ANALYSIS OF THE $\frac{3}{2}^{\pm}$ PENTAQUARK STATES IN THE DIQUARK MODEL WITH QCD SUM RULES*

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(Received August 16, 2017; accepted November 9, 2017)

In this article, we construct the scalar–diquark–axialvector–diquark–antiquark-type interpolating currents, and study the masses and pole residues of the $J^P = \frac{3}{2}^{\pm}$ hidden-charmed pentaquark states with the QCD sum rules. In calculations, we use the formula $\mu = \sqrt{M_P^2 - (2\mathbb{M}_c)^2}$ to determine the energy scales of the QCD spectral densities. We obtain the masses of the hidden-charm pentaquark states with the strangeness $S = -1$ and $S = -2$, which can be confronted with the experimental data in the future.

DOI:10.5506/APhysPolB.48.2013

1. Introduction

In 2015, the LHCb Collaboration studied the $\Lambda_b^0 \to J/\psi p$ decays, and observed two exotic hidden-charm pentaquark resonances, $P_c(4380)$ and $P_c(4450)$, in the $J/\psi p$ mass spectrum with the significance of more than 9σ [1]. They are good candidates for pentaquark states, which are made of four quarks and one antiquark. The measured masses and widths are

$$M_{P_c(4380)} = (4380 \pm 8 \pm 29)\ \text{MeV}\,, \qquad \Gamma_{P_c(4380)} = (205 \pm 18 \pm 86)\ \text{MeV}\,, \quad (1)$$

$$M_{P_c(4450)} = (4449.8 \pm 1.7 \pm 2.5)\ \text{MeV}\,, \quad \Gamma_{P_c(4450)} = (39 \pm 5 \pm 19)\ \text{MeV}\,. \quad (2)$$

The preferred spin-parity assignments of the $P_c(4380)$ and $P_c(4450)$ are $J^P = \frac{3}{2}^-$ and $\frac{5}{2}^+$, respectively [1]. The $P_c(4380)$ and $P_c(4450)$ have attracted much attention of the theoretical physicists, several possible assignments were suggested, such as the $\Sigma_c\bar{D}^*$, $\Sigma_c^*\bar{D}^*$, $J/\psi N(1440)$, $J/\psi N(1520)$

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molecule-like pentaquark states [2] (or not the molecular pentaquark states [3]), the diquark–diquark–antiquark-type pentaquark states [4–9], the diquark–triquark-type pentaquark states [10], the re-scattering effects [11], *etc.*

The QCD sum rules have been applied extensively to study the hidden-charm (bottom) tetraquark or molecular states [12–15] and pentaquark states [5–8]. We constructed the diquark–diquark–antiquark-type interpolating currents to study the $P_c(4380)$ and $P_c(4450)$ with QCD sum rules by calculating the contributions of the vacuum condensates up to dimension 10 in the operator product expansion and using the energy scale formula to determine the ideal energy scales of the QCD spectral densities [5], then we study other hidden-charm pentaquark states with $J^P = \frac{1}{2}^\pm$, $\frac{3}{2}^\pm$ in Refs. [6–8]. In summary, we have studied the $S_{\rm L}$–$A_{\rm H}$–$\bar{c}$-type hidden-charm pentaquark states with $J^P = \frac{3}{2}^-$, $\frac{5}{2}^+$ and strangeness $S=0$ [5], the $S_{\rm L}$–$S_{\rm H}$–$\bar{c}$-type, $S_{\rm L}$–$A_{\rm H}$–$\bar{c}$-type hidden-charm pentaquark states with $J^P = \frac{1}{2}^\pm$ and strangeness $S=0$ [6], the $A_{\rm L}$–$A_{\rm H}$–$\bar{c}$-type, $A_{\rm L}$–$S_{\rm H}$–$\bar{c}$-type hidden-charm pentaquark states with $J^P = \frac{1}{2}^\pm$ and strangeness $S = 0, -1, -2, -3$ [7], the $A_{\rm L}$–$A_{\rm H}$–$\bar{c}$-type, $A_{\rm L}$–$S_{\rm H}$–$\bar{c}$-type hidden-charm pentaquark states with $J^P = \frac{3}{2}^\pm$ and strangeness $S = 0, -1, -2, -3$ [8], where the $S_{\rm L/H}$ denote the light and heavy scalar diquark states, the $A_{\rm L/H}$ denote the light and heavy axialvector diquark states.

In this article, we extend our previous work to study the masses and pole residues of the $S_{\rm L}$–$A_{\rm H}$–$\bar{c}$ type hidden-charm pentaquark states with $J^P = \frac{3}{2}^-$ and strangeness $S = -1, -2$.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the $J^P = \frac{3}{2}^\pm$ hidden-charm pentaquark states with strangeness $S = -1, -2$ in Sect. 2; in Sect. 3, we present the numerical results and discussions; and Sect. 4 is reserved for our conclusion.

2. QCD sum rules for the $\frac{3}{2}^\pm$ hidden-charm pentaquark states

Now, we write down the two-point correlation functions $\Pi^i_{\mu\nu}(p)$ with $i=1,2$

$$\Pi^i_{\mu\nu}(p) = i\int d^4x e^{ip\cdot x}\left\langle 0|T\left\{J^i_\mu(x)\bar{J}^i_\nu(0)\right\}|0\right\rangle , \tag{3}$$

where

$$J^1_\mu(x) = \varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}u^T_j(x)C\gamma_5 s_k(x)u^T_m(x)C\gamma_\mu c_n(x)C\bar{c}^T_a(x)\,, \tag{4}$$

$$J^2_\mu(x) = \varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}u^T_j(x)C\gamma_5 s_k(x)s^T_m(x)C\gamma_\mu c_n(x)C\bar{c}^T_a(x)\,, \tag{5}$$

the i, j, k, l, m, n and a are color indices, the C is the charge conjugation matrix.

At the hadron side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $J^i_\mu(x)$ into the correlation functions $\Pi^i_{\mu\nu}(p)$ to obtain the hadronic representation [16, 17]. After isolating the pole terms of the lowest states of the hidden-charm pentaquark states with spin $J=\frac{3}{2}$, we get the following result:

$$\begin{aligned}
\Pi^i_{\mu\nu}(p) = {} & \lambda_i^{-2}\frac{\not{p}+M^i_-}{M^{i2}_- - p^2}\left(-g_{\mu\nu}+\frac{\gamma_\mu\gamma_\nu}{3}+\frac{2p_\mu p_\nu}{3p^2}-\frac{p_\mu\gamma_\nu-p_\nu\gamma_\mu}{3\sqrt{p^2}}\right) \\
& +\lambda_i^{+2}\frac{\not{p}-M^i_+}{M^{i2}_+ - p^2}\left(-g_{\mu\nu}+\frac{\gamma_\mu\gamma_\nu}{3}+\frac{2p_\mu p_\nu}{3p^2}-\frac{p_\mu\gamma_\nu-p_\nu\gamma_\mu}{3\sqrt{p^2}}\right)+\cdots \\
= {} & \Pi^i(p^2)\left(-g_{\mu\nu}\right)+\cdots\,, \qquad (6)
\end{aligned}$$

where the $M^i_\pm$ are the masses of the lowest pentaquark states with the parity $\pm$, respectively, and the $\lambda^\pm_i$ are the corresponding pole residues.

We obtain the hadronic spectral densities through dispersion relation [5] as

$$\begin{aligned}
\frac{\mathrm{Im}\Pi^i(s)}{\pi} &= \not{p}\left[\lambda_i^{-2}\delta\left(s-M^{i2}_-\right)+\lambda_i^{+2}\delta\left(s-M^{i2}_+\right)\right] \\
&\quad +\left[M^i_-\lambda_i^{-2}\delta\left(s-M^{i2}_-\right)-M^i_+\lambda_i^{+2}\delta\left(s-M^{i2}_+\right)\right] \\
&= \not{p}\rho^{i1}_H(s)+\rho^{i0}_H(s)\,. \qquad (7)
\end{aligned}$$

Then we introduce the weight function $\exp(-\frac{s}{T^2})$ to obtain the QCD sum rules at the hadron side

$$\int_{4m_c^2}^{s_0} \mathrm{d}s\left[\sqrt{s}\rho^{i1}_H(s)+\rho^{i0}_H(s)\right]\exp\left(-\frac{s}{T^2}\right) = 2M^i_-\lambda_i^{-2}\exp\left(-\frac{M^{i2}_-}{T^2}\right)\,, \qquad (8)$$

$$\int_{4m_c^2}^{s_0} \mathrm{d}s\left[\sqrt{s}\rho^{i1}_H(s)-\rho^{i0}_H(s)\right]\exp\left(-\frac{s}{T^2}\right) = 2M^i_+\lambda_i^{+2}\exp\left(-\frac{M^{i2}_+}{T^2}\right)\,, \qquad (9)$$

where s_0 are continuum threshold parameters and the T^2 are the Borel parameters. The contributions of the negative parity hidden-charm pentaquark states are separated from that of the positive parity.

In the following, we carry out the operator product expansion for the correlation functions $\Pi^i_{\mu\nu}(p)$ in perturbative QCD. Contracting the u, s and c quark fields in the correlation functions with Wick theorem, we obtain

$$\begin{aligned}
\Pi^{1}_{\mu\nu}(p) &= i\,\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}\varepsilon^{i'l'a'}\varepsilon^{i'j'k'}\varepsilon^{l'm'n'}\int \mathrm{d}^4x e^{ip\,x} C C^T_{a'a}(-x) C \\
&\quad \times \left\{ \mathrm{Tr}\left[\gamma_5 S_{kk'}(x)\gamma_5 C U^T_{jj'}(x) C\right]\, \mathrm{Tr}\left[\gamma_\mu C_{nn'}(x)\gamma_\nu C U^T_{mm'}(x) C\right] \right. \\
&\quad \left. -\mathrm{Tr}\left[\gamma_5 S_{kk'}(x)\gamma_5 C U^T_{mj'}(x) C\gamma_\mu C_{nn'}(x)\gamma_\nu C U^T_{jm'}(x) C\right] \right\} , \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\Pi^{2}_{\mu\nu}(p) &= i\,\varepsilon^{ila}\varepsilon^{ijk}\varepsilon^{lmn}\varepsilon^{i'l'a'}\varepsilon^{i'j'k'}\varepsilon^{l'm'n'}\int \mathrm{d}^4x e^{ip\,x} C C^T_{a'a}(-x) C \\
&\quad \times \left\{ \mathrm{Tr}\left[\gamma_5 S_{km'}(x)\gamma_5 C U^T_{jj'}(x) C\right]\, \mathrm{Tr}\left[\gamma_\mu C_{nn'}(x)\gamma_\nu C S^T_{mm'}(x) C\right] \right. \\
&\quad \left. -\mathrm{Tr}\left[C S^T_{km'}(x) C\gamma_5 U^T_{jj'}(x)\gamma_5 C S^T_{mk'}(x) C\gamma_\mu C_{nn'}(x)\gamma_\nu\right] \right\} , \qquad (11)
\end{aligned}$$

where the $U_{ij}(x)$, $S_{ij}(x)$ and $C_{ij}(x)$ are the full u, s and c quark propagators, respectively,

$$\begin{aligned}
U_{ij}(x) &= \frac{i\delta_{ij}\not{x}}{2\pi^2x^4} - \frac{\delta_{ij}\langle\bar{q}q\rangle}{12} - \frac{\delta_{ij}x^2\langle\bar{q}g_s\sigma Gq\rangle}{192} - \frac{ig_sG^a_{\alpha\beta}t^a_{ij}\left(\not{x}\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not{x}\right)}{32\pi^2x^2} \\
&\quad -\frac{1}{8}\langle\bar{q}_j\sigma^{\mu\nu}q_i\rangle\sigma_{\mu\nu} + \cdots , \\
S_{ij}(x) &= \frac{i\delta_{ij}\not{x}}{2\pi^2x^4} - \frac{\delta_{ij}m_s}{4\pi^2x^2} - \frac{\delta_{ij}\langle\bar{s}s\rangle}{12} + \frac{i\delta_{ij}\not{x}m_s\langle\bar{s}s\rangle}{48} - \frac{\delta_{ij}x^2\langle\bar{s}g_s\sigma Gs\rangle}{192} \\
&\quad + \frac{i\delta_{ij}x^2\not{x}m_s\langle\bar{s}g_s\sigma Gs\rangle}{1152} - \frac{ig_sG^a_{\alpha\beta}t^a_{ij}\left(\not{x}\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not{x}\right)}{32\pi^2x^2} \\
&\quad - \frac{\langle\bar{s}_j\sigma^{\mu\nu}s_i\rangle\sigma_{\mu\nu}}{8} + \cdots , \qquad (12) \\
C_{ij}(x) &= \frac{i}{(2\pi)^4}\int \mathrm{d}^4k e^{-ik\cdot x} \\
&\quad \times \left\{ \frac{\delta_{ij}}{\not{k}-m_c} - \frac{g_sG^n_{\alpha\beta}t^n_{ij}}{4}\frac{\sigma^{\alpha\beta}(\not{k}+m_c)+(\not{k}+m_c)\sigma^{\alpha\beta}}{(k^2-m_c^2)^2} \right. \\
&\quad \left. - \frac{g_s^2\left(t^at^b\right)_{ij}G^a_{\alpha\beta}G^b_{\mu\nu}\left(f^{\alpha\beta\mu\nu}+f^{\alpha\mu\beta\nu}+f^{\alpha\mu\nu\beta}\right)}{4\left(k^2-m_c^2\right)^5} + \cdots \right\} , \\
f^{\alpha\beta\mu\nu} &= (\not{k}+m_c)\gamma^\alpha(\not{k}+m_c)\gamma^\beta(\not{k}+m_c)\gamma^\mu(\not{k}+m_c)\gamma^\nu(\not{k}+m_c) , \qquad (13)
\end{aligned}$$

where $t^n = \frac{\lambda^n}{2}$, the λ^n is the Gell-Mann matrix [17]. Then we compute the integrals both in the coordinate and momentum spaces, and obtain the correlation functions $\Pi^i_{\mu\nu}(p)$, therefore the QCD spectral densities $\rho^{i1}_{\mathrm{QCD}}(s)$ and $\tilde{\rho}^{i0}_{\mathrm{QCD}}(s)$ at the quark level through dispersion relation.

Finally, we take the quark–hadron duality below the continuum thresholds s_0 and introduce the weight function $\exp(-\frac{s}{T^2})$ to obtain the following QCD sum rules:

$$2M_-^i\lambda_i^{-2}\exp\left(-\frac{M_-^{i2}}{T^2}\right)=\int\limits_{4m_c^2}^{s_0}\mathrm{d}s\left[\sqrt{s}\rho_{\rm QCD}^{i1}(s)+m_c\tilde{\rho}_{\rm QCD}^{i0}(s)\right]\exp\left(-\frac{s}{T^2}\right)\,, \quad (14)$$

$$2M_+^i\lambda_i^{+2}\exp\left(-\frac{M_+^{i2}}{T^2}\right)=\int\limits_{4m_c^2}^{s_0}\mathrm{d}s\left[\sqrt{s}\rho_{\rm QCD}^{i1}(s)-m_c\tilde{\rho}_{\rm QCD}^{i0}(s)\right]\exp\left(-\frac{s}{T^2}\right)\,, \quad (15)$$

where

$$\begin{aligned}\rho_{\rm QCD}^{i1}(s)&=\rho_0^{i1}(s)+\rho_3^{i1}(s)+\rho_4^{i1}(s)+\rho_5^{i1}(s)+\rho_6^{i1}(s)+\rho_8^{i1}(s)+\rho_9^{i1}(s)+\rho_{10}^{i1}(s)\,,\\ \tilde{\rho}_{\rm QCD}^{i0}(s)&=\tilde{\rho}_0^{i0}(s)+\tilde{\rho}_3^{i0}(s)+\tilde{\rho}_4^{i0}(s)+\tilde{\rho}_5^{i0}(s)+\tilde{\rho}_6^{i0}(s)+\tilde{\rho}_8^{i0}(s)+\tilde{\rho}_9^{i0}(s)+\tilde{\rho}_{10}^{i0}(s)\,,\end{aligned} \quad (16)$$

the explicit expressions of the QCD spectral densities $\rho_j^{i1}(s)$ and $\tilde{\rho}_j^{i0}(s)$ with $j=0,3,4,5,6,8,9,10$ are given in the appendix.

We differentiate Eqs. (14) and (15) with respect to $\frac{1}{T^2}$, then eliminate the pole residues $\lambda_i^{\mp}$, and obtain the QCD sum rules for the masses of the hidden-charm pentaquark states

$$M_-^{i2} = -\frac{\frac{\mathrm{d}}{\mathrm{d}(1/T^2)}\int_{4m_c^2}^{s_0}\mathrm{d}s\left[\sqrt{s}\rho_{\rm QCD}^{i1}(s)+m_c\tilde{\rho}_{\rm QCD}^{i0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{s_0}\mathrm{d}s\left[\sqrt{s}\rho_{\rm QCD}^{i1}(s)+m_c\tilde{\rho}_{\rm QCD}^{i0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}\,, \quad (17)$$

$$M_+^{i2} = -\frac{\frac{\mathrm{d}}{\mathrm{d}(1/T^2)}\int_{4m_c^2}^{s_0}\mathrm{d}s\left[\sqrt{s}\rho_{\rm QCD}^{i1}(s)-m_c\tilde{\rho}_{\rm QCD}^{i0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{s_0}\mathrm{d}s\left[\sqrt{s}\rho_{\rm QCD}^{i1}(s)-m_c\tilde{\rho}_{\rm QCD}^{i0}(s)\right]\exp\left(-\frac{s}{T^2}\right)}\,. \quad (18)$$

Once the masses M_-^i (M_+^i) are obtained, we can take them as input parameters and obtain the pole residues λ_i^- (λ_i^+) from the QCD sum rules in Eqs. (14) and (15).

3. Numerical results and discussions

The input parameters are taken to be the standard values $\langle\bar{q}q\rangle=-(0.24\pm 0.01\,\text{GeV})^3$, $\langle\bar{s}s\rangle=(0.8\pm0.1)\langle\bar{q}q\rangle$, $\langle\bar{q}g_s\sigma Gq\rangle=m_0^2\langle\bar{q}q\rangle$, $\langle\bar{s}g_s\sigma Gs\rangle=m_0^2\langle\bar{s}s\rangle$, $m_0^2=(0.8\pm0.1)\,\text{GeV}^2$, $\langle\frac{\alpha_s GG}{\pi}\rangle=(0.33\,\text{GeV})^4$ at the energy scale $\mu=1\,\text{GeV}$ [16–18], $m_c(m_c)=(1.275\pm0.025)\,\text{GeV}$ and $m_s(\mu=2\,\text{GeV})=(0.095\pm 0.005)\,\text{GeV}$ from the Particle Data Group [19]. Furthermore, we set $m_u =$

$m_d = 0$ due to the small current quark masses. We take into account the energy-scale dependence of the input parameters from the renormalization group equation

$$
\begin{aligned}
\langle\bar{q}q\rangle(\mu) &= \langle\bar{q}q\rangle(Q)\left[\frac{\alpha_{\rm s}(Q)}{\alpha_{\rm s}(\mu)}\right]^{\frac{4}{9}}\,, \\
\langle\bar{s}s\rangle(\mu) &= \langle\bar{s}s\rangle(Q)\left[\frac{\alpha_{\rm s}(Q)}{\alpha_{\rm s}(\mu)}\right]^{\frac{4}{9}}\,, \\
\langle\bar{q}g_s\sigma Gq\rangle(\mu) &= \langle\bar{q}g_s\sigma Gq\rangle(Q)\left[\frac{\alpha_{\rm s}(Q)}{\alpha_{\rm s}(\mu)}\right]^{\frac{2}{27}}\,, \\
\langle\bar{s}g_s\sigma Gs\rangle(\mu) &= \langle\bar{s}g_s\sigma Gs\rangle(Q)\left[\frac{\alpha_{\rm s}(Q)}{\alpha_{\rm s}(\mu)}\right]^{\frac{2}{27}}\,, \\
m_c(\mu) &= m_c(m_c)\left[\frac{\alpha_{\rm s}(\mu)}{\alpha_{\rm s}(m_c)}\right]^{\frac{12}{25}}\,, \\
m_s(\mu) &= m_s(2\,{\rm GeV})\left[\frac{\alpha_{\rm s}(\mu)}{\alpha_{\rm s}(2\,{\rm GeV})}\right]^{\frac{4}{9}}\,, \\
\alpha_{\rm s}(\mu) &= \frac{1}{b_0 t}\left[1-\frac{b_1}{b_0^2}\frac{\log t}{t}+\frac{b_1^2\left(\log^2 t-\log t-1\right)+b_0 b_2}{b_0^4 t^2}\right]\,, \qquad (19)
\end{aligned}
$$

where $t = \log\frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_{\rm f}}{12\pi}$, $b_1 = \frac{153-19n_{\rm f}}{24\pi^2}$, $b_2 = \frac{2857-\frac{5033}{9}n_{\rm f}+\frac{325}{27}n_{\rm f}^2}{128\pi^3}$, $\Lambda = 213\,{\rm MeV}$, $296\,{\rm MeV}$ and $339\,{\rm MeV}$ for the flavors $n_{\rm f} = 5$, 4 and 3, respectively [19], and evolve all the input parameters to the optimal energy scales μ to extract the masses of the hidden-charm pentaquark states.

In previous works, we studied the energy-scale dependence of the QCD sum rules for the hidden-charm (hidden-bottom) tetraquark states and molecular states X, Y, Z in details for the first time, and suggested an energy scale formula $\mu = \sqrt{M^2_{X/Y/Z} - (2\mathbb{M}_Q)^2}$ with the effective heavy quark masses $\mathbb{M}_Q$ to determine the ideal energy scales of the QCD spectral densities [13, 14], then we extended the energy scale formula to study the hidden-charm pentaquark states [5–8] and obtained satisfactory results. In this article, we use the energy scale formula $\mu = \sqrt{M^2_P - (2\mathbb{M}_c)^2}$ to determine the energy scales of the QCD spectral densities, and take the updated value of the effective c-quark mass $\mathbb{M}_c = 1.82\,{\rm GeV}$ [15]. For detailed discussions about the energy scale formula, one can consult Ref. [20].

In Refs. [5–8], we take the continuum threshold parameters as $\sqrt{s_0} = M_P + (0.6 - 0.8)\,\text{GeV}$, which works well for the hidden-charm pentaquark states. In this article, we also take the continuum threshold parameters $\sqrt{s_0} = M_P + (0.6 - 0.8)\,\text{GeV}$ as an additional constraint.

In the present QCD sum rules, we choose the Borel parameters T^2 and continuum threshold parameters s_0 to satisfy the following four criteria:

I. Pole dominance at the phenomenological side;

II. Convergence of the operator product expansion;

III. Appearance of the Borel platforms;

IV. Satisfying the energy scale formula.

Now, we search for the optimal Borel parameters T^2 and continuum threshold parameters s_0 by try and error. The resulting Borel parameters T^2, continuum threshold parameters s_0, pole contributions, and contributions of the vacuum condensates of dimension 9 and 10 are shown explicitly in Table I, where the quantum numbers of the hidden-charm pentaquark states are shown explicitly. From Table I, we can see that the criteria I and II of the QCD sum rules are satisfied.

TABLE I

The Borel parameters, continuum threshold parameters, pole contributions, contributions of the vacuum condensates of dimension 9 and dimension 10 of the hidden-charm pentaquark states.

	T^2 [GeV^2]	$\sqrt{s_0}$ [GeV]	Pole	D_9	D_{10}
$P_{uusc\bar{c}}\left(\frac{3}{2}^-\right)$	3.4–3.8	5.20 ± 0.10	(40–61)%	(8–11)%	(1–2)%
$P_{ussc\bar{c}}\left(\frac{3}{2}^-\right)$	3.6–4.0	5.30 ± 0.10	(42–62)%	(10–14)%	$\sim 1\%$
$P_{uusc\bar{c}}\left(\frac{3}{2}^+\right)$	3.3–3.7	5.30 ± 0.10	(40–62)%	(4–6)%	(2–3)%
$P_{ussc\bar{c}}\left(\frac{3}{2}^+\right)$	3.4–3.8	5.40 ± 0.10	(42–63)%	(5–7)%	(1–2)%

We take into account all uncertainties of the input parameters, and obtain the values of the masses and pole residues of the hidden-charm pentaquark states, which are shown explicitly in Table II and Figs. 1–2.

From Table II and Figs. 1–2, we can see that the criteria III and IV of the QCD sum rules are also satisfied. Now, the four criteria of the QCD sum rules are all satisfied, and we expect to make reasonable predictions. The present predictions can be confronted to the experimental data in the future.

TABLE II

The energy scales, masses and pole residues of the hidden-charm pentaquark states.

	μ [GeV]	M_P [GeV]	λ_P [GeV^6]
$P_{uus c\bar{c}}\left(\frac{3}{2}^-\right)$	2.65	4.49 ± 0.04	$(1.85 \pm 0.14) \times 10^{-3}$
$P_{ussc\bar{c}}\left(\frac{3}{2}^-\right)$	2.80	4.60 ± 0.04	$(2.33 \pm 0.18) \times 10^{-3}$
$P_{uusc\bar{c}}\left(\frac{3}{2}^+\right)$	2.80	4.61 ± 0.08	$(0.80 \pm 0.10) \times 10^{-3}$
$P_{ussc\bar{c}}\left(\frac{3}{2}^+\right)$	3.00	4.72 ± 0.04	$(1.03 \pm 0.09) \times 10^{-3}$

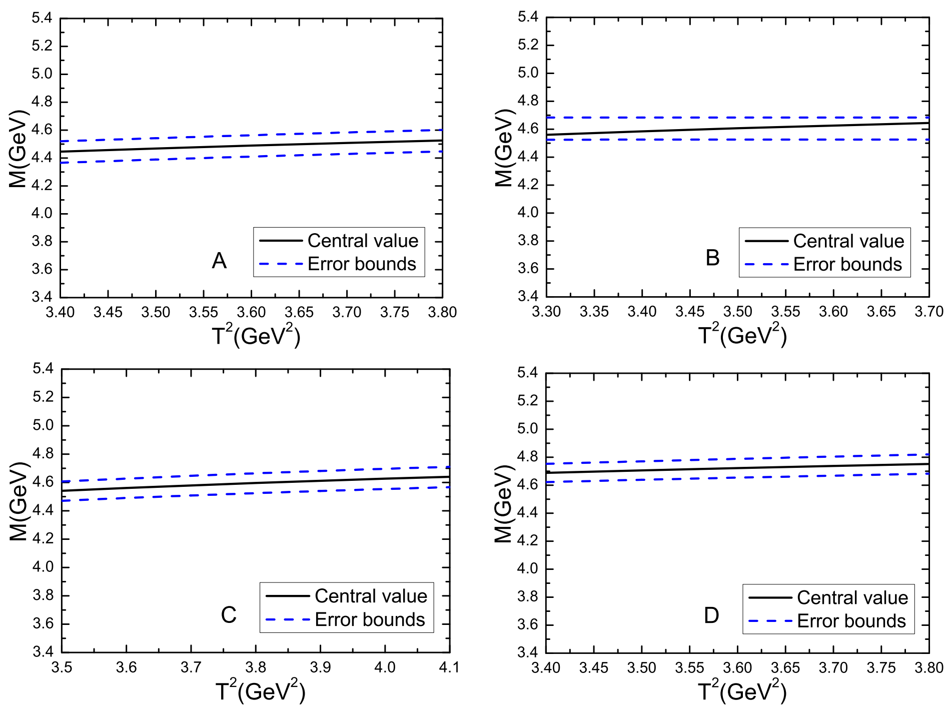

Fig. 1. The masses of the hidden-charm pentaquark states with variations of the Borel parameters T^2, where the A, B, C and D denote the pentaquark states $P_{uusc\bar{c}}(\frac{3}{2}^-)$, $P_{uusc\bar{c}}(\frac{3}{2}^+)$, $P_{ussc\bar{c}}(\frac{3}{2}^-)$ and $P_{ussc\bar{c}}(\frac{3}{2}^+)$, respectively.

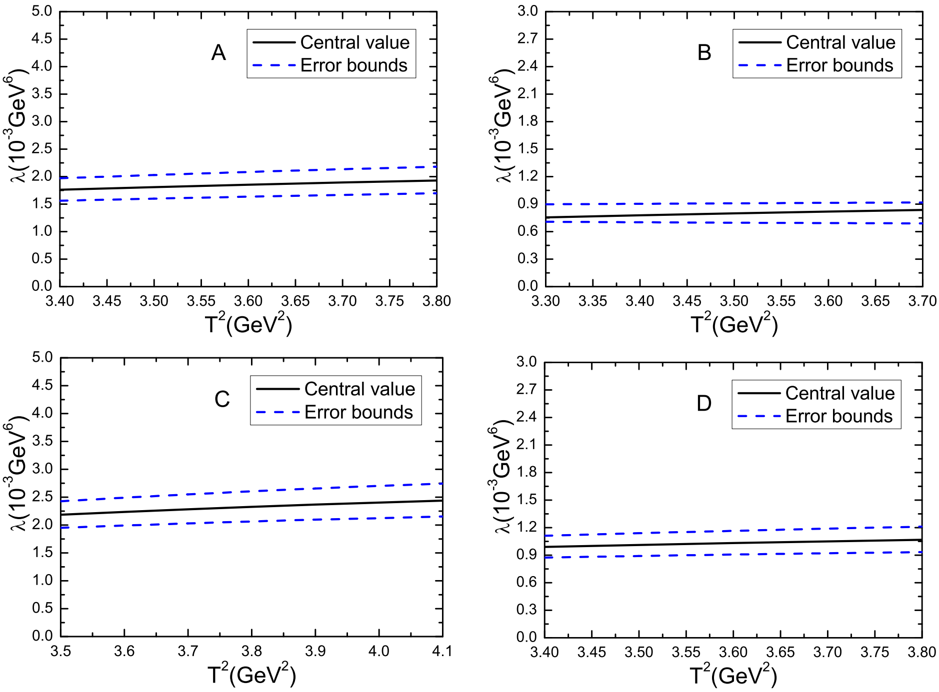

Fig. 2. The pole residues of the hidden-charm pentaquark states with variations of the Borel parameters T^2, where the A, B, C and D denote the pentaquark states $P_{uusc\bar{c}}(\frac{3}{2}^-)$, $P_{uusc\bar{c}}(\frac{3}{2}^+)$, $P_{ussc\bar{c}}(\frac{3}{2}^-)$ and $P_{ussc\bar{c}}(\frac{3}{2}^+)$, respectively.

The following two-body strong decays are the Okubo–Zweig–Iizuka superallowed:

$$P_{uusc\bar{c}}\left(\frac{3}{2}^{\pm}\right) \rightarrow \Sigma^+ J/\psi\,, \qquad \Xi_c^+ \bar{D}^0\,, \qquad \Sigma_c^{++} D_s^-\,, \tag{20}$$

$$P_{ussc\bar{c}}\left(\frac{3}{2}^{\pm}\right) \rightarrow \Xi^0 J/\psi\,, \qquad \Omega_c^+ \bar{D}^-\,, \qquad \Xi_c^+ D_s^-\,. \tag{21}$$

We can search for those P_c states in the $\Sigma^+ J/\psi$, $\Xi_c^+ \bar{D}^0$, $\Sigma_c^{++} D_s^-$, $\Xi^0 J/\psi$, $\Omega_c^+ \bar{D}^-$, $\Xi_c^+ D_s^-$ mass spectrum in the future.

4. Conclusion

In this article, we construct the $S_{\rm L}$–$A_{\rm H}$–$\bar{c}$ type interpolating currents to study the hidden-charm pentaquark states with $J^P = \frac{3}{2}^{\pm}$ and strangeness $S = -1, -2$ by calculating the contributions of the vacuum condensates up to dimension 10 in the operator product expansion. In calculations, we use the formula $\mu = \sqrt{M_P^2 - (2\mathbb{M}_c)^2}$ to determine the ideal energy scales of the QCD spectral densities in a consistent way. We obtain the masses and pole residues of the hidden-charm pentaquark states with the strangeness $S = -1, -2$, the predicted masses can be confronted to the experimental data in the future. On the other hand, we can take the pole residues as basic input parameters to study relevant processes of the hidden-charm pentaquark states with the three-point QCD sum rules.

This work is supported by the National Natural Science Foundation, grant number 11375063 and the Fundamental Research Funds for the Central Universities, grant number 2016MS155.

Appendix

The QCD spectral densities $\rho_j^{i1}(s)$ and $\widetilde{\rho}_j^{i0}(s)$ of the hidden-charm pentaquark states with $j = 0, 3, 4, 5, 6, 8, 9, 10$ are shown as follows:

$$\begin{aligned}
\rho_0^{11}(s) &= \frac{1}{491520\pi^8}\int \mathrm{d}y\mathrm{d}z\, yz(1-y-z)^4\left(s-\hat{m}_c^2\right)^4\left(7s-2\hat{m}_c^2\right)\,, \\
\widetilde{\rho}_0^{10}(s) &= \frac{1}{983040\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^4\left(s-\hat{m}_c^2\right)^4\left(6s-\hat{m}_c^2\right)\,,
\end{aligned} \tag{22}$$

$$\begin{aligned}
\rho_3^{11}(s) &= -\frac{m_c\langle\bar{q}q\rangle}{3072\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2\left(s-\hat{m}_c^2\right)^3 \\
&-\frac{2m_s\langle\bar{q}q\rangle-m_s\langle\bar{s}s\rangle}{3072\pi^6}\int \mathrm{d}y\mathrm{d}z\, yz(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(5s-2\hat{m}_c^2\right)\,, \\
\widetilde{\rho}_3^{10}(s) &= -\frac{m_c\langle\bar{q}q\rangle}{1536\pi^6}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^2\left(s-\hat{m}_c^2\right)^3 \\
&-\frac{2m_s\langle\bar{q}q\rangle-m_s\langle\bar{s}s\rangle}{6144\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(4s-\hat{m}_c^2\right)\,,
\end{aligned} \tag{23}$$

$$
\begin{aligned}
\rho_4^{11}(s) &= -\frac{m_c^2}{73728\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{z^3+y^3}{y^2z^2}(1-y-z)^4\left(s-\hat{m}_c^2\right) \\
&\times\left(2s-\hat{m}_c^2\right)-\frac{19}{7077888\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(y+z)(1-y-z)^3 \\
&\times\left(s-\hat{m}_c^2\right)^2\left(7s-4\hat{m}_c^2\right)+\frac{13}{393216\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,yz(1-y-z)^2 \\
&\times\left(s-\hat{m}_c^2\right)^2\left(5s-2\hat{m}_c^2\right)-\frac{3m_sm_c}{131072\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(y+z) \\
&\times(1-y-z)\left(s-\hat{m}_c^2\right)^2\,, \\
\widetilde{\rho}_4^{10}(s) &= \frac{1}{147456\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{z^3+y^3}{y^2z^2}(1-y-z)^4\left(s-\hat{m}_c^2\right) \\
&\times\left(2s^2-4\hat{m}_c^2s+\hat{m}_c^4\right)-\frac{19}{1179648\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(1-y-z)^3 \\
&\times\left(s-\hat{m}_c^2\right)^2\left(2s-\hat{m}_c^2\right)+\frac{13}{786432\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(y+z) \\
&\times(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(4s-\hat{m}_c^2\right) \\
&-\frac{3m_sm_c}{65536\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(1-y-z)\left(s-\hat{m}_c^2\right)^2\,, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
\rho_5^{11}(s) &= \frac{m_c\langle\bar{s}g_s\sigma Gs\rangle}{196608\pi^6}\int {\rm d}y{\rm d}z\,\frac{z^2+y^2}{yz}(1-y-z)^2(1+2y+2z)\left(s-\hat{m}_c^2\right)^2 \\
&+\frac{19m_c\langle\bar{q}g_s\sigma Gq\rangle}{32768\pi^6}\int {\rm d}y{\rm d}z\,(y+z)(1-y-z)\left(s-\hat{m}_c^2\right)^2 \\
&+\frac{57m_s\langle\bar{q}g_s\sigma Gq\rangle-16m_s\langle\bar{s}g_s\sigma Gs\rangle}{24576\pi^6}\int {\rm d}y{\rm d}z\,yz(1-y-z) \\
&\times\left(s-\hat{m}_c^2\right)\left(2s-\hat{m}_c^2\right)\,, \\
\widetilde{\rho}_5^{10}(s) &= \frac{m_c\langle\bar{s}g_s\sigma Gs\rangle}{196608\pi^6}\int {\rm d}y{\rm d}z\,\frac{y+z}{yz}(1-y-z)^2(1+2y+2z)\left(s-\hat{m}_c^2\right)^2 \\
&+\frac{19m_c\langle\bar{q}g_s\sigma Gq\rangle}{16384\pi^6}\int {\rm d}y{\rm d}z\,(1-y-z)\left(s-\hat{m}_c^2\right)^2 \\
&+\frac{57m_s\langle\bar{q}g_s\sigma Gq\rangle-16m_s\langle\bar{s}g_s\sigma Gs\rangle}{98304\pi^6}\int {\rm d}y{\rm d}z\,(y+z)(1-y-z) \\
&\times\left(s-\hat{m}_c^2\right)\left(3s-\hat{m}_c^2\right)\,, \qquad (25)
\end{aligned}
$$

$$\rho_6^{11}(s) = \frac{\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{96\pi^4}\int \mathrm{d}y\mathrm{d}z\, yz(1-y-z)\left(s-\hat{m}_c^2\right)\left(2s-\hat{m}_c^2\right)$$
$$+\frac{2m_sm_c\langle\bar{q}q\rangle^2 - m_sm_c\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{384\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)\left(s-\hat{m}_c^2\right)\,,$$
$$\widetilde{\rho}_6^{10}(s) = \frac{\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{384\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)\left(s-\hat{m}_c^2\right)\left(3s-\hat{m}_c^2\right)$$
$$+\frac{2m_sm_c\langle\bar{q}q\rangle^2 - m_sm_c\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{192\pi^4}\int \mathrm{d}y\mathrm{d}z\,\left(s-\hat{m}_c^2\right)\,, \tag{26}$$

$$\rho_8^{11}(s) = -\frac{16\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle + 19\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{6144\pi^4}\int \mathrm{d}y\mathrm{d}z\, yz\left(3s-2\hat{m}_c^2\right)$$
$$-\frac{\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle}{12288\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)\left(5s-4\hat{m}_c^2\right)$$
$$-\frac{192m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{q}q\rangle - 32m_sm_c\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle - 57m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{73728\pi^4}$$
$$\times\sqrt{1-4m_c^2/s}\,,$$
$$\widetilde{\rho}_8^{10}(s) = -\frac{16\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle + 19\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{12288\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)\left(2s-\hat{m}_c^2\right)$$
$$-\frac{192m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{q}q\rangle - 32m_sm_c\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle - 57m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{36864\pi^4}$$
$$\times\sqrt{1-4m_c^2/s} - \frac{\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle}{6144\pi^4}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)\left(4s-3\hat{m}_c^2\right)\,, \tag{27}$$

$$\rho_9^{11}(s) = -\frac{m_c\langle\bar{s}s\rangle\langle\bar{q}q\rangle^2}{144\pi^2}\sqrt{1-4m_c^2/s}\,,$$
$$\widetilde{\rho}_9^{10}(s) = -\frac{m_c\langle\bar{s}s\rangle\langle\bar{q}q\rangle^2}{72\pi^2}\sqrt{1-4m_c^2/s}\,, \tag{28}$$
$$\rho_{10}^{11}(s) = \frac{19\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{24576\pi^4}\int \mathrm{d}y\, y(1-y)\left\{2+s\delta\left(s-\widetilde{m}_c^2\right)\right\}$$
$$+\frac{17\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{442368\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)\left\{4+s\delta\left(s-\hat{m}_c^2\right)\right\}$$
$$+\frac{48m_sm_c\langle\bar{q}g_s\sigma Gq\rangle^2 - 11m_sm_c\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{147456\pi^4}$$
$$\int \mathrm{d}y\,\left(3+\frac{s}{T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)\,,$$

$$
\begin{aligned}
\widetilde{\rho}_{10}^{10}(s) &= \frac{19\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{49152\pi^4}\int \mathrm{d}y\,\left\{1+s\delta\left(s-\widetilde{m}_c^2\right)\right\} \\
&+\frac{17\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{221184\pi^4}\int \mathrm{d}y\mathrm{d}z\,\left\{3+s\delta\left(s-\hat{m}_c^2\right)\right\} \\
&+\frac{48m_sm_c\langle\bar{q}g_s\sigma Gq\rangle^2-19m_sm_c\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{73728\pi^4} \\
&\times\int \mathrm{d}y\,\left(2+\frac{s}{T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)\,, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\rho_0^{21}(s) &= \frac{1}{491520\pi^8}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^4\left(s-\hat{m}_c^2\right)^4\left(7s-2\hat{m}_c^2\right) \\
&+\frac{m_sm_c}{49152\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^3\left(s-\hat{m}_c^2\right)^4\,, \\
\widetilde{\rho}_0^{20}(s) &= \frac{1}{983040\pi^8}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^4\left(s-\hat{m}_c^2\right)^4\left(6s-\hat{m}_c^2\right) \\
&+\frac{m_sm_c}{24576\pi^8}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^3\left(s-\hat{m}_c^2\right)^4\,, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\rho_3^{21}(s) &= -\frac{m_c\langle\bar{q}q\rangle}{3072\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2\left(s-\hat{m}_c^2\right)^3 \\
&+\frac{m_s\langle\bar{s}s\rangle-m_s\langle\bar{q}q\rangle}{1536\pi^6}\int \mathrm{d}y\mathrm{d}z\,yz(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(5s-2\hat{m}_c^2\right)\,, \\
\widetilde{\rho}_3^{20}(s) &= -\frac{m_c\langle\bar{q}q\rangle}{1536\pi^6}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)^2\left(s-\hat{m}_c^2\right)^3 \\
&+\frac{m_s\langle\bar{s}s\rangle-m_s\langle\bar{q}q\rangle}{3072\pi^6}\int \mathrm{d}y\mathrm{d}z\,(y+z)(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(4s-\hat{m}_c^2\right)\,, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\rho_4^{21}(s) &= -\frac{m_c^2}{73728\pi^6}\left\langle\frac{\alpha_{\mathrm{s}}GG}{\pi}\right\rangle\int \mathrm{d}y\mathrm{d}z\,\frac{y^3+z^3}{y^2z^2} \\
&\times(1-y-z)^4\left(s-\hat{m}_c^2\right)\left(2s-\hat{m}_c^2\right) \\
&-\frac{m_sm_c}{73728\pi^6}\left\langle\frac{\alpha_{\mathrm{s}}GG}{\pi}\right\rangle\int \mathrm{d}y\mathrm{d}z\,\frac{y^2+z^2}{yz}(1-y-z)^3\left(s-\hat{m}_c^2\right)^2\left(5s-3\hat{m}_c^2\right) \\
&+\frac{m_sm_c}{36864\pi^6}\left\langle\frac{\alpha_{\mathrm{s}}GG}{\pi}\right\rangle\int \mathrm{d}y\mathrm{d}z\,\frac{y^3+z^3}{y^2z^2}(1-y-z)^3\left(s-\hat{m}_c^2\right)^2 \\
&-\frac{m_sm_c^3}{36864\pi^6}\left\langle\frac{\alpha_{\mathrm{s}}GG}{\pi}\right\rangle\int \mathrm{d}y\mathrm{d}z\,\frac{y^2+z^2}{y^2z^2}(1-y-z)^3\left(s-\hat{m}_c^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{19}{7077888\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(y+z)(1-y-z)^3\left(s-\hat{m}_c^2\right)^2\left(7s-4\hat{m}_c^2\right)\\
&-\frac{5}{98304\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,yz(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(5s-2\hat{m}_c^2\right)\\
&+\frac{m_sm_c}{8192\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(y+z)(1-y-z)\left(s-\hat{m}_c^2\right)^2\\
&-\frac{m_sm_c}{131072\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y^2+z^2}{yz}(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\\
&+\frac{m_sm_c}{98304\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y^2+z^2}{yz}(1-y-z)^3\left(s-\hat{m}_c^2\right)\left(5s-3\hat{m}_c^2\right)\,,\\
&\widetilde{\rho}_4^{20}(s)=\frac{1}{147456\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y^3+z^3}{y^2z^2}\\
&\times(1-y-z)^4\left(s-\hat{m}_c^2\right)\left(2s^2-4\hat{m}_c^2+\hat{m}_c^4\right)\\
&+\frac{5m_sm_c}{73728\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y^2+z^2}{y^2z^2}(1-y-z)^3\left(s-\hat{m}_c^2\right)^2\\
&-\frac{m_sm_c}{36864\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y+z}{yz}(1-y-z)^3\left(s-\hat{m}_c^2\right)^2\\
&-\frac{m_sm_c}{36864\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(1-y-z)^3(2-y-z)\left(s-\hat{m}_c^2\right)s\\
&-\frac{m_sm_c^3}{36864\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y^3+z^3}{y^3z^3}(1-y-z)^3\left(s-\hat{m}_c^2\right)\\
&-\frac{19}{1179648\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(1-y-z)^3\left(s-\hat{m}_c^2\right)^2\left(2s-\hat{m}_c^2\right)\\
&+\frac{1}{73728\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(y+z)(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(4s-\hat{m}_c^2\right)\\
&+\frac{m_sm_c}{4096\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,(1-y-z)\left(s-\hat{m}_c^2\right)^2\\
&-\frac{3}{393216\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,yz(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\left(4s-\hat{m}_c^2\right)\\
&-\frac{m_sm_c}{131072\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y+z}{yz}(1-y-z)^2\left(s-\hat{m}_c^2\right)^2\\
&+\frac{m_sm_c}{49152\pi^6}\left\langle\frac{\alpha_{\rm s}GG}{\pi}\right\rangle\int {\rm d}y{\rm d}z\,\frac{y+z}{yz}(1-y-z)^3\left(s-\hat{m}_c^2\right)\left(2s-\hat{m}_c^2\right)\,,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\rho_5^{21}(s) &= \frac{19m_c \langle\bar{s}g_s\sigma Gs\rangle}{32768\pi^6}\int dydz\,(y+z)(1-y-z)\left(s-\hat{m}_c^2\right)^2 \\
&+\frac{3m_s\langle\bar{q}g_s\sigma Gq\rangle-2m_s\langle\bar{s}g_s\sigma Gs\rangle}{1536\pi^6}\int dydz\,yz(1-y-z)\left(s-\hat{m}_c^2\right)\left(2s-\hat{m}_c^2\right) \\
&+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{32768\pi^6}\int dydz\,(y+z)(1-y-z)^2\left(s-\hat{m}_c^2\right)\left(3s-\hat{m}_c^2\right) \\
&+\frac{m_c\langle\bar{q}g_s\sigma Gq\rangle}{196608\pi^6}\int dydz\,\frac{y^2+z^2}{yz}(1-y-z)^2(1+2y+2z)\left(s-\hat{m}_c^2\right)^2\,, \\
\widetilde{\rho}_5^{20}(s) &= \frac{19m_c\langle\bar{s}g_s\sigma Gs\rangle}{32768\pi^6}\int dydz\,(1-y-z)\left(s-\hat{m}_c^2\right)^2 \\
&+\frac{3m_s\langle\bar{q}g_s\sigma Gq\rangle-2m_s\langle\bar{s}g_s\sigma Gs\rangle}{6144\pi^6}\int dydz\,(y+z) \\
&\times(1-y-z)\left(s-\hat{m}_c^2\right)\left(3s-\hat{m}_c^2\right) \\
&+\frac{m_s\langle\bar{q}g_s\sigma Gq\rangle}{16384\pi^6}\int dydz\,(1-y-z)^2\left(s-\hat{m}_c^2\right)\left(5s-3\hat{m}_c^2\right) \\
&-\frac{m_c\langle\bar{q}g_s\sigma Gq\rangle}{196608\pi^6}\int dydz\,\frac{y+z}{yz}(1-y-z)^2(1-4y-4z)\left(s-\hat{m}_c^2\right)^2\,, \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
\rho_6^{21}(s) &= \frac{\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{96\pi^4}\int dydz\,yz(1-y-z)\left(s-\hat{m}_c^2\right)\left(2s-\hat{m}_c^2\right) \\
&+\frac{4m_sm_c\langle\bar{s}s\rangle\langle\bar{q}q\rangle-m_sm_c\langle\bar{s}s\rangle^2}{384\pi^4}\int dydz\,(y+z)\left(s-\hat{m}_c^2\right)\,, \\
\widetilde{\rho}_6^{20}(s) &= \frac{\langle\bar{s}s\rangle\langle\bar{q}q\rangle}{384\pi^4}\int dydz\,(y+z)(1-y-z)\left(s-\hat{m}_c^2\right)\left(3s-\hat{m}_c^2\right) \\
&+\frac{4m_sm_c\langle\bar{s}s\rangle\langle\bar{q}q\rangle-m_sm_c\langle\bar{s}s\rangle^2}{192\pi^4}\int dydz\,\left(s-\hat{m}_c^2\right)\,,
\end{aligned}
$$

$$
\begin{aligned}
\rho_8^{21}(s) &= -\frac{19\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle+16\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{6144\pi^4}\int dydz\,yz\left(3s-2\hat{m}_c^2\right) \\
&+\frac{5m_sm_c\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{s}s\rangle-12m_sm_c\langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}q\rangle-12m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{4608\pi^4} \\
&\times\sqrt{1-4m_c^2/s}-\frac{\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{12288\pi^4}\int dydz\,(y+z)(1-y-z)\left(5s-4\hat{m}_c^2\right) \\
&-\frac{m_sm_c\langle\bar{q}g_s\sigma Gq\rangle\langle\bar{s}s\rangle}{12288\pi^4}\int dydz\,\frac{y^2+z^2}{yz}(1-2y-2z)\,, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\rho}_8^{20}(s) &= -\frac{19\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}q\right\rangle+16\left\langle\bar{q}g_s\sigma Gq\right\rangle\left\langle\bar{s}s\right\rangle}{12288\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)\left(2s-\widehat{m}_c^2\right)\\
&+\frac{5m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{s}s\right\rangle-12m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}q\right\rangle-12m_sm_c\left\langle\bar{q}g_s\sigma Gq\right\rangle\left\langle\bar{s}s\right\rangle}{2304\pi^4}\\
&\times\sqrt{1-4m_c^2/s}-\frac{\left\langle\bar{q}g_s\sigma Gq\right\rangle\left\langle\bar{s}s\right\rangle}{6144\pi^4}\int \mathrm{d}y\mathrm{d}z\,(1-y-z)\left(4s-3\widehat{m}_c^2\right)\\
&-\frac{m_sm_c\left\langle\bar{q}g_s\sigma Gq\right\rangle\left\langle\bar{s}s\right\rangle}{12288\pi^4}\int \mathrm{d}y\mathrm{d}z\,\frac{y+z}{yz}(1-2y-2z)\,, \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
\rho_9^{21}(s) &= -\frac{m_c\left\langle\bar{s}s\right\rangle^2\left\langle\bar{q}q\right\rangle}{144\pi^2}\sqrt{1-4m_c^2/s}\\
&+\frac{m_s\left\langle\bar{s}s\right\rangle^2\left\langle\bar{q}q\right\rangle}{144\pi^2}\int \mathrm{d}y\,y(1-y)\left\{2+s\delta\left(s-\widetilde{m}_c^2\right)\right\}\,,\\
\widetilde{\rho}_9^{20}(s) &= -\frac{m_c\left\langle\bar{s}s\right\rangle^2\left\langle\bar{q}q\right\rangle}{72\pi^2}\sqrt{1-4m_c^2/s}\\
&+\frac{m_s\left\langle\bar{s}s\right\rangle^2\left\langle\bar{q}q\right\rangle}{288\pi^2}\int \mathrm{d}y\left\{1+s\delta\left(s-\widetilde{m}_c^2\right)\right\}, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\rho_{10}^{21}(s) &= \frac{19\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{24576\pi^4}\int \mathrm{d}y\,y(1-y)\left\{2+s\delta\left(s-\widetilde{m}_c^2\right)\right\}\\
&+\frac{6m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle-m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle^2}{9216\pi^4}\int \mathrm{d}y\left(2+\frac{s}{T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)\\
&+\frac{17\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{442368\pi^4}\int \mathrm{d}y\mathrm{d}z\,(y+z)\left\{4+s\delta\left(s-\widehat{m}_c^2\right)\right\}\\
&-\frac{m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{36864\pi^4}\int \mathrm{d}y\,\frac{1-y}{y}\delta\left(s-\widetilde{m}_c^2\right)\\
&+\frac{m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{36864\pi^4}\int \mathrm{d}y\mathrm{d}z\,\frac{y^2+z^2}{yz}\delta\left(s-\widehat{m}_c^2\right)\,,\\
\widetilde{\rho}_{10}^{20}(s) &= \frac{19\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{49152\pi^4}\int \mathrm{d}y\,y(1-y)\left\{1+s\delta\left(s-\widetilde{m}_c^2\right)\right\}\\
&+\frac{6m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle-m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle^2}{4608\pi^4}\int \mathrm{d}y\left(1+\frac{s}{T^2}\right)\delta\left(s-\widetilde{m}_c^2\right)\\
&+\frac{17\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{221184\pi^4}\int \mathrm{d}y\mathrm{d}z\,\left\{3+s\delta\left(s-\widehat{m}_c^2\right)\right\}\\
&-\frac{m_sm_c\left\langle\bar{s}g_s\sigma Gs\right\rangle\left\langle\bar{q}g_s\sigma Gq\right\rangle}{36864\pi^4}\int \mathrm{d}y\,\frac{1}{y}\delta\left(s-\widetilde{m}_c^2\right)
\end{aligned}
$$

$$+\frac{m_s m_c \langle\bar{s}g_s\sigma Gs\rangle\langle\bar{q}g_s\sigma Gq\rangle}{36864\pi^4}\int \mathrm{d}y\mathrm{d}z\,\frac{y+z}{yz}\delta\left(s-\hat{m}_c^2\right)\,, \qquad (37)$$

$\int \mathrm{d}y\mathrm{d}z = \int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z$, $y_f = \frac{1+\sqrt{1-4m_c^2/s}}{2}$, $y_i = \frac{1-\sqrt{1-4m_c^2/s}}{2}$, $z_i = \frac{ym_c^2}{ys-m_c^2}$, $\hat{m}_c^2 = \frac{(y+z)m_c^2}{yz}$, $\widetilde{m}_c^2 = \frac{m_c^2}{y(1-y)}$, $\int_{y_i}^{y_f}\mathrm{d}y \to \int_0^1 \mathrm{d}y$, $\int_{z_i}^{1-y}\mathrm{d}z \to \int_0^{1-y}\mathrm{d}z$, when the δ functions $\delta(s-\hat{m}_c^2)$ and $\delta(s-\widetilde{m}_c^2)$ appear.

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