SYMMETRY ANALYSIS AND SOME NEW EXACT SOLUTIONS OF THE (2+1)-DIMENSIONAL BURGERS EQUATIONS

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In this paper, the Lie point symmetry analysis method is used to investigate the (2+1)-dimensional Burgers equations. We have obtained the optimal system of Lie subalgebras. Some new exact solutions for the (2+1)-dimensional Burgers equations are obtained.

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1. Introduction

The solutions of nonlinear differential equations are an essential tool for many physical and engineering applications. There are many methods to solve nonlinear partial differential equations (PDEs) such as the Weierstrass function method [1], Jacobi elliptic function method [2, 3], Hirota bilinear method [4], the inverse scattering method [5], the tanh method [6], the extended mapping transformation method [7], the truncated expansion method [8], the simplest equation method [9], the bifurcation method [10] and Lie symmetry method [11–14]. The latter is considered as the most powerful method for getting exact solutions of PDEs.

In this paper, we use the Lie symmetry method to investigate the (2+1)-dimensional Burgers equations [15]

$$\begin{aligned} u_t &= uu_y + \lambda v u_x + \mu u_{yy} + \lambda\mu u_{xx}\,, \\ u_x &= v_y\,, \end{aligned} \tag{1.1}$$

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where $u = u(x,y,t)$ and $v = v(x,y,t)$, μ and λ are real constants. When $x = y$ and $u = v$, Eq. (1.1) degenerates to the famous one-dimensional Burgers equation

$$u_t = muu_x + nu_{xx}\,, \tag{1.2}$$

where $m = \lambda+1$ and $n = \mu(\lambda+1)$. Burgers' equation (1.2) is widely used for describing physical phenomena in fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. For example, it is considered as the lowest order approximation for the one-dimensional propagation of weak shock waves in fluids [16]. Burgers' equations (1.1) are a generalization of Burgers' equation (1.2) and its equivalent form is derived from the Painleve integrability classification in [17].

Many types of exact solutions for Eq. (1.1) are obtained in [15, 18–25]. Soliton and soliton-like solutions are obtained in [15, 18]. Periodic and doubly periodic solutions are obtained in [19–21]. In [22, 23], variable separation solutions are obtained. Interaction between kink solitary wave and rogue wave is investigated in [24]. Residual symmetry analysis is investigated in [25]. In this paper, we concentrate on finding new similarity solutions of Eq. (1.1).

The sequence of this paper is as follows: In Section 2, we use the symmetry analysis of (1.1) to find all Lie algebra of symmetry generators. In Section 3, we obtain the optimal one-dimensional system of these subalgebras. In Section 4, we obtain exact solutions of the reduced equation that is produced from the infinitesimal transformations.

2. Symmetry analysis of Eq. (1.1)

The infinitesimal generator Γ of the Lie point transformations is given by

$$\Gamma = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}\,. \tag{2.1}$$

We use Maple to obtain the infinitesimal symmetry generators by solving the determining equations that are produced from the invariant condition $\Gamma^{(2)}\Delta\,|_{\Delta=0} = 0$ (this condition is defined in [26]). We obtain

$$\begin{aligned}
T &= \frac{1}{2}c_5t^2 + c_4t + c_1\,,\\
X &= F(t) + \frac{1}{2}(c_5t + c_4)x\,,\\
Y &= \frac{1}{2}(c_5y + 2c_3)t + \frac{1}{2}c_4y + c_2\,,\\
U &= -\frac{1}{2}c_5(tu + y) - \frac{1}{2}c_4u - c_3\,,\\
V &= -\frac{1}{2}(c_5t + c_4)v - \frac{1}{\lambda}F'(t) - \frac{1}{2\lambda}c_5x\,,
\end{aligned} \tag{2.2}$$

where c_1, c_2, c_3, c_4 and c_5 are constants, $F(t)$ is an arbitrary function and $F'(t)$ is its derivative with respect to t. In [27], the nonlocal symmetry analysis of Eq. (1.1) is investigated. The authors in [27] have obtained infinitesimals (2.2), however, they considered only the case of $F(t) = c_6t + c_7$. Here, we consider $F(t)$ as an arbitrary function of t in order to obtain some new similarity solutions of Eq. (1.1). The Lie algebra of infinitesimal symmetry generators is spanned by five-dimensional and the infinite-dimensional subalgebras

$$\begin{aligned}
v_1 &= \partial_t\,,\\
v_2 &= \partial_y\,,\\
v_3 &= t\partial_y - \partial_u\,,\\
v_4 &= 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v\,,\\
v_5 &= t^2\partial_t + tx\partial_x + ty\partial_y - (tu + y)\partial_u - \left(tv + \frac{x}{\lambda}\right)\partial_v\,,\\
v_f &= F(t)\partial_x - \frac{F'(t)}{\lambda}\partial_v\,.
\end{aligned} \tag{2.3}$$

3. Classification of group invariant solutions

To find the one-dimensional optimal system of the five-dimensional subalgebras (2.3), we follow the procedure described in [28]. To achieve this task, we use the commutator table shown in Table I and the table of adjoint shown in Table II.

Let $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5$ be an arbitrary element of the subalgebra (2.3). The invariant function such that $\phi(\mathrm{Ad}_g(v)) = \phi(v)$, where $v \in \mathcal{G}$ ($\mathcal{G}$ is a five-dimensional Lie algebra of (1.1) generated by $v_1, \cdots, v_5$), $g \in G$ (G is the corresponding symmetry group of $\mathcal{G}$) and the adjoint action defined in [28] is given by

$$\phi(v) = a_4^2 - a_1a_5\,. \tag{3.1}$$

TABLE I

Table of commutators.

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5
v_1	0	0	v_2	$2v_1$	v_4
v_2	0	0	0	v_2	v_3
v_3	$-v_2$	0	0	$-v_3$	0
v_4	$-2v_1$	$-v_2$	v_3	0	$2v_5$
v_5	$-v_4$	$-v_3$	0	$-2v_5$	0

TABLE II

Table of adjoint.

$\mathrm{Ad}(e^{\epsilon v_i})v_j$	v_1	v_2	v_3	v_4	v_5
v_1	v_1	v_2	$v_3-\epsilon v_2$	$v_4-2\epsilon v_1$	$v_5-\epsilon v_4+\epsilon^2 v_1$
v_2	v_1	v_2	v_3	$v_4-\epsilon v_2+\frac{1}{2}\epsilon^2 v_2$	$v_5-\epsilon v_3$
v_3	$v_1+\epsilon v_2$	v_2	v_3	$v_4+\epsilon v_3$	v_5
v_4	$e^{2\epsilon}v_1$	$e^{\epsilon}v_2$	$e^{-\epsilon}v_3$	v_4	$e^{-2\epsilon}v_5$
v_5	$v_1+\epsilon v_4+\epsilon^2 v_5$	$v_2+\epsilon v_3$	v_3	$v_4+2\epsilon v_5$	v_5

There are three cases to be considered: $\phi(v) > 0$, $\phi(v) < 0$ and $\phi(v) = 0$.

— *Case 1*: If $\phi(v) > 0$, then we put $a_1 = a_5 = 0$ and $a_4 = 1$. By adjoint action $\mathrm{Ad}(e^{\epsilon v_3} v)$, if we take $\epsilon = -a_3$, then v is equivalent to

$$v = a_2 v_2 + v_4 \,. \tag{3.2}$$

— *Case 2*: If $\phi(v) < 0$, then we put $a_4 = 0$ and $a_1 = a_5 = 1$. By adjoint action $\mathrm{Ad}(e^{\epsilon_1 v_2} v)$, if we take $\epsilon_1 = a_3$, then v is equivalent to $v = v_1 + a_2 v_2 + v_5$. Another adjoint action $\mathrm{Ad}(e^{\epsilon_2 v_3} v)$, if we take $\epsilon_2 = -a_2$, then v is equivalent to

$$v = v_1 + v_5 \,. \tag{3.3}$$

— *Case 3*: If $\phi(v) = 0$, then there are three subcases.

— *Subcase 1*: We put $a_4 = a_5 = 0$ and $a_1 = 1$. By adjoint action $\mathrm{Ad}(e^{\epsilon v_1} v)$, if we take $\epsilon = \frac{a_2}{a_3}$, then v is equivalent to

$$v = v_1 + a_3 v_3 \,. \tag{3.4}$$

— *Subcase 2*: We put $a_4 = a_1 = 0$ and $a_5 = 1$. By adjoint action $\mathrm{Ad}(e^{\epsilon v_5} v)$, if we take $\epsilon = -\frac{a_3}{a_2}$, then v is equivalent to

$$v = a_2 v_2 + v_5 \,. \tag{3.5}$$

— *Subcase 3*: We put $a_1 = a_4 = a_5 = 0$. By adjoint action $\mathrm{Ad}(e^{\epsilon v_5} v)$, if we take $\epsilon = -\frac{a_3}{a_2}$, then v is equivalent to

$$v = v_2 \,, \tag{3.6}$$

or by adjoint action $\mathrm{Ad}(e^{\epsilon v_1} v)$, if we take $\epsilon = \frac{a_2}{a_3}$, then v is equivalent to

$$v = v_3 \,. \tag{3.7}$$

The optimal system of one-dimensional subalgebras is as follows:

$$\begin{aligned} & a_2 v_2 + v_4 \,, \\ & v_1 + v_5 \,, \\ & v_1 + a_3 v_3 \,, \\ & a_2 v_2 + v_5 \,, \\ & v_2 \,, \\ & v_3 \,. \end{aligned} \tag{3.8}$$

We apply these subalgebras to (1.1) and find exact solutions.

4. Group invariant solutions

4.1. Exact solution using the generator $v_1 + a_3 v_3$

In this subsection, we consider the subalgebra $v_1 + a_3 v_3$ and take $a_3 = 1$. In this case, the invariant surface conditions are given by

$$\begin{aligned} t u_y + u_t &= -1 \,, \\ t v_y + v_t &= 70 \,. \end{aligned} \tag{4.1}$$

Solving (4.1), we find

$$\begin{aligned} u &= -t + g(x, r) \,, \\ v &= h(x, r) \,, \end{aligned} \tag{4.2}$$

where $r = t^2 - 2y$. By substituting (4.2) into (1.1), we get

$$\begin{aligned} 1 - 2 g g_r + 4\mu g_{rr} + \lambda h g_x + \lambda \mu g_{xx} &= 0 \,, \\ 2 h_r + g_x &= 0 \,. \end{aligned} \tag{4.3}$$

To obtain the travelling wave solutions of (4.3), we suppose that

$$g = G(z) \,, \qquad h = H(z) \,, \qquad z = x + ar \,. \tag{4.4}$$

Substituting (4.4) into (4.3), we obtain

$$1 + (\lambda H - 2aG)G' + \left(4\mu a^2 + \lambda\mu\right) G'' = 0\,, \tag{4.5a}$$

$$2aH' + G' = 0\,. \tag{4.5b}$$

Integrating (4.5b) with respect to z, one finds

$$H = -\frac{1}{2a}G + c_1\,. \tag{4.6}$$

Let us substitute (4.6) into (4.5a), to get

$$1 - \left(\frac{\lambda}{2a} + 2a\right) GG' + \left(4\mu a^2 + \lambda\mu\right) G'' + c_1\lambda G' = 0\,. \tag{4.7}$$

By integrating (4.7) with respect to z, we find

$$z - \left(\frac{\lambda}{2a} + 2a\right) \frac{G^2}{2} + \left(4\mu a^2 + \lambda\mu\right) G' + \lambda c_1 G = c_2\,, \tag{4.8}$$

where c_1 and c_2 are integration constants. The solution of (4.8) is given by

$$G(z) = \frac{C}{2A} - \left(\frac{B}{A^2}\right)^{\frac{1}{3}} \frac{Bi'(k) + c_3 Ai'(k)}{Bi(k) + c_3 Ai(k)}\,, \tag{4.9}$$

where $A = \frac{\lambda}{4a} + a$, $B = 4\mu a^2 + \lambda\mu$, $C = \lambda c_1$, $k = \frac{1}{4(AB)^{\frac{2}{3}}}\left(C^2 - 4Ac_2 + 4Az\right)$, c_3 is the integration constant, $Ai(k)$ and $Bi(k)$ are the Airy functions defined by [29]

$$\begin{aligned} Ai(k) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + kt\right) \mathrm{d}t\,, \\ Bi(k) &= \frac{1}{\pi}\int_0^\infty \left[e^{-\frac{1}{3}t^3 + kt} + \sin\left(\frac{1}{3}t^3 + kt\right)\right] \mathrm{d}t\,. \end{aligned} \tag{4.10}$$

In this case, the solution of (1.1) is given by

$$\begin{aligned} u(x,y,t) &= -t + \frac{C}{2A} - \left(\frac{B}{A^2}\right)^{\frac{1}{3}} \frac{Bi'(k) + c_3 Ai'(k)}{Bi(k) + c_3 Ai(k)}\,, \\ v(x,y,t) &= -\frac{a}{2b}\left(\frac{C}{2A} - \left(\frac{B}{A^2}\right)^{\frac{1}{3}} \frac{Bi'(k) + c_3 Ai'(k)}{Bi(k) + c_3 Ai(k)}\right) + c_1\,. \end{aligned} \tag{4.11}$$

4.2. Exact solution using the generator $a_2v_2 + v_5$

In this subsection, we consider the subalgebra $a_2v_2+v_5$ and take $a_2 = 0$. In this case, the invariant surface conditions are given by

$$\begin{aligned} txu_x + tyu_y + t^2u_t &= -(tu+y)\,, \\ txv_x + tyv_y + t^2v_t &= -\left(tv + \frac{x}{\lambda}\right) . \end{aligned} \tag{4.12}$$

Solving (4.12), we find

$$\begin{aligned} u &= -\frac{r}{s} + \frac{1}{x}g(r,s)\,, \\ v &= -\frac{1}{\lambda s} + \frac{1}{x}h(r,s)\,, \end{aligned} \tag{4.13}$$

where $r = \frac{y}{x}$, $s = \frac{t}{x}$. Substituting (4.13) into (1.1), we get

$$\begin{aligned} &g(2\lambda\mu - \lambda h + g_r) \\ &+\lambda\left[(4\mu - h)(sg_s + rg_r) + \mu\left(s^2g_{ss} + r^2g_{rr} + 2rsg_{rs}\right)\right] + \mu g_{rr} = 0\,, \\ &sg_s + rg_r + g + h_r = 0\,. \end{aligned} \tag{4.14}$$

To obtain the travelling wave solutions of (4.14), we assume that

$$h = H(z)\,, \qquad g = G(z)\,, \qquad z = r + s\,. \tag{4.15}$$

Substituting (4.15) into (4.14), we find

$$G(2\lambda\mu - \lambda H + G') + \lambda\left[(4\mu - H)zG' + \mu z^2G''\right] + \mu G'' = 0\,, \tag{4.16a}$$

$$zG' + G + H' = 0\,, \tag{4.16b}$$

where G', G'', H' and H'' are the derivative with respect to z. By integrating (4.16b) with respect to z and considering the integration constant to be zero, we obtain

$$H = -zG\,. \tag{4.17}$$

Let us substitute (4.17) into (4.16a), to obtain

$$2\lambda\mu\left[G + zG'\right] + \lambda\mu\left[2zG' + z^2G''\right] + \lambda\left[zG^2 + z^2GG'\right] + GG' + \mu G'' = 0\,. \tag{4.18}$$

By integrating (4.18) with respect to z and considering the integration constant to be zero, we find

$$2\lambda\mu zG + \lambda\mu z^2G' + \frac{\lambda}{2}z^2G^2 + \frac{1}{2}G^2 + \mu G' = 0\,. \tag{4.19}$$

Equation (4.19) has the following solution:

$$G(z)=\frac{2\mu\sqrt{\lambda}}{(1+\lambda z^2)\left(2c_1\mu\sqrt{\lambda}+\tan^{-1}\sqrt{\lambda}z\right)}\,, \tag{4.20}$$

where c_1 is the integration constant. Finally, the solution of (1.1) is given by

$$\begin{aligned} u(x,y,t) &= -\frac{y}{t}+\frac{2\mu\sqrt{\lambda}}{x\left((1+\lambda z^2)\left(2c_1\mu\sqrt{\lambda}+\tan^{-1}\sqrt{\lambda}z\right)\right)}\,,\\ v(x,y,t) &= -\frac{x}{\lambda t}-\frac{2\mu\sqrt{\lambda}z}{x\left((1+\lambda z^2)\left(2c_1\mu\sqrt{\lambda}+\tan^{-1}\sqrt{\lambda}z\right)\right)}\,. \end{aligned} \tag{4.21}$$

Now, we are interested in the infinite-dimensional subalgebra that is ignored in optimal system calculations.

4.3. Exact solution using the generator v_f

The invariant surface conditions, in this case, are given by

$$\begin{aligned} F(t)u_x &= 0\,,\\ F(t)v_x &= -\frac{F'(t)}{\lambda}\,. \end{aligned} \tag{4.22}$$

Solving (4.22), we find

$$\begin{aligned} u &= g(y,t)\,,\\ v &= -\frac{F'(t)}{\lambda F(t)}x+h(y,t)\,. \end{aligned} \tag{4.23}$$

Substituting (4.23) into (1.1), we get

$$g_t=gg_y+\mu g_{yy}\,, \tag{4.24a}$$

$$h_y=0\,. \tag{4.24b}$$

Equation (4.24a) is the one-dimensional Burgers equation. This equation has many famous solutions (see, for example, [28]) and we will not list them here.

4.4. Exact solution using a linear combination of v_f, v_1, v_2

In this case, the invariant surface conditions are given by

$$\begin{aligned} F(t)u_x + u_y + u_t &= 0\,, \\ F(t)v_x + v_y + v_t &= -\frac{F'(t)}{\lambda}\,. \end{aligned} \tag{4.25}$$

Solving (4.25), we obtain

$$\begin{aligned} u &= g(r,s)\,, \\ v &= -\frac{F(t)}{\lambda} + h(r,s)\,, \end{aligned} \tag{4.26}$$

where $r = -y + t$ and $s = -\int F(t)\,\mathrm{d}t + x$. Substituting (4.26) into (1.1), we get

$$\begin{aligned} \lambda h g_s + \lambda\mu g_{ss} - (1+g)g_r + \mu g_{rr} &= 0\,, \\ g_s + h_r &= 0\,. \end{aligned} \tag{4.27}$$

To obtain exact solutions of (4.27), we apply the Lie symmetry analysis to it. In this case, we obtain the following infinitesimal generators:

$$\begin{aligned} \Gamma_1 &= -(g+1)\partial_g - h\partial_h + r\partial_r + s\partial_s\,, \\ \Gamma_2 &= \partial_r\,, \\ \Gamma_3 &= \partial_s\,. \end{aligned} \tag{4.28}$$

We use these generators in the following subsection to obtain exact solutions of (4.27).

4.4.1. The infinitesimal generator Γ_1 of (4.27)

The solution of the invariant surface conditions, in this case, are given by

$$\begin{aligned} g(r,s) &= -1 + \frac{1}{r}G(z)\,, \\ h(r,s) &= \frac{1}{r}H(z)\,, \end{aligned} \tag{4.29}$$

where $z = \frac{s}{r}$. Substituting (4.29) into (4.27), we find

$$G(2\mu + G) + (4\mu z + zG + \lambda H)G' + \mu(z^2 + \lambda)G'' = 0\,, \tag{4.30a}$$

$$-H + G' - zH' = 0\,. \tag{4.30b}$$

Integrate (4.30b) and consider the constant of integration equal $-\mu$ to get

$$H=\frac{G+\mu}{z}\,. \tag{4.31}$$

Substituting (4.31) into (4.30a), we obtain

$$\begin{aligned}&\left(2\mu zG+\mu z^2G'\right)+\tfrac{1}{2}\left(2zG^2+2z^2GG'\right)\\&+\left(3\mu z^2G'+\mu z^3G''\right)+\left(\mu\lambda zG''+\mu\lambda G'\right)+\lambda GG'=0\,.\end{aligned} \tag{4.32}$$

By integrating (4.32) and considering the integration constant to be zero, we find

$$\mu z^2G+\frac{1}{2}z^2G^2+\mu z^3G'+\mu\lambda zG'+\frac{\lambda}{2}G^2=0\,. \tag{4.33}$$

Equation (4.33) has the following solution:

$$G(z)=\frac{2\mu\sqrt{\lambda}}{\left(\sqrt{z^2+\lambda}\right)\left(2\mu c_1\sqrt{\lambda}+\ln\frac{z}{\lambda+\sqrt{\lambda z^2+\lambda^2}}\right)}\,, \tag{4.34}$$

where c_1 is the integration constant. In this case, the solution of (1.1) is given by

$$\begin{aligned}u(x,y,t)&=-1+\frac{2\mu\sqrt{\lambda}}{r\left(\sqrt{z^2+\lambda}\right)\left(2\mu c_1\sqrt{\lambda}+\ln\frac{z}{\lambda+\sqrt{\lambda z^2+\lambda^2}}\right)}\,,\\v(x,y,t)&=-\frac{F(t)}{\lambda}+\frac{2\mu\sqrt{\lambda}}{s\left(\sqrt{z^2+\lambda}\right)\left(2\mu c_1\sqrt{\lambda}+\ln\frac{z}{\lambda+\sqrt{\lambda z^2+\lambda^2}}\right)}+\frac{\mu}{s}\,.\end{aligned} \tag{4.35}$$

4.4.2. Travelling wave infinitesimal generator of (4.27)

In this subsection, we consider a linear combination of Γ_2 and Γ_3. In this case, the solution of the invariant surface conditions is given by

$$h=H(z)\,,\qquad g=G(z)\,,\qquad z=r+as\,. \tag{4.36}$$

Substituting (4.36) into (4.27), we find

$$\lambda aHG'+\left(\lambda\mu a^2+\mu\right)G''-(1+G)G'=0\,, \tag{4.37a}$$

$$aG'+H'=0\,. \tag{4.37b}$$

Integrating (4.37b) with respect to z, we get

$$H=-aG+c_1\,, \tag{4.38}$$

where c_1 is the integration constant. Substituting (4.38) into (4.37a), we obtain

$$-\left(\lambda a^2+1\right) GG' + \mu\left(\lambda a^2+1\right) G'' + \left(\lambda a c_1 - 1\right) G' = 0\,. \tag{4.39}$$

Let us integrate (4.39) with respect to z, to obtain

$$G' = A + BG + \frac{1}{2\mu}G^2\,, \tag{4.40}$$

where $A = \frac{c_2}{\mu(\lambda a^2+1)}$ and $B = \frac{1-\lambda a c_1}{\mu(\lambda a^2+1)}$. Equation (4.40) is the Riccati equation that has the following solutions:

— *Case 1*: the first solution is given by

$$\begin{aligned} G(z) &= -B\mu - \sqrt{\mu}\sqrt{B^2\mu - 2A} \\ &\quad \times \tanh\left(\frac{\sqrt{B^2\mu-2A}}{2\sqrt{\mu}}z + c_3\sqrt{\mu}\sqrt{B^2\mu-2A}\right)\,, \end{aligned} \tag{4.41}$$

and the corresponding solution of (1.1) in this case is given by

$$\begin{aligned} u(x,y,t) &= -B\mu - \sqrt{\mu}\sqrt{B^2\mu - 2A} \\ &\quad \times \tanh\left(\frac{\sqrt{B^2\mu-2A}}{2\sqrt{\mu}}z + c_3\sqrt{\mu}\sqrt{B^2\mu-2A}\right)\,, \\ v(x,y,t) &= -\frac{F(t)}{\lambda} + aB\mu + a\sqrt{\mu}\sqrt{B^2\mu-2A} \\ &\quad \times \tanh\left(\frac{\sqrt{B^2\mu-2A}}{2\sqrt{\mu}}z + c_3\sqrt{\mu}\sqrt{B^2\mu-2A}\right) + c_1\,, \end{aligned} \tag{4.42}$$

where c_2 and c_3 are arbitrary constants.

— *Case 2*: The second solution is given by (when $A = B = 0$ or $c_2 = 0$ and $c_1 = \frac{1}{a\lambda}$)

$$G(z) = -\frac{2\mu}{z+c_3}\,, \tag{4.43}$$

and the exact solution of (1.1) in this case is given by

$$\begin{aligned} u(x,y,t) &= -\frac{2\mu}{z+c_3}\,, \\ v(x,y,t) &= -\frac{F(t)}{\lambda} + \frac{2a\mu}{z+c_3} + \frac{1}{a\lambda}\,, \end{aligned} \tag{4.44}$$

where c_1, c_2 and c_3 are the integration constants.

5. Conclusion

In this paper, by using symmetry analysis method, the Lie algebra of infinitesimal symmetry generators spanned by five-dimensional and infinite-dimensional subalgebra is produced. The optimal system of the five-dimensional subalgebras is computed. These generators are applied to obtain some reduced equations and the exact solutions of the reduced equations are obtained. We get some new exact similarity solutions of Eq. (1.1) in the form of the Airy function (Eq. (4.11)), arctan function (Eq. (4.21)) and logarithmic function (Eq. (4.35)).

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