

TIME EVOLUTION OF AN UNSTABLE QUANTUM SYSTEM*

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After reviewing the description of an unstable state in the framework of Lee Hamiltonians (valid both for Quantum Mechanics (QM) and Quantum Field Theory (QFT)), we consider some theoretical aspects of non-exponential decays: the case of two decay channels, the broadening of the energy spectrum at short times, the effect of an imperfect measurement, the link to QFT, and the decay of an unstable moving particle with definite momentum. All the presented effects were not confirmed in experiments, hence they are at the present stage predictions.

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1. Introduction

Decays of unstable states take place in quite different areas of physics, which range from atomic and molecular phenomena (such as spontaneous emission) up to elementary particles (such as the Higgs bosons). The survival probability $p(t)$ for an unstable state is typically very well described by an exponential function, $e^{-\Gamma t}$. Yet, it is nowadays well-understood both in Quantum Mechanics (QM) [1, 2] and (at least partly) in Quantum Field Theory (QFT) [3, 4] that $p(t)$ is not exactly exponential: deviations at short as well as at long times appear. These deviations were verified experimentally in Ref. [5] for short times and in Ref. [6] for long times (for an indirect evidence, see also Ref. [7]). As a consequence of non-exponential decays, the famous Quantum Zeno Effect (QZE) is realized when repeated ideal measurements are performed at τ , 2τ , *etc.* (usually called bang-bang measurements) [8–11]: the survival probability approaches one for $\tau \rightarrow 0$.

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Experimentally, the QZE was measured by reducing the probability of Rabi oscillations between atomic energy levels in Ref. [12, 13] and for a genuine unstable tunneling process in Ref. [14]. Quite remarkably, as presented theoretically in Ref. [15] and verified experimentally in Ref. [16], the QZE takes place also for continuous measurements.

Here, we briefly review the mathematical treatment of an unstable state via Lee Hamiltonians [17]. For definiteness, we use a simple cutoff model in which two physical aspects, the left-hand threshold and a cutoff at high energies, are simultaneously present. This model nicely reproduces the purely exponential decay when the cutoff is sent to infinity. Then, we discuss some interesting modern developments: (i) The case of two (or more) decay channels [4]; (ii) The final-state spectrum of a decay process such as spontaneous emission [18]; (iii) The QZE induced by an imperfect detector [19]; (iv) The link to QFT [3]; (v) The decay of moving particle with a definite momentum [24].

2. Aspects of non-exponential decay

Lee Hamiltonian(s): The Lee Hamiltonian $H = H_0 + H_1$ couples an unstable quantum state $|S\rangle$ to final states $|k\rangle$ [4, 17, 18]

$$H_0 = M_0 |S\rangle \langle S| + \int_{-\infty}^{+\infty} dk \omega(k) |k\rangle \langle k|, \quad H_1 = \int_{-\infty}^{+\infty} dk \frac{gf(k)}{\sqrt{2\pi}} (|S\rangle \langle k| + \text{h.c.}) . \quad (1)$$

Usually, $|k\rangle$ represents a two-particle state emitted back-to-back (for instance, $|S\rangle$ can be a neutral pion and $|k\rangle$ the final two-photon state). The survival probability amplitude of $|S\rangle$ is

$$a(t) = \langle S | e^{-iHt} | S \rangle = \int_{-\infty}^{\infty} dm d_S(m) e^{-imt}, \quad (2)$$

where $d_S(m)$ is the energy distribution of the unstable state. The survival probability is $p(t) = |a(t)|^2$. Here, we work with a simplified model in which $\omega(k) = k$ and $f(k) = \theta(\Lambda - k)\theta(k - E_{\text{th}})$ [19] ($E_{\text{th}} < \Lambda$). In this way, the unstable state $|S\rangle$ couples in a limited energy range to the final states. The general outcome of the time evolution is $e^{-iHt} |S\rangle = a(t) |S\rangle + \int_{-\infty}^{+\infty} dk b(k, t) |k\rangle$. When E_{th} and Λ are finite, deviations both at short and long times occur, see the explicit numerical results in [18] and, for a particular illustrative numerical choice, Fig. 1 (parameters chosen to visualize better the effect). In the limit $E_{\text{th}} \rightarrow -\infty$, $\Lambda \rightarrow \infty$, the model reduces exactly to the exponential decay [18]: $a(t) = e^{-i(M_0 - i\Gamma/2)t}$ (with $\Gamma = g^2$).

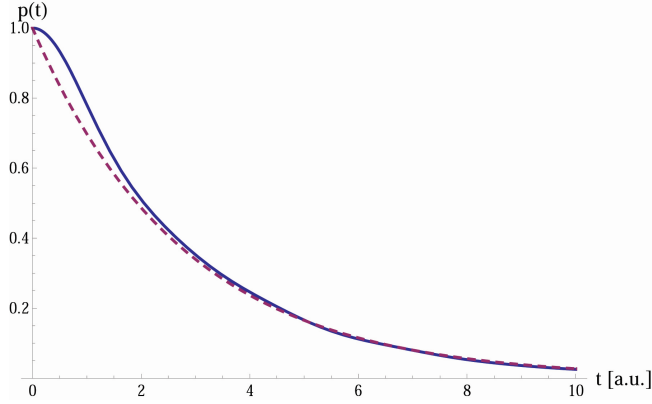


Fig. 1. Survival probability $p(t)$ (solid line) in the cutoff model upon using $E_{\text{th}} = 0$, $\Lambda = 5$, $M_0 = 3$, $g^2 = 0.6^2$ in a.u. of the energy. The dashed line refers to the corresponding exponential case, $e^{-\Gamma t}$ with $\Gamma = g^2$.

Two decay channels [4]: The Lee Hamiltonian is easily generalized to the case of two decay channels. In particular, we shall consider $h_1(t)dt$ as the probability that $|S\rangle$ decays in the first channel between t and $t+dt$ ($h_2(t)$ is the same object in the second channel). Then, it is useful to study the ratio $R(t) = h_1(t)/h_2(t)$, which reduces to a constant $R(t) = \Gamma_1/\Gamma_2$ (ratio of decay widths) in the exponential limit. As shown in Fig. 2, this ratio shows interesting fluctuations. Moreover, it deviates from the constant limit for a quite long time (it does not flatten on it), thus it is potentially interesting to be measured in future experiments.

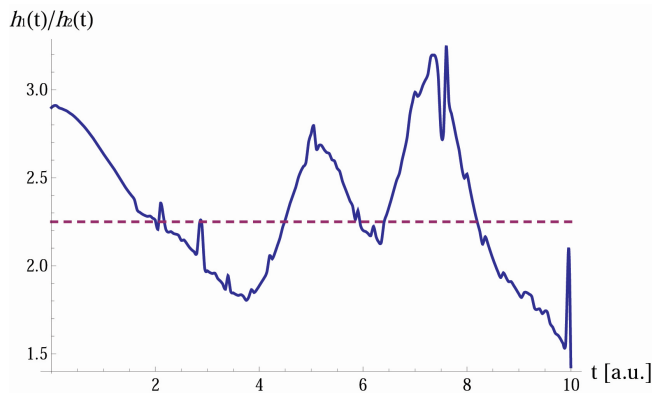


Fig. 2. Ratio $R(t) = h_1(t)/h_2(t)$ (solid line) upon using $M_0 = 3$, $E_{\text{th},1} = 0$, $\Lambda_1 = 5$, $g_1^2 = 0.6^2$, $E_{\text{th},2} = 0.5$, $\Lambda_2 = 4$, $g_2^2 = 0.4^2$ in a.u. of the energy. The dashed line refers to the exponential case, Γ_1/Γ_2 with $\Gamma_k = g_k^2$.

Energy spreading of the final state [18]: One studies the function $\eta(t, \omega)$ defined as the probability that, by measuring the final state at the time t , it has an energy between ω and $\omega + d\omega$. In the case of spontaneous photon emission, $\eta(t, \omega)d\omega$ is the energy distribution of the photon at the time t . In Fig. 3, we show this function for various values of t . When t is small, this spectrum is large. Hence, if it could be possible to measure emitted photons soon enough, they should show a larger spectrum than the simple decay width Γ .

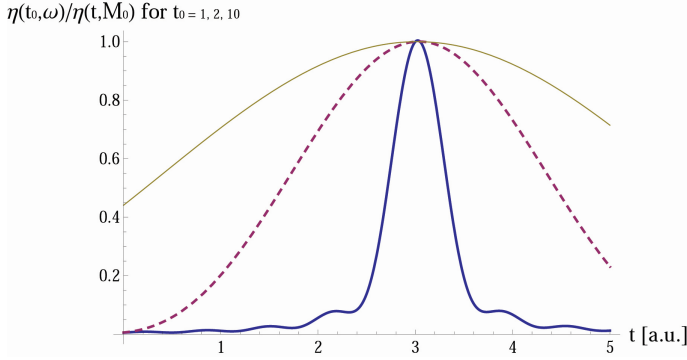


Fig. 3. $\eta(t_0, \omega)/\eta(t_0, M_0)$ for the parameters of Fig. 1 and for $t_0 = 1$ (upper), $t_0 = 2$ (dashed), $t_0 = 10$ (lower curve). Note the broadening for small t_0 .

QZE induced by an imperfect measurements [19]: We assume that the detector can only detect $|k\rangle$ if $M_0 - \lambda \leq k \leq M_0 + \lambda$. This means that the probability to “hear” the click of the detector for a measurement at τ is $p_{\text{click}}(\tau) = \int_{M_0 - \lambda}^{M_0 + \lambda} |b(k, \tau)|^2 dk$. Then, one performs a second measurement at the time 2τ , and so on. Finally, the no-click probability at the instant $t = n\tau$ is $p_{\text{no-click}}(t = n\tau) = 1 - w_\lambda(\tau) \frac{1 - p(\tau)^n}{1 - p(\tau)}$ [21]. Heaving $w_\lambda(\tau \rightarrow 0) = 0$, one obtains a QZE (no-click). For the link to continuous measurements, see [11, 20]. A side-effect of Ref. [19] is that the QZE obtained for bang-bang measurements and the QZE through continuous measurements are in general different, hence one could, in principle, check if the collapse of the wave function is a real physical process as proposed in Ref. [23].

Link to QFT [3]: In the framework of relativistic QFT, one obtains a picture similar to the QM case. In QFT, one uses the scalar fields S and φ embedded in the Lagrangian [3, 4, 22]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_0^2 S^2 + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 \varphi^2 + gS\varphi^2. \quad (3)$$

The decay process $S \rightarrow \varphi\varphi$ is analogous to the transition $|S\rangle \rightarrow |k\rangle$ described previously. Upon taking into account proper relativistic expressions (instead

of non-relativistic ones), the QFT results can be obtained in the framework of the Lee Hamiltonian by a due choice of the vertex function $f(k)$ [4]. Hence, the non-exponential nature of decay and all the other phenomena described in this section apply also for QFT (Eq. (2) is valid also in QFT). It would be interesting in the future to go beyond the use of the Lee Hamiltonian's matching and evaluate the time evolution in QFT in the interaction picture. While it is not expected to invalid previous results, it would be anyhow an important theoretical achievement. This project is left for the future.

Decay of a moving particle [24]: Finally, we mention the survival probability of a state with non-vanishing momentum q (here, we follow [24]; for previous works, see Ref. [25–28]). The survival probability amplitude reads $a(t, q) = \int_{-\infty}^{\infty} dm d_S(m) e^{-i\sqrt{m^2+q^2}t}$ (as in Eq. (2)) which implies that $p(t, q) = |a(t, q)|^2 \neq p\left(tM_0/\sqrt{p^2+M_0^2}\right)$, hence the usual Einstein's dilatation formula does not hold [26–28]. In the exponential limit, the non-decay (survival) probability reads $e^{-\Gamma_q t}$ with

$$\Gamma_q = \sqrt{2} \sqrt{\left[\left(M_0^2 - \frac{\Gamma^2}{4} + q^2 \right)^2 + M_0^2 \Gamma^2 \right]^{1/2} - \left(M_0^2 - \frac{\Gamma^2}{4} + q^2 \right)} \quad (4)$$

which *differs* from the standard formula $\Gamma M_0 / \sqrt{q^2 + M_0^2}$. One can easily prove that for $\Gamma/M_0 \ll 1$, the Einstein formula represents a very good approximation. Indeed, the maximal deviation is obtained for $q_{\max} = \sqrt{2/3}M_0$, for which (normalized to M_0) reads $\sim (\Gamma/M)^3$. This is in almost all cases a ridiculously small number. On the contrary, a boost transforms an unstable state into its decay products. A boosted neutral pion is a two-photon state.

3. Conclusions

In this work, we have reviewed some recent theoretical works on non-exponential decay and the QZE which were not confirmed yet experimentally. The non-exponential decay when two (or more) decay channels are present seems promising. Others, such as the measurement of decay products soon after their emission and the QZE induced by detectors are appealing but probably difficult to measure. The measurement of deviations from the Einstein's formula is at present not possible. Theoretically, the firm understanding of these issues in QFT on a solid mathematical basis (and without using Lee Hamiltonian's analogy) is an important outlook of the present work.

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