + V_0^{\star}\,,\\
V_1(\zeta,t) &= -I^{\mu} v_0 - I^{\mu}\left(H_0 - h(\zeta,r)\right)\,,\\
&\cdots\\
V_{n+1}(\zeta,\tau) &= -I^{\mu}\left(H_n\right)\,.
\end{aligned}$$

Theorem 3.3. *NHPM that was used in the solution of (2.10) is equivalent to the sequence that comes next*

$$z_n = V_1 + V_2 + \cdots + V_n\,, \qquad z_0 = 0\,,$$

by using the iterative formula

$$z_{n+1} = -I^{\mu} N_n\left(z_n + V_0\right) - V_0^{\star} + I^{\mu}\left(h(\zeta,r)\right)\,, \tag{3.1}$$

where $V_0^{\star} = I^{\mu} v_0$, and

$$N_n\left(\sum_{i=0}^{n} V_i\right) = \sum_{i=0}^{n} H_i,\;\; n = 0,1,2,\cdots\,. \tag{3.2}$$

Proof. For $n = 0$ from (3.1), we have

$$z_1 = -I^{\mu} N_0\left(z_0 + V_0\right) - V_0^{\star} + I^{\mu}\left(h(\zeta,r)\right) = -I^{\mu}(H_0) - V_0^{\star} + I^{\mu}\left(h(\zeta,r)\right)\,.$$

Then,

$$V_1 = -I^{\mu}(H_0) - V_0^{\star} + I^{\mu}\left(h(\zeta,r)\right)\,.$$

For $n = 1$,

$$\begin{aligned}
z_2 &= -I^{\mu} N_1\left(z_1 + V_0\right) - V_0^{\star} + I^{\mu}\left(h(\zeta,r)\right)\\
&= -I^{\mu}(H_0 + H_1) - V_0^{\star} + I^{\mu}\left(h(\zeta,r)\right)\\
&= -I^{\mu}(H_1) + V_1\,.
\end{aligned}$$

According to $z_2 = V_1 + V_2$, we get $V_2 = -I^{\mu}(H_1)$.

This theorem will be proved by a convincing induction. Let us assume that $V_{k+1} = -I^{\mu}(H_k)$ for $k = 1, 2, \cdots, n-1$, hence

$$\begin{aligned} z_{n+1} &= -I^{\mu} N_n \left(z_n + V_0\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) \\ &= -I^{\mu}\left(\sum_{i=0}^{n} H_i\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) \\ &= -\sum_{i=0}^{n} I^{\mu}(H_i) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) \\ &= V_1 + V_2 + \cdots + V_n - I^{\mu}(H_n)\,. \end{aligned}$$

So, $V_{n+1} = -I^{\mu}(H_n)$. □

Theorem 3.4. *Suppose that B is a Banach space.*

(i) $\sum_{k=0}^{\infty} V_k$ *gained convergence to* $s \in B$, *if* $\exists \lambda$ $(0 \leq \lambda < 1)$, *subject to* $\forall n \in \mathbb{N}$, $\|V_n\| \leq \lambda \|V_{n-1}\|$,

(ii) $z = \sum_{k=1}^{\infty} V_k$ *is satisfied in*

$$z = -I^{\mu} N\left(s + V_0\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right)\,.$$

Proof. *(i)* The result will be

$$\|z_{n+1} - z_n\| = \|V_{n+1}\| \leq \lambda \|V_n\| \leq \lambda^2 \|V_{n-1}\| \leq \cdots \leq \lambda^{n+1} \|V_0\|\,.$$

For any $n, m \in \mathbb{N}$, $n \geq m$, we derive

$$\begin{aligned} \|z_n - z_m\| &= \|(z_n - z_{n-1}) + (z_{n-1} - z_{n-2}) + \cdots + (z_{m+1} - z_m)\| \\ &\leq \|z_n - z_{n-1}\| + \|z_{n-1} - z_{n-2}\| + \cdots + \|z_{m+1} - z_m\| \\ &\leq \lambda^n \|V_0\| + \lambda^{n-1} \|V_0\| + \cdots + \lambda^{m+1} \|V_0\| \\ &\leq (\lambda^n + \lambda^{n-1} + \cdots + \lambda^{m+1}) \|V_0\| \\ &\leq \lambda^{m+1}(1 + \lambda + \cdots + \lambda^n) \|V_0\| \\ &\leq \frac{\lambda^{m+1}}{1-\lambda} \|V_0\|\,. \end{aligned}$$

Therefore, $\lim\limits_{n,m\to\infty} \|z_n - z_m\| = 0$. Then, z_n is a Cauchy sequence in the Banach space and so, it is convergence, *i.e.* $\exists\ z \in B$ subject to $\lim\limits_{n\to\infty} z_n = \sum_{n=1}^{\infty} V_n = z$.

(ii) From (3.1), yields to

$$\begin{aligned}
\lim_{n\to\infty} z_{n+1} &= -I^{\mu} \lim_{n\to\infty} N_n \left(z_n + V_0\right) + I^{\mu}\left(h(\zeta, r)\right) \\
&= -I^{\mu} \lim_{n\to\infty} N_n \left(\sum_{k=0}^{n} V_k\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) .
\end{aligned}$$

Thus,

$$\begin{aligned}
z &= -I^{\mu} \lim_{n\to\infty} N_n \left(\sum_{k=0}^{n} H_k\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) \\
&= -I^{\mu} \lim_{n\to\infty} N_n \left(\sum_{k=0}^{\infty} H_k\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) .
\end{aligned}$$

By Eq. (3.2) and $N\left(\sum\limits_{k=0}^{\infty} V_k q^k\right) = \sum\limits_{k=0}^{\infty} H_k q^k$ for $q = 1$, we derive $\sum\limits_{k=0}^{\infty} H_k = N\left(\sum\limits_{k=0}^{\infty} V_k\right)$.

Hence,

$$\begin{aligned}
z &= -I^{\mu} N \left(\sum_{k=0}^{\infty} V_k\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) \\
&= -I^{\mu} N \left(s + V_0\right) - V_0^{\star} + I^{\mu}\left(h(\zeta, r)\right) .
\end{aligned}$$

□

4. Test examples

Now, we apply NHPM based on the multiple fractional power series to solve heat- and wave-like equation featuring time fractional derivative. All computations were done using Mathematica. In this section, the following symbols will be defined:

$$c^{(i,0,0)} = \frac{\partial^i}{\partial x^i} c \,, \qquad c^{(0,j,0)} = \frac{\partial^j}{\partial y^j} c \,, \qquad c^{(0,0,k)} = \frac{\partial^k}{\partial z^k} c \,.$$

Test example 4.1. We purpose the $2D$ fractional heat-like equation [8]:

$$\begin{aligned}
D_t^{\mu} v(x,t) &= \tfrac{1}{2}{}_{xx}(x,t)\,, \qquad 0 < x < 1\,, \qquad 0 < \mu \le 1\,, \qquad t > 0\,, && (4.1)\\
v(0,t) &= 0\,, \qquad v(1,t) = E_{\mu}(t^{\mu})\,, \qquad v(x,0) = x^2\,. && (4.2)
\end{aligned}$$

Assume $v_0(x,\ t) = \sum_{n=0}^{m} c_n(x)t^{n\mu}$, $V(x,0) = g(x)$. Solving Eqs. (4.1)–(4.2) for $V_1(x,t)$, one will obtain as follows:

$$\begin{aligned} V_1(x,t) = {} & t^{\mu}\left(\frac{x^2}{\mu\Gamma(\mu)} - \frac{c_0(x)}{\mu\Gamma(\mu)}\right) + t^{2\mu}\left(\frac{\sqrt{\pi}2^{-2\mu-1}x^2c_0''(x)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)} - \frac{\sqrt{\pi}2^{-2\mu}c_1(x)}{\Gamma\left(\mu+\frac{1}{2}\right)}\right) \\ & + t^{3\mu}\left(\frac{x^2\Gamma(\mu+1)c_1''(x)}{2\Gamma(3\mu+1)} - \frac{c_2(x)\Gamma(2\mu+1)}{\Gamma(3\mu+1)}\right) \\ & + \frac{x^2\Gamma(2\mu+1)t^{4\mu}c_2''(x)}{2\Gamma(4\mu+1)} + \cdots . \end{aligned} \tag{4.3}$$

By vanishing of $V_1(x,t)$, the coefficients $c_n(x,y)$, $n = 1,2,3,\ldots$, lead to the following result:

$$c_0(x) = x^2\,, \qquad c_1(x) = \frac{x^2}{\Gamma(\mu+1)}\,, \qquad c_2(x) = \frac{x^2}{\Gamma(2\mu+1)} \cdots .$$

This implies that

$$\begin{aligned} v(x,t) & = V_0(x,t) \\ & = x^2 + \frac{x^2t^{\mu}}{\Gamma(\mu+1)} + \frac{x^2t^{2\mu}}{\Gamma(2\mu+1)} + \frac{x^2t^{3\mu}}{\Gamma(3\mu+1)} + \cdots = x^2E_{\mu}(t^{\mu})\,. \end{aligned} \tag{4.4}$$

We can see the estimate solutions featuring $\mu = 1$ and $y = 1$ in figure 1. The estimate solutions featuring $\mu = 1$ acquired for several amounts of x, y and t applying NHPM can be seen in Table I.

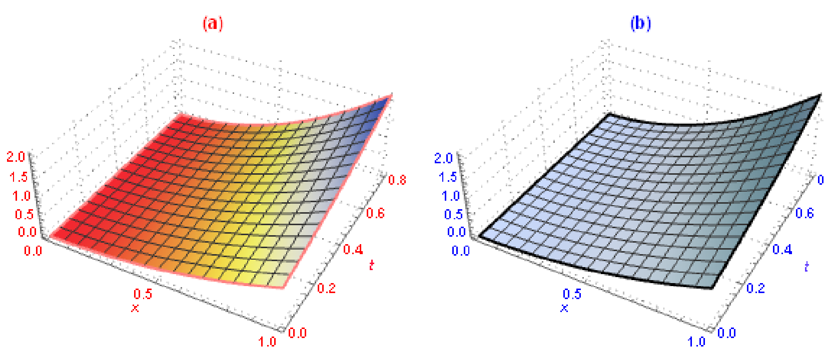

Fig. 1. (a) The analytical solution of Eqs. (4.1)–(4.2) for various x and t, when $\mu = 1$ and $y = 1$. (b) The accurate solution.

TABLE I

Approximate result of test example 4.1.

t	x	v_{NHPM}	v_{Exact}	Absolute error
0.1	0.50	0.276292	0.276293	1.06285×10^{-6}
	0.75	0.380378	0.380408	0.0000297766
	1.00	1.10517	1.10517	4.25141×10^{-6}
0.3	0.50	0.337375	0.337465	0.0000897019
	0.75	0.759094	0.759296	0.000201829
	1.00	1.3495	1.34986	0.000358808
0.5	0.50	0.411458	0.41218	0.000721984
	0.75	0.925781	0.927406	0.00162446
	1.00	1.64583	1.64872	0.00288794

Test example 4.2. We purpose the $2D$ fractional heat-like equation [8]:

$$D_t^{\mu} v(x,y,t) = v_{xx}(x,y,t) + v_{yy}(x,y,t), \qquad 0 < x,y < 2\pi,\ \ 0 < \mu \le 1,\ \ t > 0, \tag{4.5}$$

subject to the boundary and primary conditions

$$\begin{aligned} v(0,y,t) &= 0\,, \qquad v(2\pi,y,t) = 0\,, \qquad v(x,0,t) = 0\,, \qquad v(x,2\pi,t) = 0\,, \\ v(x,y,0) &= \sin(x)\sin(y)\,. \end{aligned} \tag{4.6}$$

Assume $v_0(x,\ y,\ t) = \sum\limits_{n=0}^{m} c_n(x,y) t^{n\mu}$, $V(x,y,0) = g(x,y)$. Solving Eqs. (4.5)–(4.6) for $V_1(x,y,t)$ leads to the following result:

$$\begin{aligned} V_1(x,y,t) &= t^{\mu}\left(-\frac{c_0(x,y)}{\mu\Gamma(\mu)} - \frac{2\sin(x)\sin(y)}{\mu\Gamma(\mu)}\right) \\ &+ t^{2\mu}\left(-\frac{\sqrt{\pi}4^{-\mu}c_1(x,y)}{\Gamma\left(\mu+\frac{1}{2}\right)} + \frac{\sqrt{\pi}4^{-\mu}{c_0}^{(0,2)}(x,y)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)} + \frac{\sqrt{\pi}4^{-\mu}{c_0}^{(2,0)}(x,y)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}\right) \\ &+ t^{3\mu}\left(-\frac{\Gamma(2\mu+1)c_2(x,y)}{\Gamma(3\mu+1)} + \frac{\Gamma(\mu+1){c_1}^{(0,2)}(x,y)}{\Gamma(3\mu+1)} + \frac{{c_1}^{(2,0)}(x,y)}{\Gamma(3\mu+1)}\right) \\ &+ t^{4\mu}\left(\frac{\Gamma(2\mu+1){c_2}^{(0,2)}(x,y)}{\Gamma(4\mu+1)} + \frac{\Gamma(2\mu+1){c_2}^{(2,0)}(x,y)}{\Gamma(4\mu+1)}\right) + \cdots\,. \end{aligned} \tag{4.7}$$

By vanishing of $V_1(x,t)$, the coefficients $c_n(x,y)$, $n = 1, 2, 3, \ldots$, will be obtained as follows:

$$\begin{aligned} c_0(x,y) &= -2\sin(x)\sin(y)\,, \\ c_1(x,y) &= \frac{4\sin(x)\sin(y)}{\Gamma(\mu+1)}\,, \\ c_2(x,y) &= -\frac{8\sin(x)\sin(y)}{\Gamma(2\mu+1)}\cdots\,. \end{aligned}$$

This implies that

$$\begin{aligned} v(x,y,t) &= V_0(x,y,t) \\ &= \sin(x)\sin(y)\left(-\frac{2t^\mu}{\Gamma(\mu+1)}+4t^{2\mu}\left(\frac{1}{\Gamma(2\mu+1)}-\frac{2t^\mu}{\Gamma(3\mu+1)}\right)+1\right) \\ &= E_\mu(-2t^\mu)\sin(x)\sin(y)\,. \end{aligned} \tag{4.8}$$

In figure 2, we can see analytical solutions featuring $\mu = 1$ and $y = 1$ what is concluded for various amounts of x, y and t utilizing NHPM. The analytical solutions featuring $\mu = 1$ acquired for various amounts of x, y and t applying NHPM are shown in Table II.

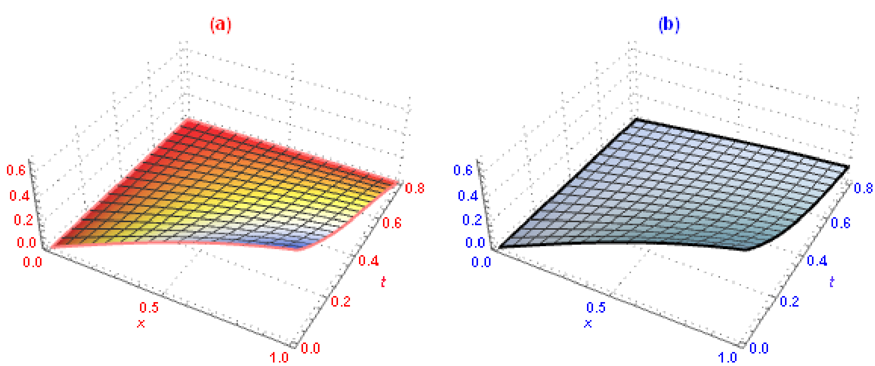

Fig. 2. (a) The analytical solution of Eqs. (4.5)–(4.6) for various x and t, when $\mu = 1$ and $y = 1$. (b) The accurate solution.

TABLE II

Approximate result of test example 4.2.

t	x	y	v_{NHPM}	v_{Exact}	Absolute error
0.1	0.50	0.50	0.188170	0.188184	0.0000147302
	0.75	0.75	0.380378	0.380408	0.0000297766
	1.00	1.00	0.579676	0.579721	0.0000453779
0.3	0.50	0.50	0.125038	0.126144	0.0011059500
	0.75	0.75	0.252759	0.254995	0.0022356400
	1.00	1.00	0.385192	0.388599	0.0034069900
0.5	0.50	0.50	0.0766163	0.0845567	0.0079403800
	0.75	0.75	0.1548770	0.1709280	0.0160512000
	1.00	1.00	0.2360240	0.2604860	0.0244612000

Test example 4.3. Next, consider the $2D$ wave-like equation featuring fractional derivative

$$D_t^{\mu} v(x,y,t) = \frac{1}{2}\left(x^2 v_{xx}(x,y,t) + y^2 v_{yy}(x,y,t)\right) , \quad 0 < x,y \leq 1 , \qquad 1 < \mu \leq 2 , \qquad t > 0 , \tag{4.9}$$

subject to

$$\begin{aligned} v(0,y,t) &= 0 , & v(1,y,t) &= y^2 E_{\mu}(2t^{\mu}) , \\ v(x,0,t) &= 0 , & v(x,1,t) &= x^2 E_{\mu}(2t^{\mu}) , \\ v(x,y,0) &= x^2 y^2 , & v_t(x,y,0) &= 0 . \end{aligned} \tag{4.10}$$

With considering $V_1(x,y,t)$, we obtain

$$\begin{aligned} V_1(x,y,t) = {} & t^{\mu}\left(\frac{2x^2y^2}{\mu\Gamma(\mu)} - \frac{c_0(x,y)}{\mu\Gamma(\mu)}\right) \\ & + t^{2\mu}\left(\frac{x^2 {c_0}^{(2,0)}(x,y)}{2\Gamma(2\mu+1)} + \frac{y^2 {c_0}^{(0,2)}(x,y)}{2\Gamma(2\mu+1)} - \frac{\sqrt{\pi}4^{-\mu}c_1(x,y)}{\Gamma\left(\mu+\frac{1}{2}\right)}\right) \\ & + t^{3\mu}\left(\frac{x^2\Gamma(\mu+1){c_1}^{(2,0)}(x,y)}{2\Gamma(3\mu+1)} + \frac{y^2\Gamma(\mu+1){c_1}^{(0,2)}(x,y)}{2\Gamma(3\mu+1)} - \frac{c_2(x,y)}{\Gamma(3\mu+1)}\right) \\ & + t^{4\mu}\left(\frac{x^2\Gamma(2\mu+1){c_2}^{(2,0)}(x,y)}{2\Gamma(4\mu+1)} + \frac{y^2\Gamma(2\mu+1){c_2}^{(0,2)}(x,y)}{2\Gamma(4\mu+1)}\right) + \cdots . \end{aligned} \tag{4.11}$$

Considering the hypothesis $V_1(x,y,t)=0$, coefficients $c_n(x,y)$, $n=1,2,3,\ldots$ will be determined as follows:

$$c_0(x,y) = 2x^2y^2 , \quad c_1(x,y) = \frac{4x^2y^2}{\Gamma(\mu+1)} , \quad c_2(x,y) = \frac{8x^2y^2}{\Gamma(2\mu+1)} , \cdots .$$

Therefore, the solution of (4.9) is obtained

$$\begin{aligned} v(x,t) &= x^2y^2 + \frac{2x^2y^2t^\mu}{\Gamma(\mu+1)} + \frac{\sqrt{\pi}4^{1-\mu}x^2y^2t^{2\mu}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\mu+1)} + \frac{8x^2y^2t^{3\mu}}{\Gamma(3\mu+1)} + \dots \\ &= x^2y^2E_\mu(2t^\mu)\,. \end{aligned}$$

In figure 3, we can see the precise solutions featuring $y = 1$ and $\mu = 2$. In Table III, we may view the analytical solutions featuring $\mu = 2$, which is derived for several amounts of x, y and t utilizing NHPM.

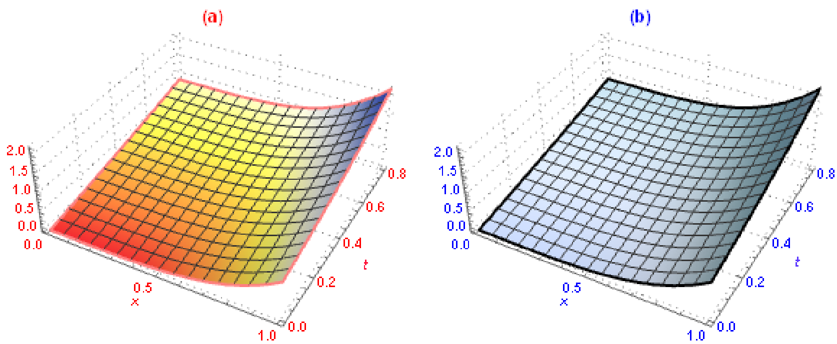

Fig. 3. (a) The analytical solution of Eqs. (4.9)–(4.10) for various x and t, when $\mu = 2$ and $y = 1$. (b) The accurate solution.

TABLE III

Approximate result of test example 4.3.

t	x	z	v_{NHPM}	v_{Exact}	Absolute error
0.1	0.50	0.50	0.0690638	0.0690732	9.37986×10^{-6}
	0.75	0.75	0.3496360	0.3496830	0.0000474856
	1.00	1.00	1.1050200	1.1051700	0.0001500780
0.3	0.50	0.50	0.0841683	0.0843662	0.00019789
	0.75	0.75	0.426102	0.4271040	0.00100182
	1.00	1.00	1.346690	1.3498600	0.00316625
0.5	0.50	0.50	0.102383	0.103045	0.000661996
	0.75	0.75	0.518314	0.521666	0.003351350
	1.00	1.00	1.638130	1.648720	0.010591900

Test example 4.4. In this example, we choose the $3D$ heat-like equation featuring fractional derivative

$$\begin{aligned} D_t^\mu v(x,y,z,t) = x^4y^4z^4 + \tfrac{1}{36}\left(x^2 v_{xx}(x,y,z,t)\right. &+ y^2 v_{yy}(x,y,z,t) \\ &+ \left. z^2 v_{zz}(x,y,z,t)\right) , \\ 0 < x,y,z \leq 1\,, \qquad 0<\mu\leq 1\,, \qquad t>0\,, \end{aligned} \tag{4.12}$$

subject to

$$\begin{aligned} &v(0,y,z,t)=0\,, \qquad v(1,y,z,t)=y^4z^4(E_\mu(t^\mu)-1)\,, \\ &v(x,0,z,t)=0\,, \qquad v(x,1,z,t)=x^4z^4(E_\mu(t^\mu)-1)\,, \\ &v(x,y,0,t)=0\,, \qquad v(x,y,1,t)=x^4y^4(E_\mu(t^\mu)-1)\,, \\ &v(x,y,z,0)=0\,. \end{aligned} \tag{4.13}$$

Assume $v_0(x,y,z,t) = \sum\limits_{n=0}^{m} c_n(x,y,z)t^{n\mu}$, $V(x,y,z,0)=g(x,y,z)$. Considering $V_1(x,y,z,t)$, the equation gives

$$\begin{aligned} V_1(x,y,z,t) &= t^\mu\left(\frac{x^4y^4z^4}{\mu\Gamma(\mu)} - \frac{c_0(x,y,z)}{\mu\Gamma(\mu)}\right) + t^{2\mu}\left(-\frac{\sqrt{\pi}4^{-\mu}c_1(x,y,z)}{\Gamma\left(\mu+\frac{1}{2}\right)}\right. \\ &+\frac{\sqrt{\pi}4^{-\mu-1}z^2{c_0}^{(0,0,2)}(x,y,z)}{9\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)} + \frac{\sqrt{\pi}4^{-\mu-1}y^2{c_0}^{(0,2,0)}(x,y,z)}{9\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)} \\ &+\left.\frac{\sqrt{\pi}4^{-\mu-1}x^2{c_0}^{(2,0,0)}(x,y,z)}{9\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}\right) + t^{3\mu}\left(-\frac{\Gamma(2\mu+1)c_2(x,y,z)}{\Gamma(3\mu+1)}\right. \\ &+\frac{z^2\Gamma(\mu+1){c_1}^{(0,0,2)}(x,y,z)}{36\Gamma(3\mu+1)} + \frac{y^2\Gamma(\mu+1){c_1}^{(0,2,0)}(x,y,z)}{36\Gamma(3\mu+1)} \\ &+\left.\frac{x^2\Gamma(\mu+1){c_1}^{(2,0,0)}(x,y,z)}{36\Gamma(3\mu+1)}\right) + t^{4\mu}\left(\frac{z^2\Gamma(2\mu+1){c_2}^{(0,0,2)}(x,y,z)}{36\Gamma(4\mu+1)}\right. \\ &+\left.\frac{x^2\Gamma(2\mu+1){c_2}^{(2,0,0)}(x,y,z)}{36\Gamma(4\mu+1)} + \frac{y^2\Gamma(2\mu+1){c_2}^{(0,2,0)}(x,y,z)}{36\Gamma(4\mu+1)}\right)\,, \end{aligned} \tag{4.14}$$

so

$$c_0(x,y,z) = x^4y^4z^4,\ c_1(x,y,z) = \frac{x^4y^4z^4}{\Gamma(\mu+1)},\ c_2(x,y,z) = \frac{x^4y^4z^4}{\Gamma(2\mu+1)}\cdots\,.$$

Therefore, we obtain analytical solution of Eq. (4.12)

$$\begin{aligned} v(x,y,z,t) &= \frac{x^4y^4z^4t^\mu}{\mu\Gamma(\mu)} + \frac{\sqrt{\pi}4^{-\mu}x^4y^4z^4t^{2\mu}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\mu+1)} + \frac{x^4y^4z^4t^{3\mu}}{\Gamma(3\mu+1)} + \ldots \\ &= x^4y^4z^4(E_\mu(t^\mu)-1)\,. \end{aligned} \tag{4.15}$$

We can see the precise solutions featuring $\mu = 1$, $y = 1$ and $z = 1$, in figure 4. The analytical solutions featuring $\mu = 1$ acquired for several amounts of x, y, z and t applying NHPM can be seen in Table IV.

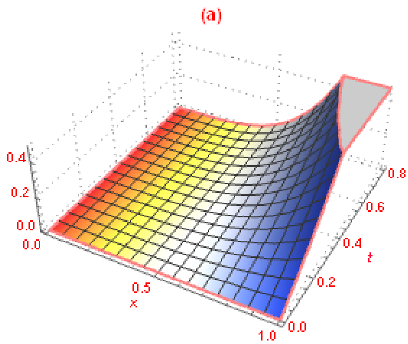

Fig. 4. (a) The analytical solution of Eqs. (4.12)–(4.13) for various x and t, when $\mu = 1$, $y = 1$ and $z = 1$. (b) The accurate solution.

TABLE IV

Approximate result of test example 4.4.

t	x	y	z	v_{NHPM}	v_{Exact}	Absolute error
0.1	0.50	0.50	0.50	0.0000256755	0.0000256765	1.03794×10^{-9}
	0.75	0.75	0.75	0.0033313000	0.0033314300	1.34669×10^{-7}
	1.00	1.00	1.00	0.1051670000	0.1051710000	4.25141×10^{-6}
0.3	0.50	0.50	0.50	0.0000853271	0.0000854147	8.75995×10^{-8}
	0.75	0.75	0.75	0.01107092	0.0110823	0.0000113657
	1.00	1.00	1.00	0.34950000	0.3498590	0.000358808
0.5	0.50	0.50	0.50	0.000157674	0.000158379	7.05063×10^{-7}
	0.75	0.75	0.75	0.020457600	0.020549100	0.00009147930
	1.00	1.00	1.00	0.645833000	0.648721000	0.00288794000

Test example 4.5. We choose the $3D$ heat-like equation featuring fractional derivative

$$D_t^{\mu} v(x,y,z,t) = x^2y^2z^2 + \frac{1}{2}\left(x^2 v_{xx}(x,y,z,t) \;+\; y^2 v_{yy}(x,y,z,t) \right.$$
$$\left. +\; z^2 v_{zz}(x,y,z,t)\right),$$
$$0 < x,y,z \le 1\,, \qquad 0 < \mu \le 1\,, \qquad t > 0\,, \tag{4.16}$$

subject to

$$
\begin{aligned}
&v(0,y,z,t) = \left(y^2+z^2\right) E_\mu(t^\mu)\,,\\
&v(x,0,z,t) = \left(x^2+z^2\right) E_\mu(t^\mu)\,,\\
&v(x,y,0,t) = \left(x^2+y^2\right) E_\mu(t^\mu)\,,\\
&v(1,y,z,t) = \left(1+y^2+z^2\right) E_\mu(t^\mu) + \tfrac{1}{3}\left(y^2z^2\right)\,,\\
&v(x,1,z,t) = \left(x^2+1+z^2\right) E_\mu(t^\mu) + \tfrac{1}{3}\left(x^2z^2\right)\,,\\
&v(x,y,1,t) = \left(x^2+y^2+1\right) E_\mu(t^\mu) + \tfrac{1}{3}\left(x^2y^2\right)\,,\\
&v(x,y,z,0) = 0\,.
\end{aligned}
\tag{4.17}
$$

Assume $v_0(x,y,z,t) = \sum_{n=0}^{m} c_n(x,y,z)t^{n\mu}$, $V(x,y,z,0) = g(x,y,z)$. Solving Eq. (4.16) for $V_1(x,y,z,t)$, one obtains the following result:

$$
\begin{aligned}
V_1(x,y,z,t) &= t^\mu\left(-\frac{c_0(x,y,z)}{\mu\Gamma(\mu)} + \frac{x^2}{\mu\Gamma(\mu)} + \frac{y^2}{\mu\Gamma(\mu)} + \frac{z^2}{\mu\Gamma(\mu)}\right)\\
&+t^{2\mu}\left(-\frac{\sqrt{\pi}2^{-2\mu}c_1(x,y,z)}{\Gamma\left(\mu+\frac{1}{2}\right)} + \frac{\sqrt{\pi}2^{-2\mu-1}z^2{c_0}^{(0,0,2)}(x,y,z)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}\right.\\
&\left.+\frac{\sqrt{\pi}2^{-2\mu-1}x^2{c_0}^{(2,0,0)}(x,y,z)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)} + \frac{\sqrt{\pi}2^{-2\mu-1}y^2{c_0}^{(0,2,0)}(x,y,z)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}\right) + \cdots\,.
\end{aligned}
\tag{4.18}
$$

By vanishing of $V_1(x,y,z,t)$, the coefficients $c_n(x,y,z)$, $n=1,2,3,\cdots$ will be obtained as follows:

$$
\begin{aligned}
c_0(x,y,z) &= x^2+y^2+z^2\,,\\
c_1(x,y,z) &= \frac{x^2+y^2+z^2}{\Gamma(\mu+1)}\,,\\
c_2(x,y,z) &= \frac{x^2+y^2+z^2}{\Gamma(2\mu+1)}\,.
\end{aligned}
$$

Therefore, we obtain the analytical solution of Eq. (4.16)

$$
\begin{aligned}
v(x,y,z,t) =\ & x^2+y^2+z^2 - \tfrac{1}{3}x^2y^2z^2 + \frac{t^\mu\left(x^2+y^2+z^2\right)}{\Gamma(\mu+1)}\\
&+\frac{t^{2\mu}\left(x^2+y^2+z^2\right)}{\Gamma(2\mu+1)} + \frac{t^{3\mu}\left(x^2+y^2+z^2\right)}{\Gamma(3\mu+1)} + \cdots\\
=\ & E_\mu(t^\mu)\left(x^2+y^2+z^2\right) - \tfrac{1}{3}x^2y^2z^2\,.
\end{aligned}
\tag{4.19}
$$

We can see the precise solutions featuring $\mu = 1$, $y = 1$ and $z = 1$, in figure 5. The analytical solutions featuring $\mu = 1$ acquired for several amounts of x, y, z and t utilizing NHPM can be seen in Table V.

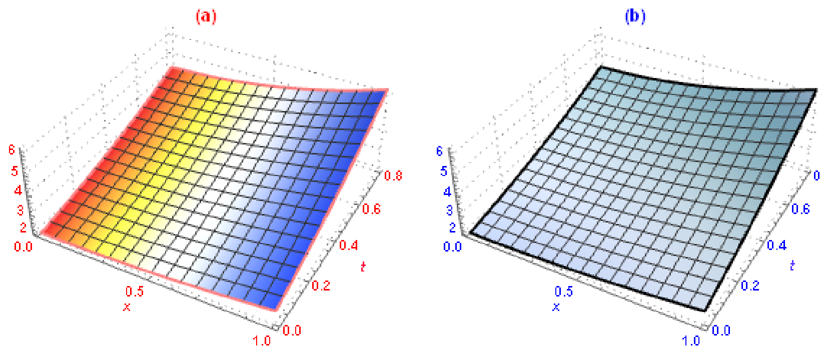

Fig. 5. (a) The analytical solution of Eqs. (4.16)–(4.17) for various x and t, when $\mu = 1$ and $y = 1$, $z = 1$. (b) The accurate solution.

TABLE V

Approximate result of test example 4.5.

t	x	y	z	v_{NHPM}	v_{Exact}	Absolute error
	0.50	0.50	0.50	2.40329	2.4033	9.56567×10^{-6}
0.1	0.75	0.75	0.75	2.64449	0.0647883	0.0000108942
	1.00	1.00	1.00	2.98217	2.98218	0.0000127542
	0.50	0.50	0.50	1.00692	1.00719	0.000269106
0.3	0.75	0.75	0.75	2.21796	2.21856	0.000605488
	1.00	1.00	1.00	3.71517	3.71624	0.00107642
	0.50	0.50	0.50	1.22917	1.23133	0.00216595
0.5	0.75	0.75	0.75	2.71802	2.72289	0.00487339
	1.00	1.00	1.00	4.60417	4.61283	0.00866381

Test example 4.6. We choose the $3D$ fractional wave-like equation

$$D_t^{\mu} v(x,y,z,t) = \frac{1}{2}\left(x^2 v_{xx}(x,y,z,t) + y^2 v_{yy}(x,y,z,t) + z^2 v_{zz}(x,y,z,t)\right) ,$$
$$0 < x,y,z \leq 1\,, \qquad 1 < \mu \leq 2\,, \qquad t > 0\,, \tag{4.20}$$

subject to

$$\begin{aligned}
&v(0,y,z,t)=\cosh(t)\left(y^2-z^2\right), && v(x,0,z,t)=\cosh(t)\left(x^2-z^2\right),\\
&v(x,y,0,t)=\cosh(t)\left(x^2+y^2\right), && v(1,y,z,t)=\cosh(t)\left(1+y^2-z^2\right),\\
&v(x,1,z,t)=\cosh(t)\left(x^2+1-z^2\right), && v(x,y,1,t)=\cosh(t)\left(x^2+y^2-1\right),\\
&v(x,y,z,0)=x^2+y^2-z^2\,, && v_t(x,y,z,0)=0\,.
\end{aligned} \tag{4.21}$$

Assume $v_0(x,y,z,t)=\sum_{n=0}^{m} c_n(x,y,z)t^{n\mu}$, $V(x,y,z,0)=g(x,y,z)$. Solving Eq. (4.20) for $V_1(x,y,z,t)$, one obtains the following result:

$$\begin{aligned}
V_1(x,y,z,t)&=t^{\mu}\left(-\frac{c_0(x,y,z)}{\mu\Gamma(\mu)}+\frac{x^2}{\mu\Gamma(\mu)}+\frac{y^2}{\mu\Gamma(\mu)}-\frac{z^2}{\mu\Gamma(\mu)}\right)\\
&+t^{2\mu}\left(-\frac{\sqrt{\pi}2^{-2\mu}c_1(x,y,z)}{\Gamma\left(\mu+\frac{1}{2}\right)}+\frac{\sqrt{\pi}2^{-2\mu-1}z^2{c_0}^{(0,0,2)}(x,y,z)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}\right.\\
&\left.+\frac{\sqrt{\pi}2^{-2\mu-1}x^2{c_0}^{(2,0,0)}(x,y,z)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}+\frac{\sqrt{\pi}2^{-2\mu-1}y^2{c_0}^{(0,2,0)}(x,y,z)}{\mu\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)}\right)+\cdots\,.
\end{aligned} \tag{4.22}$$

By vanishing of $V_1(x,y,z,t)$, the coefficients $c_n(x,y,z)$, $n=1,2,3,\cdots$ will be obtained as follows:

$$\begin{aligned}
c_0(x,y,z)&=x^2+y^2-z^2\,,\\
c_1(x,y,z)&=\frac{x^2+y^2-z^2}{\Gamma(\mu+1)}\,,\\
c_2(x,y,z)&=\frac{x^2+y^2-z^2}{\Gamma(2\mu+1)}\,.
\end{aligned}$$

Therefore, we obtain the analytical solution of Eq. (4.20)

$$\begin{aligned}
v(x,y,z,t)&=x^2+y^2-z^2+\frac{t^{\mu}\left(x^2+y^2-z^2\right)}{\Gamma(\mu+1)}\\
&\qquad+\frac{t^{2\mu}\left(x^2+y^2-z^2\right)}{\Gamma(2\mu+1)}+\frac{t^{3\mu}\left(x^2+y^2-z^2\right)}{\Gamma(3\mu+1)}+\cdots\\
&=\left(x^2+y^2-z^2\right)E_{\mu}(t^{\mu})\,.
\end{aligned} \tag{4.23}$$

We can view the precise solutions featuring $\mu=2$, $y=1$ and $z=1$, in figure 6. The analytical solutions featuring $\mu=2$ acquired for several amounts of x, y, z and t applying NHPM can be seen in Table VI.

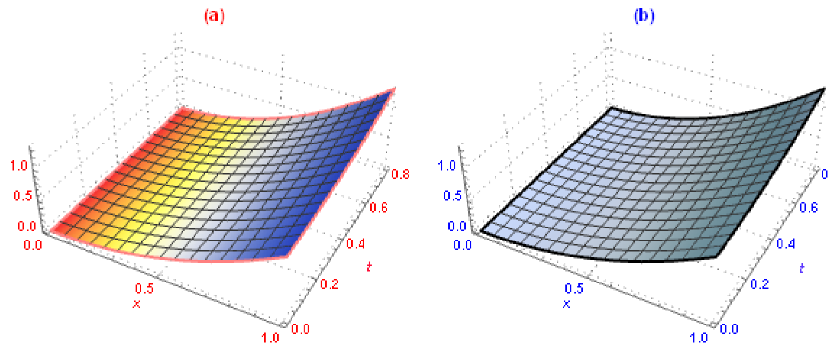

Fig. 6. (a) The analytical solution of Eqs. (4.20)–(4.21) for various x and t, when $\mu = 2$, $y = 1$ and $z = 1$. (b) The accurate solution.

TABLE VI

Approximate result of test example 4.6.

t	x	y	z	v_{NHPM}	v_{Exact}	Absolute error
0.1	0.50	0.50	0.50	0.251251	0.251251	6.19504×10^{-6}
	0.75	0.75	0.75	0.565315	0.565315	1.39444×10^{-13}
	1.00	1.00	1.00	1.005	1.005	2.47802×10^{-13}
0.3	0.50	0.50	0.50	0.261335	0.261335	4.07215×10^{-10}
	0.75	0.75	0.75	0.588003	0.588003	9.16234×10^{-10}
	1.00	1.00	1.00	1.04534	1.04534	1.62886×10^{-10}
0.5	0.50	0.50	0.50	0.281906	0.281906	2.42877×10^{-8}
	0.75	0.75	0.75	0.63429	0.63429	5.46473×10^{-8}
	1.00	1.00	1.00	1.12763	1.12763	9.71508×10^{-8}

5. Conclusion

In this work, NHPM has been successfully employed to obtain a solution of the time-fractional wave- and heat-like equations. This manner appeared obviously a very effectual and potent technique in acquiring the solutions of the offered equations. The result illustrates that a few iteration of NHPM may outcome a good solution. Finally, this approach can be utilized to solve other similar nonlinear problems in PDEs featuring fractional derivative.

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