SOLUTION OF THE EINSTEIN EQUATIONS FOR AXIALLY SYMMETRIC CLOSED NULL STRING, CONSERVING ITS SHAPE

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In this work, a solution of the Einstein equations for axially symmetric closed null string has been found. At each moment of time, the direction of its motion is orthogonal to the plane in which null string is located. An absence of limit transition between the found solution and a solution for a closed null string of constant radius has been shown. This may indicate a stability of configuration of a closed null string in shape of a circle during motion in an external gravitational field. An influence of this field for such a null string may be reduced to a change of its motion as a whole (without change of its shape and size) or to a change of its radius. It has been noted that a part of characteristics of null string gas, such as an ability to form a domain structure and an existence of polarized states (multi-string systems), does not depend on a shape of a null string. However, a dynamics of a test null string in a field of such a multi-string system will have peculiarities.

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1. Introduction

Gauge Grand Unified Theories (GUT) predict a possibility of a formation of one-dimensional topological defect in a process of phase transitions in the early Universe. These defects are called cosmic strings [1–7]. In work [8], it was shown that a presence of such objects in the Universe does not contradict the observable cosmic microwave background. It is also not excluded that cosmic strings can be preserved until modern era and can be observable [9, 10].

A cosmic string is characterized by the following parameters: a linear density ρ_{l} and a cross-section radius r_{s}. For strings, appearing in GUT models, these parameters are connected with typical mass scale of theory m_{GUT} and the Higgs constant λ by the following equations:

$$\frac{G}{c^2}\rho_{\rm l} = \lambda^{-1}\left(\frac{m_{\rm GUT}}{m_{\rm Pl}}\right)^2 , \qquad r_{\rm s} = l_{\rm Pl}\left(\frac{m_{\rm Pl}}{m_{\rm GUT}}\right) ,$$

where $m_{\rm Pl}$ and $l_{\rm Pl}$ are Planck mass and length respectively; G is the gravitational constant, c is the speed of light. A cosmic string tension is proportional to a linear density $\rho_{\rm l}$ and, as follows from these equations, is measured by negative powers of the Planck mass.

If in these equations $m_{\rm GUT} \approx 10^{15}$ GeV, $\lambda \approx 10^{-2}$, then a cross-section radius of a cosmic string is estimated as

$$r_{\rm s} \approx 10^{-31} m \, .$$

For a description of a string motion, an approximation is used in which a position of a string is determined by a line in D-dimensional spacetime. Then, a trajectory of a string is a two-dimensional world surface which is mathematically defined by functions $x_m(\tau, \sigma)$, where τ and σ are parameters on a world surface of a string. σ is a space-like parameter, marking points along a string. τ is a time-like parameter, being proper time of an observer located on a string in a point with a coordinate σ.

Null strings realize a limit case of zero tension for the Nambu–Goto strings [7, 11–15]. They describe a limit case in which points of a string can interact only with a surrounding (external) gravitational field, but not with each other. Wherein, velocities of all points of a string are equal to zero. Since a string tension is measured by negative powers of the Planck mass $m_{\rm Pl}$, then a limit of a zero tension corresponds to asymptotically large scales of energy $E \gg m_{\rm Pl}$ [12]. With this point of view, null strings realize a high-temperature phase of a string theory [14, 15], *i.e.* they could have been formed in early stages of the Universe evolution. Thus, it is possible that they were taking part in processes of formation of the observable Universe structure.

Thus, for example, it is interesting to consider a possibility of participation of a string net (gas) in a formation of the "dark" matter [17–19], and also, a null string mechanism of the inflation scenario, suggested in the work [16].

Previous works covered an investigation of a motion of a test null string in a gravitational field of a closed null string of a constant (invariable in time) radius [20, 21], and also, in a gravitational field of a closed null string, radially expanding or radially collapsing in a plane [22, 23]. The results obtained allow assuming a possibility of existence of some interesting in terms of cosmology properties of null string gas. Thus, for example, it was noted that for a test null string, there is only a "narrow" region ("interaction zone") in which a test null string can interact with a null string generating the gravitational field. This indicates an opportunity of realization of a "granular" structure of space filled with null string gas.

Studies also show a presence of anomalous regions of a trajectory for each test null string got in an "interaction zone". In these regions, a test null string for a very short period of time is either rapidly pushed to infinity or rapidly attracted from infinity. This may confirm indirectly the hypothesis of possible string nature of the Universe inflation mechanism suggested in work [16].

An analysis of solutions of test null string motion equations provided in works [21–23] also shows a possibility of realization of stable polarized states (phase) of null string gas. It also illustrates an opportunity of a formation of a domain structure in space filled with null string gas. Thus, work [23] covers an investigation of a motion of a test null string in a gravitational field of a secluded closed null string, radially collapsing in a plane (source). It showed a possibility of realization of a state (phase) of null string gas in which closed null strings in shape of circle are located in parallel planes and are simultaneously radially expanding or collapsing without change of their shape. In work [24], the simplest model of such null string "gas" (secluded multi-string system) has been considered. Specifically, a case of multi-string system with a layered structure has been considered. This system consists of m equidistant layers (surfaces) orthogonal to the z-axis. There are n closed coaxial null strings on each layer simultaneously radially expanding or collapsing. Moreover, at each moment of time, a position of closed null strings on each layer is the same.

An analysis of a test null string motion in a gravitational field of such a multi-string system showed the following [24]. Depending on the value of test null string points initial momentum, an influence of a multi-string system gravitational field may lead to stable in time oscillations of a test null string in a vicinity of a fixed (motionless) point of space. As a consequence, stable in time and limited in space regions in which oscillations of a test null string occur can be considered as localized in space particles with effective nonzero restmass. It is also interesting to consider a conclusion that for null string gas a formation of particles with effective nonzero restmass is possible on early stages of the Universe evolution. Mutual penetrations of null strings belonging to adjacent domains could have appeared inevitably and massively in this era as a consequence of high pressures and a domain structure of such a gas. In future, as a consequence of the Universe expansion, such an opportunity could have disappeared.

Problems about an influence of a null string shape on its gravitational properties may become one of directions in an investigation of null string gas. First of all, it is interesting to study a question about an influence of a null string shape on the characteristics of null string gas found in works [21–24].

The suggested work dedicated to a search of a solution of the Einstein equations for a "smeared" axially symmetrical closed null string which direction of motion is orthogonal to its position plane.

2. The Einstein equations

The components of energy-momentum tensor for a null string, moving in the pseudo-Riemann spacetime, are determined by the equalities [16]

$$T^{mn}\sqrt{-g} = \varrho \int \mathrm{d}\tau\, \mathrm{d}\sigma x^m_{,\tau} x^n_{,\tau} \delta^4 \left(x^l - x^l(\tau,\sigma)\right) , \tag{1}$$

where indexes m, n, l take values 0, 1, 2, 3; functions $x^m = x^m(\tau,\sigma)$ define a trajectory of a null string motion (world surface); τ and σ are parameters on a null string world surface; $x^m_{,\tau} = \partial x^m/\partial\tau$; $g = |g_{mn}|$; g_{mn} is the metric tensor of external space-time; $\varrho = \mathrm{const}$.

In the cylindrical coordinate system

$$x^0 = t\,, \qquad x^1 = \rho\,, \qquad x^2 = \theta\,, \qquad x^3 = z\,,$$

functions $x^\alpha(\tau,\sigma)$, determining the trajectories of motion of closed null strings considered in the work, have the form of

$$t = \tau\,, \qquad \rho = R(\sigma)\,, \qquad \theta = \sigma\,, \qquad z = \pm\tau\,, \tag{2}$$

where $\tau \in (-\infty, +\infty)$, $\sigma \in [0; 2\pi]$. The function $R(\theta)$ satisfies the conditions

$$R(\theta)|_{\theta=0} = R(\theta)|_{\theta=2\pi}$$

(a condition of closeness of a null string) and

$$R(\theta) = R(-\theta)$$

(a condition of an invariance under an inversion of θ to $-\theta$). A sign $\pm$ corresponds to a choice of a direction along the z-axis. In the following, for definitiveness let us choose in (2) a sign "$-$". Let us note that trajectory (2) describes a motion of a closed null string along the z-axis. During this motion, at each moment of time, a null string is completely located in a plane orthogonal to this axis. Wherein, a shape of a closed null string determined by the function $R(\theta)$ does not change during motion.

Since for trajectory (2) a shape of a null string is invariant relatively to an inversion of θ to $-\theta$ and does not change during motion, then the quadratic form of spacetime should be invariant to an inversion of θ to $-\theta$, then

$$g_{mn}(t,\rho,\theta,z) = g_{mn}(t,\rho,-\theta,z)\,. \tag{3}$$

A consequence of (3) is $g_{02} = g_{12} = g_{32} = 0$.

It can be also noted that for trajectory (2), the quadratic form should be invariant relatively to a simultaneous inversion of $t \to -t$, $z \to -z$, then

$$g_{mn}(t,\rho,\theta,z) = g_{mn}(-t,\rho,\theta,-z)\,. \tag{4}$$

A consequence of (2) is $g_{01} = g_{31} = 0$. Finally, using a freedom of choice of a coordinate system in GTR, we partially fix it by choosing $g_{03} = 0$. Thus, the quadratic form for the considered task may be represented in the form of

$$\mathrm{d}S^2 = e^{2\nu}(\mathrm{d}t)^2 - A(\mathrm{d}\rho)^2 - B(\mathrm{d}\theta)^2 - e^{2\mu}(\mathrm{d}z)^2\,, \tag{5}$$

where ν, μ, A, B are functions of the variables t, ρ, θ, z.

A motion of a null string in the pseudo-Riemann spacetime is defined by the equation system [16]

$$x^m_{,\tau\tau} + \mathit{\Gamma}^m_{pq} x^p_{,\tau} x^q_{,\tau} \;=\; 0\,, \tag{6}$$

$$g_{mn} x^m_{,\tau} x^n_{,\tau} \;=\; 0\,, \qquad g_{mn} x^m_{,\tau} x^n_{,\sigma} = 0\,, \tag{7}$$

where $\mathit{\Gamma}^m_{pq}$ are the Christoffel symbols. Since the trajectory of null string motion (2) must be a particular solution of motion equations, the analysis of these equations could give additional limitations on the quadratic form functions (5). Writing out the null string motion equations (6), (7) for (5), it can be directly shown that for trajectory (2), equation (7) leads to the equation

$$e^{2\nu} - e^{2\mu} = 0\,, \tag{8}$$

from where

$$\nu = \mu\,. \tag{9}$$

Equation (6), considering equality (9), leads to the equation

$$\nu_{,t} - \nu_{,z} = 0\,, \tag{10}$$

from where

$$\nu = \nu(q,\rho,\theta), \tag{11}$$

where

$$q = t + z\,. \tag{12}$$

An analysis of the Einstein equations system for (1), (2), (5), (9), (11) allows to determine a dependence of functions of quadratic form (5), specifically

$$A = A(q,\rho,\theta)\,, \qquad B = B(q,\rho,\theta)\,. \tag{13}$$

Wherein, the Einstein system itself takes the form of

$$\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)_{,q}-2\nu_{,q}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)$$
$$+\frac{1}{2}\left(\left(\frac{A_{,q}}{A}\right)^2+\left(\frac{B_{,q}}{B}\right)^2\right)=-2\chi T_{00}\,, \tag{14}$$

$$\frac{1}{A}\left\{-\frac{B_{,\rho\rho}}{B}+\frac{1}{2}\left(\frac{B_{,\rho}}{B}\right)^2+\frac{1}{2}\frac{B_{,\rho}}{B}\frac{A_{,\rho}}{A}-2\nu_{,\rho\rho}-2\left(\nu_{,\rho}\right)^2\right.$$
$$\left.+\nu_{,\rho}\left(\frac{A_{,\rho}}{A}-\frac{B_{,\rho}}{B}\right)\right\}+\frac{1}{B}\left\{-\frac{A_{,\theta\theta}}{A}+\frac{1}{2}\left(\frac{A_{,\theta}}{A}\right)^2+\frac{1}{2}\frac{B_{,\theta}}{B}\frac{A_{,\theta}}{A}\right.$$
$$\left.-2\nu_{,\theta\theta}-2(\nu_{,\theta})^2+\nu_{,\theta}\left(\frac{B_{,\theta}}{B}-\frac{A_{,\theta}}{A}\right)\right\}=0\,, \tag{15}$$

$$\frac{1}{A}\left\{-2\nu_{,\rho\rho}-2(\nu_{,\rho})^2+\nu_{,\rho}\left(\frac{A_{,\rho}}{A}+\frac{B_{,\rho}}{B}\right)\right\}$$
$$+\frac{1}{B}\left\{2\nu_{,\theta\theta}+2(\nu_{,\theta})^2-\nu_{,\theta}\left(\frac{B_{,\theta}}{B}+\frac{A_{,\theta}}{A}\right)\right\}=0\,, \tag{16}$$

$$\frac{\nu_{,\rho}}{A}\left\{\frac{\partial}{\partial\rho}\ln\left[\nu_{,\rho}e^{2\nu}\sqrt{\frac{B}{A}}\right]\right\}+\frac{\nu_{,\theta}}{B}\left\{\frac{\partial}{\partial\theta}\ln\left[\nu_{,\theta}e^{2\nu}\sqrt{\frac{A}{B}}\right]\right\}=0\,, \tag{17}$$

$$\left(\frac{B_{,\rho}}{B}+2\nu_{,\rho}\right)_{,q}-\nu_{,\rho}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)-\frac{1}{2}\frac{B_{,\rho}}{B}\left(\frac{A_{,q}}{A}-\frac{B_{,q}}{B}\right)=0\,, \tag{18}$$

$$\left(\frac{A_{,\theta}}{A}+2\nu_{,\theta}\right)_{,q}-\nu_{,\theta}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)-\frac{1}{2}\frac{A_{,\theta}}{A}\left(\frac{B_{,q}}{B}-\frac{A_{,q}}{A}\right)=0\,, \tag{19}$$

$$-2\nu_{,\rho\theta}-2\nu_{,\rho}\nu_{,\theta}+\nu_{,\rho}\frac{A_{,\theta}}{A}+\nu_{,\theta}\frac{B_{,\rho}}{B}=0\,, \tag{20}$$

where $T_{00}=\varrho\frac{e^{2\nu}}{\sqrt{AB}}\delta(q)\delta(\rho-R(\theta))$, $\chi=8\pi G$, G is the gravitational constant.

For (9), (11), (13), the quadratic form (5) takes the form of

$$\mathrm{d}S^2=e^{2\nu}\left((\mathrm{d}t)^2-(\mathrm{d}z)^2\right)-A(\mathrm{d}\rho)^2-B(\mathrm{d}\theta)^2\,, \tag{21}$$

where ν, B, A are the functions of the variables q, ρ, θ.

It can be also noted that in (21), according to (3), (4), (12), the functions ν, A and B are even in the variables q and θ.

As it follows from equations system (14)–(20), outside the string, *i.e.* at $q\neq 0$, $\rho\neq R(\theta)$, $\theta=0\ldots 2\pi$, all components of the string energy-momentum tensor are identically equal to zero, but they are not equal to zero on the string (tend to infinity). It gives an opportunity to investigate the Einstein equations system in two directions:

1. To be limited by the analysis of the "external" problem, *i.e.* in the region ($q \neq 0$, $\rho \neq R(\theta)$, $\theta = 0 \ldots 2\pi$) for which the components of the energy-momentum tensor (r.h.s. of the Einstein equations) are equal to zero.

2. To consider the components of the string energy-momentum tensor as a limit of some "smeared" distribution and to conduct an analysis of the Einstein equations for this "smeared" distribution.

It can be shown that an analysis of the "external" problem leads to huge amount of vacuum solutions of the Einstein equations which satisfy symmetries of the task. However, it is unclear which criteria allow choosing the one solution which describes a gravitational field of a null string from the entire set of solutions. Under an attempt to consider the components of the energy-momentum tensor as a limit of some "smeared" distribution, inaccuracies may occur. It may be possible, for example, during a substitution of delta functions in the energy-momentum tensor with corresponding delta-functional sequences. These inaccuracies are connected with the fact that it is unclear how to consider possible appearances of the summands (multipliers) which tend to zero (constant) during a contraction of this "smeared" distribution into a one-dimensional object. That is why it is easier to consider some "well-defined" "smeared" distribution from the beginning, such as, for example, real massless scalar field (since we consider scalar null object). Then, we contract it into a string of necessary configuration, demanding that the components of the energy-momentum tensor of the scalar field asymptotically coincide with the components of the null string energy-momentum tensor.

3. The Einstein equations system for the smeared distribution

Components of the energy-momentum tensor for a real massless scalar field have the form of [2]

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta} L \,, \tag{22}$$

where $L = g^{\omega\lambda}\varphi_{,\omega}\varphi_{,\lambda}$, $\varphi_{,\alpha} = \partial\varphi/\partial x^{\alpha}$, φ is a distribution function of a scalar field; indexes α, β, ω, λ take values 0, 1, 2, 3. In order to provide self-consistency of the Einstein equations for (21), (22), we demand

$$T_{\alpha\beta} = T_{\alpha\beta}(q,\rho,\theta) \Rightarrow \varphi = \varphi(q,\rho,\theta) \,. \tag{23}$$

The Einstein equations system for (21)–(23) may be presented in the form of

$$\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)_{,q}-2\nu_{,q}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)$$
$$+\frac{1}{2}\left(\left(\frac{A_{,q}}{A}\right)^2+\left(\frac{B_{,q}}{B}\right)^2\right)=-2\chi(\varphi_{,q})^2\,, \tag{24}$$

$$\frac{1}{A}\left\{-\frac{B_{,\rho\rho}}{B}+\frac{1}{2}\left(\frac{B_{,\rho}}{B}\right)^2+\frac{1}{2}\frac{B_{,\rho}}{B}\frac{A_{,\rho}}{A}-2\nu_{,\rho\rho}-2(\nu_{,\rho})^2\right.$$
$$\left.+\,\nu_{,\rho}\left(\frac{A_{,\rho}}{A}-\frac{B_{,\rho}}{B}\right)\right\}+\frac{1}{B}\left\{-\frac{A_{,\theta\theta}}{A}+\frac{1}{2}\left(\frac{A_{,\theta}}{A}\right)^2+\frac{1}{2}\frac{B_{,\theta}}{B}\frac{A_{,\theta}}{A}\right.$$
$$\left.-\,2\nu_{,\theta\theta}-2(\nu_{,\theta})^2+\nu_{,\theta}\left(\frac{B_{,\theta}}{B}-\frac{A_{,\theta}}{A}\right)\right\}=\chi\left\{\frac{(\varphi_{,\rho})^2}{A}+\frac{(\varphi_{,\theta})^2}{B}\right\}, \tag{25}$$

$$\frac{1}{A}\left\{-2\nu_{,\rho\rho}-2(\nu_{,\rho})^2+\nu_{,\rho}\left(\frac{A_{,\rho}}{A}+\frac{B_{,\rho}}{B}\right)\right\}$$
$$+\frac{1}{B}\left\{2\nu_{,\theta\theta}+2(\nu_{,\theta})^2-\nu_{,\theta}\left(\frac{B_{,\theta}}{B}+\frac{A_{,\theta}}{A}\right)\right\}$$
$$=\chi\left\{\frac{(\varphi_{,\rho})^2}{A}+\frac{(\varphi_{,\theta})^2}{B}\right\}, \tag{26}$$

$$\frac{\nu_{,\rho}}{A}\left\{\frac{\partial}{\partial\rho}\ln\left[\nu_{,\rho}e^{2\nu}\sqrt{\frac{B}{A}}\,\right]\right\}+\frac{\nu_{,\theta}}{B}\left\{\frac{\partial}{\partial\theta}\ln\left[\nu_{,\theta}e^{2\nu}\sqrt{\frac{A}{B}}\,\right]\right\}=0\,, \tag{27}$$

$$\left(\frac{B_{,\rho}}{B}+2\nu_{,\rho}\right)_{,q}-\nu_{,\rho}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)-\frac{1}{2}\frac{B_{,\rho}}{B}\left(\frac{A_{,q}}{A}-\frac{B_{,q}}{B}\right)=-2\chi\varphi_{,q}\varphi_{,\rho}\,, \tag{28}$$

$$\left(\frac{A_{,\theta}}{A}+2\nu_{,\theta}\right)_{,q}-\nu_{,\theta}\left(\frac{A_{,q}}{A}+\frac{B_{,q}}{B}\right)-\frac{1}{2}\frac{A_{,\theta}}{A}\left(\frac{B_{,q}}{B}-\frac{A_{,q}}{A}\right)=-2\chi\varphi_{,q}\varphi_{,\theta}\,, \tag{29}$$

$$-2\nu_{,\rho\theta}-2\nu_{,\rho}\nu_{,\theta}+\nu_{,\rho}\frac{A_{,\theta}}{A}+\nu_{,\theta}\frac{B_{,\rho}}{B}=\chi\varphi_{,\rho}\varphi_{,\theta}\,. \tag{30}$$

Let us consider system of equations (24)–(30) for a scalar field distribution concentrated inside a "thin region" (a smeared null string). The variables q (defined by equality (12)) and ρ take values in the interval

$$q\in[-\Delta q,+\Delta q]\,,\qquad \rho\in[R(\theta)-\Delta\rho,R(\theta)+\Delta\rho]\,,\qquad \theta=0\ldots2\pi\,, \tag{31}$$

where the function $R(\theta)$ defines a shape of a closed null string. Δq and $\Delta\rho$ are small positive constants, defining a "thickness" of a smeared null string, *i.e.*

$$\Delta q\ll1\,,\qquad \Delta\rho\ll1\,. \tag{32}$$

With further contraction of such a "thin ring" into a one-dimensional object (null string)

$$\Delta q \to 0\,, \qquad \Delta\rho \to 0\,. \tag{33}$$

The space in which such a "smeared" null string is located and for which the variables q and ρ take values in the intervals $q \in (-\infty, +\infty)$, $\rho \in [0, +\infty]$ can be divided into three regions:

— Region I for which

$$q \in (-\infty, -\Delta q) \cup (+\Delta q, +\infty)\,, \qquad \rho \in [0, +\infty)\,, \qquad \theta = 0 \ldots 2\pi\,. \tag{34}$$

— Region II for which

$$\begin{aligned} &q \in (-\Delta q, +\Delta q)\,, \\ &\rho \in [0, R(\theta) - \Delta\rho) \cup (R(\theta) + \Delta\rho, +\infty)\,, \qquad \theta = 0 \ldots 2\pi\,. \end{aligned} \tag{35}$$

— Region III for which

$$q \in [-\Delta q, +\Delta q]\,, \qquad \rho \in [R(\theta) - \Delta\rho, R(\theta) + \Delta\rho]\,, \qquad \theta = 0 \ldots 2\pi\,. \tag{36}$$

Comparing the Einstein equation system for a closed null string (14)–(20) with system (24)–(30), it can be seen that during contraction of the scalar field into a string, *i.e.* at $\Delta q \to 0, \Delta\rho \to 0$

$$\begin{aligned} (\varphi_{,\rho})^2|_{q\to 0,\rho\to R(\theta)} &\to 0\,, & (\varphi_{,\theta})^2|_{q\to 0,\rho\to R(\theta)} &\to 0\,, \\ (\varphi_{,q})^2|_{q\to 0,\rho\to R(\theta)} &\to \infty\,, & (\varphi_{,q}\varphi_{,\rho})|_{q\to 0,\rho\to R(\theta)} &\to 0\,, \\ (\varphi_{,q}\varphi_{,\theta})^2|_{q\to 0,\rho\to R(\theta)} &\to 0\,, & (\varphi_{,\rho}\varphi_{,\theta})^2|_{q\to 0,\rho\to R(\theta)} &\to 0\,. \end{aligned} \tag{37}$$

Outside the region in which the scalar field is concentrated (regions I and II)

$$\varphi \to 0\,, \qquad \varphi_{,q} \to 0\,, \qquad \varphi_{,\rho} \to 0\,, \qquad \varphi_{,\theta} \to 0\,. \tag{38}$$

4. The distribution of the scalar field

For provided conditions (37), (38), it is convenient to present the distribution function of the scalar field in the form of

$$\varphi(q, \rho, \theta) = \ln\left(\frac{1}{(\alpha(q) + \lambda(q) f(\eta))^{\gamma}}\right)\,, \tag{39}$$

where $\eta = \eta(\rho, \theta) = \rho - R(\theta)$; γ is some positive constant; the function $\alpha(q) + \lambda(q) f(\eta)$ is limited

$$0 < \alpha(q) + \lambda(q) f(\eta) \leq 1 \,. \tag{40}$$

Function (39), according to (40), can take values from

$$\varphi \to 0 \,, \qquad \text{at} \quad \alpha(q) + \lambda(q) f(\eta) \to 1 \,, \tag{41}$$

to

$$\varphi \to \infty \,, \qquad \text{at} \quad \alpha(q) + \lambda(q) f(\eta) \to 0 \,. \tag{42}$$

Note that the form of the distribution function (39) is not general and its choice should be considered as one of the possible ways of "spreading" a null string. The form of the distribution function of the scalar field must influence the gravitational properties of the model of the string in the form of a scalar field tube. However, since the null string corresponds to the case in which the scalar field is contracted into a one-dimensional object, then the "smearing" method in limiting cases (32), (33) cannot be significant.

It can be shown that for distribution (39), conditions (37), (38) lead to the following limitations of the functions $\alpha(q)$, $\lambda(q)$ and $f(\eta)$:

1. The functions $\lambda(q)$ and $\alpha(q)$ are connected by the relation

$$\lambda(q) = \frac{1}{f_0}(1 - \alpha(q)) \,, \qquad f_0 = \text{const} \tag{43}$$

2. The functions $\alpha(q)$ and $f(\eta)$ are limited for all $q \in (-\infty, +\infty)$, $\rho \in [0, +\infty)$, $\theta = 0 \ldots 2\pi$ and take values in the interval

$$0 < \alpha(q) < 1 \,, \qquad 0 < f(\eta) < f_0 \,, \tag{44}$$

moreover,

$$\alpha(q)|_{q\in(-\infty,-\Delta q)\cup(+\Delta q,+\infty)} \to 1 \,, \qquad \alpha(q)|_{q\to 0} \to 0 \,, \tag{45}$$

$$f(\eta)|_{\rho\in[0,R(\theta)-\Delta\rho)\cup(R(\theta)+\Delta\rho,+\infty)} \to f_0 \,, \qquad f(\eta)|_{\eta\to 0} \to 0 \,. \tag{46}$$

Also, in a limit of contraction into a one-dimensional object (null string), the conditions (at $\Delta q \to 0, \Delta\rho \to 0$) must be satisfied

$$\left|\frac{\alpha_{,q}}{\alpha(q)}\right|_{q\to 0} \to \infty \,, \qquad \left.\frac{f_{,\eta}}{f(\eta)}\right|_{\eta\to 0} \to 0 \,, \qquad \left.\frac{\alpha_{,q}}{\alpha(q)} \times \frac{f_{,\eta}}{f(\eta)}\right|_{q\to 0,\eta\to 0} \to 0 \,. \tag{47}$$

One of examples of the functions $\alpha(q)$ and $f(\eta)$, satisfying the found conditions, is presented here

$$\alpha(q) = \exp\left(\frac{-1}{\epsilon + (\xi q)^2}\right), \tag{48}$$

$$f(\eta) = f_0 \exp\left(-\mu\left(1 - \exp\left(\frac{-1}{(\zeta\eta)^2}\right)\right)\right). \tag{49}$$

Constants ξ and ζ define a size ("thickness") of a "ring" inside which a scalar field is concentrated in variables q and ρ respectively. Specifically, as it follows from (48), (49), at $\Delta q \to 0$ $\Delta\rho \to 0$

$$\xi \to \infty, \qquad \zeta \to \infty. \tag{50}$$

Positive constants ϵ and μ provide a validation of conditions (45)–(47) at $\Delta\rho \to 0$, $\Delta q \to 0$, $\rho \to R(\theta)$, $\theta = 0\ldots 2\pi$, $q \to 0$. Specifically, at $\Delta q \gg 1$, $\Delta\rho \ll 1$

$$\epsilon \ll 1, \qquad \mu \gg 1. \tag{51}$$

During following constriction of a scalar field into a one-dimensional object (null string), *i.e.* at $\Delta\rho \to 0$, $\Delta q \to 0$

$$\epsilon \to 0, \qquad \mu \to \infty. \tag{52}$$

Using (43), (48), (49) for (39), we obtain an expression of one of possible distributions of a massless scalar field. During the constriction, its energy-momentum tensor components asymptotically coincide with energy-momentum tensor components of the closed null string, moving in trajectory (2).

Figures 1–4 present distributions of the scalar field (39) in the variable ρ ($\rho \in [0; 10]$), $\theta = 0\ldots 2\pi$ at the fixed value of the variable $q = 0.01$ for the functions $\alpha(q)$ and $f(\eta)$ given by equations (48), (49). The values of constants are $\epsilon = 0.01$, $\mu = 2$, $\xi = \zeta = 1.6$. Each figure corresponds to different function $R(\theta)$. The region in which $\varphi \to 0$ is highlighted in black. Figures 5 and 6 present a change of the distribution of the scalar field (38) in variable ρ ($\rho \in [0; 10]$), $\theta = 0\ldots 2\pi$ at the fixed value of the variable $q = 0.01$ for the functions $\alpha(q)$ and $f(\eta)$ given by equalities (48), (49). The function $R(\theta) = r_0 + R_1\cos^2\theta + R_2\sin^2\theta$; the values of the constants are $r_0 = R_1 = 1$, $R_2 = 5$, $\epsilon = 0.01$, $\mu = 4$, $\xi = \zeta = 0.5$ and $\xi = \zeta = 1.3$ for figures 5 and 6, respectively. From the provided figures, it is seen that a region in which a scalar field potential is nonzero constricts with an increase of values of constants ξ and ζ. *I.e.* a "thickness" of "ring" in which a scalar field is concentrated decreases in ρ.

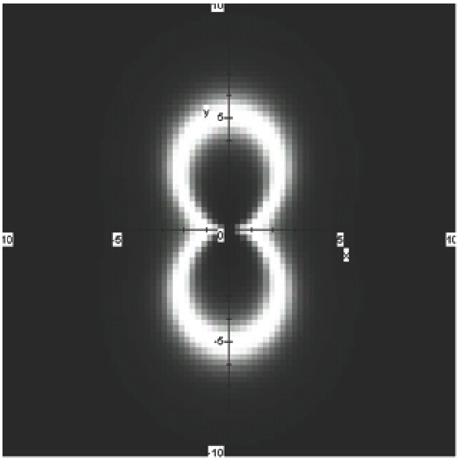

Fig. 1. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \sin^2\theta$, for $r_0 = R_1 = 1$, $R_2 = 5$.

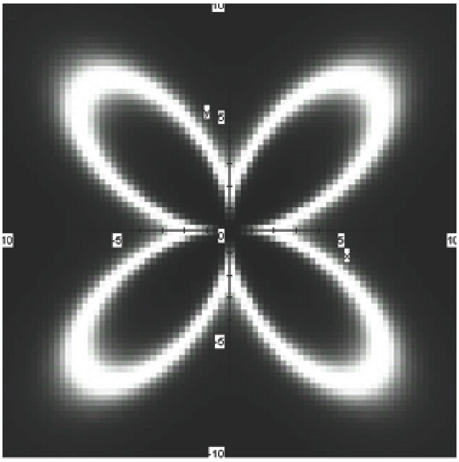

Fig. 2. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2(2\theta) + R_2 \sin^2(2\theta)$, for $r_0 = R_1 = 1$, $R_2 = 8$.

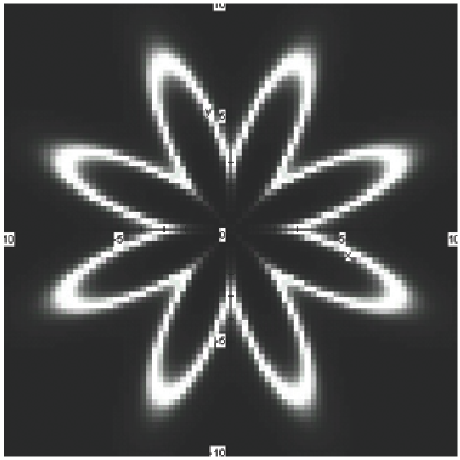

Fig. 3. The distribution of the scalar field defined by (38), (47), (48), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2(4\theta) + R_2 \sin^2(4\theta)$, for $r_0 = R_1 = 1, R_2 = 6$.

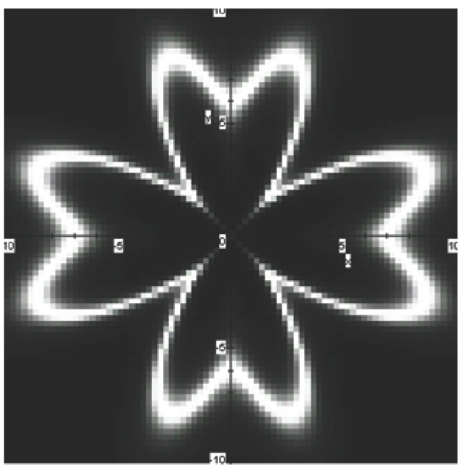

Fig. 4. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \left(\sin^2(4\theta) + \cos^2(2\theta)\right)$, for $r_0 = R_1 = 1$, $R_2 = 4$.

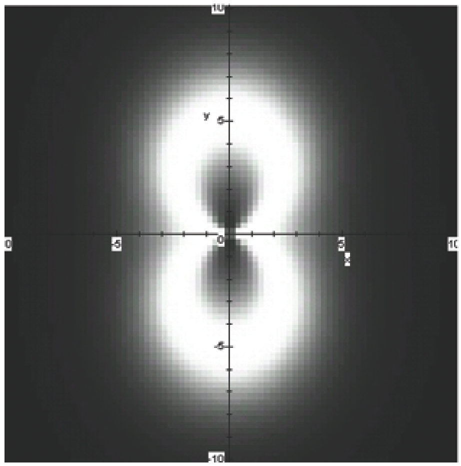

Fig. 5. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \sin^2\theta$, for $\xi = \zeta = 0.5$.

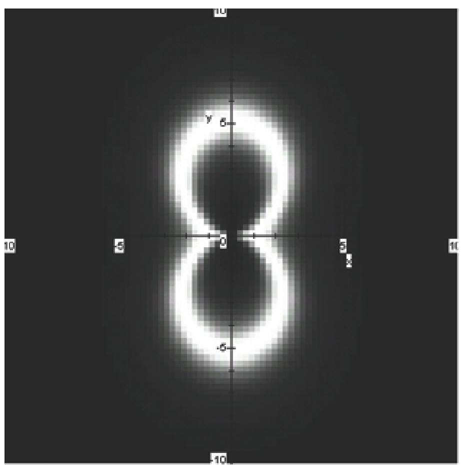

Fig. 6. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \sin^2\theta$, for $\xi = \zeta = 1.3$.

Figures 7 and 8 present a change of the distribution of the scalar field on the surface $\theta = 0$ for the functions $\alpha(q)$ and $f(\eta)$ given by equalities (47), (48). Here, $q \in [-10; 10]$, $\rho \in [0; 10]$. The function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \sin^2\theta$; the values of the constants are $r_0 = R_1 = 1$, $R_2 = 5$, $\epsilon = 0.01$, $\mu = 4$, $\xi = \zeta = 0.5$ and $\xi = \zeta = 1.3$ for figures 7 and 8, respectively. From the provided figures, it is seen that a region in which a scalar field potential is nonzero constricts with an increase of values of constants ξ and ζ. *I.e.* a "thickness" of "ring" in which a scalar field is concentrated decreases.

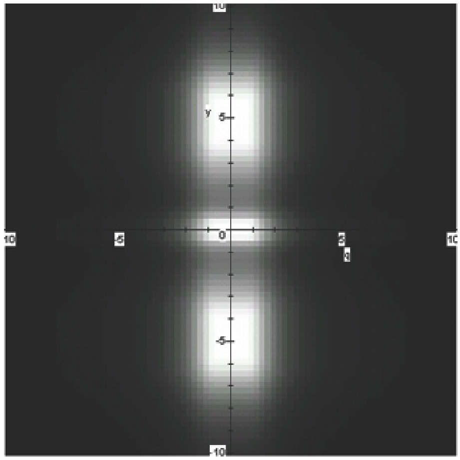

Fig. 7. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \sin^2\theta$ on the surface $\theta = 0$, for $\xi = \zeta = 0.5$.

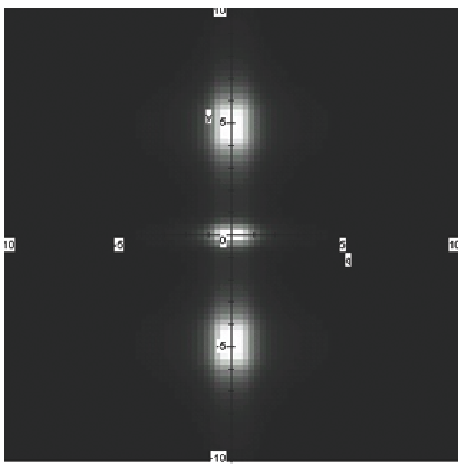

Fig. 8. The distribution of the scalar field defined by (39), (48), (49), corresponding to the function $R(\theta) = r_0 + R_1 \cos^2\theta + R_2 \sin^2\theta$ on the surface $\theta = 0$, for $\xi = \zeta = 1.3$.

5. Solution of the Einstein equations system for the "smeared" distribution

Let us supplement the Einstein equation system (24)–(30) with an equation of a scalar field which is for tensor (22)

$$\left(g^{\alpha\beta}\varphi_{,\alpha}\right)_{;\beta} = 0\,. \tag{53}$$

Here, a semicolon is the covariant derivative. For (21), (23), equation (53) takes the form of

$$\frac{\varphi_{,\rho}}{A}\left\{\frac{\partial}{\partial\rho}\ln\left[\varphi_{,\rho}e^{2\nu}\sqrt{\frac{B}{A}}\right]\right\}+\frac{\varphi_{,\theta}}{A}\left\{\frac{\partial}{\partial\theta}\ln\left[\varphi_{,\theta}e^{2\nu}\sqrt{\frac{A}{B}}\right]\right\}=0\,. \tag{54}$$

Comparing equations (27) and (54), we find

$$\nu_{,\rho}=c(q)\varphi_{,\rho}\,,\qquad \nu_{,\theta}=c(q)\varphi_{,\theta}\,, \tag{55}$$

from which

$$\nu=\nu(q,\rho,\theta)=\nu(q,\eta)=c(q)\varphi(q,\eta)+\nu_0(q)\,, \tag{56}$$

where, according to (4) and (11), the functions $c(q)$ and $\nu_0(q)$ are symmetric respectively to change of q to $-q$.

For (56), equation (30) can be represented in the form of

$$\frac{\partial}{\partial\eta}\ln\left[(\varphi_{,\eta})^2e^{2c(q)\varphi\left(1+\frac{\chi}{2c^2(q)}\right)}\right]=\frac{B_{,\rho}}{B}-\frac{1}{R_{,\theta}}\frac{A_{,\theta}}{A}\,. \tag{57}$$

It can be noted that r.h.s. of the obtained equality is a function of the variables q and η

$$\frac{B_{,\rho}}{B}-\frac{1}{R_{,\theta}}\frac{A_{,\theta}}{A}=F(q,\eta)\,. \tag{58}$$

From (58)

$$B=B(q,\rho,\theta)=B_1B_2\,,\qquad A=A(q,\rho,\theta)=A_1\alpha_1\,, \tag{59}$$

where $B_1=B_1(q,\eta)$, $B_2=B_2(q,\theta)$, $A_1=A_1(q,\eta)$, $\alpha_1=\alpha_1(q)$.

Integrating equation (57) for (59), we find

$$A_1=\frac{(c_1)^2(\varphi_{,\eta})^2}{B_1}e^{2c(q)\varphi\left(1+\frac{\chi}{2c^2(q)}\right)}\,, \tag{60}$$

where $(c_1)^2=(c_1(q))^2$ is integration “constant”.

Equation (26) for (30), (56), (59) on the condition

$$R_{,\theta}\neq 0 \tag{61}$$

takes the form of

$$\frac{B_{2,\theta}}{B_2}=2\frac{R_{,\theta\theta}}{R_{,\theta}}\,. \tag{62}$$

Integrating equation (62), we find

$$B_2(q,\theta) = \beta(R_{,\theta})^2\,, \tag{63}$$

where $\beta = \beta(q)$ is integration "constant".

For (56), (59), (63), scalar field equation (54) and equation (25) take, respectively, the form of

$$\frac{\partial}{\partial\eta}\ln\left[\varphi_{,\eta}\left(\frac{\alpha_1 A_1 + \beta B_1}{\alpha_1\beta\sqrt{A_1 B_1}}\right)e^{2\nu}\right] = 0\,, \tag{64}$$

$$\frac{\partial}{\partial\eta}\ln\left[\frac{(\alpha_1 A_1 + \beta B_1)_{,\eta}}{\sqrt{A_1 B_1}}\right] = -2\nu_{,\eta}\,. \tag{65}$$

Integrating equations (64), (65), considering (56), (60), we find

$$\alpha_1 A_1 + \beta B_1 = c_1 c_2 \alpha_1 \beta e^{-2\nu_0(q)} e^{-\left(c(q) - \frac{\chi}{2c(q)}\right)\varphi}\,, \tag{66}$$

$$(\alpha_1 A_1 + \beta B_1)_{,\eta} = c_1 c_3 \varphi_{,\eta} \alpha_1 \beta e^{-2\nu_0(q)} e^{-\left(c(q) - \frac{\chi}{2c(q)}\right)\varphi}\,, \tag{67}$$

where $c_2 = c_2(q)$ and $c_3 = c_3(q)$ are integration "constants". Differentiating equation (66) over the variable η, we find the expression for the function $c_3(q)$

$$c_3(q) = -\alpha_1(q)\beta(q)c_2(q)\left(c(q) - \frac{\chi}{2c(q)}\right)\,. \tag{68}$$

It can be noted that for (56), (59), (63), (66), equation (27) holds identically.

Applying equality (60) for (66), we obtain an algebraic equation, connecting the function B_1 and the function of the scalar field distribution φ

$$\beta(B_1)^2 - bB_1 + a = 0\,, \tag{69}$$

where

$$\begin{aligned} b &= c_1 c_2 \alpha_1 \beta e^{-2\nu_0(q)} e^{-c(q)\varphi\left(1 - \frac{\chi}{2c^2(q)}\right)}\,, \\ a &= \alpha_1 (c_1)^2 (\varphi_{,\eta})^2 e^{2c(q)\varphi\left(1 + \frac{\chi}{2c^2(q)}\right)}\,. \end{aligned} \tag{70}$$

Let us note that according to (60), (66)

$$b = (\alpha_1 A_1 + \beta B_1)\,, \qquad a = \alpha_1 A_1 B_1\,. \tag{71}$$

The discriminant of equation (69), using (70), (71), is

$$D = (\alpha_1 A_1 - \beta B_1)^2\,. \tag{72}$$

From (72), it follows that $D = 0$ in the case of

$$\alpha_1 A_1 = \beta B_1 \tag{73}$$

and $D > 0$ in the case of

$$\alpha_1 A_1 \neq \beta B_1 \,. \tag{74}$$

As it follows from (56), (60), (66), case (73) leads to an equation, connecting the function of the scalar field distribution φ and integration "constants"

$$2e^{c(q)\varphi+\nu_0(q)}\varphi_{,\eta} = c_2\sqrt{\alpha_1(q)\beta(q)}\,. \tag{75}$$

Integrating (75), we find

$$\varphi(q,\eta) = -\frac{\nu_0(q)}{c(q)} + \frac{1}{2c(q)}\ln\left(c(q)c_2(q)\sqrt{\alpha_1(q)\beta(q)}\eta + \varphi_0(q)\right)\,, \tag{76}$$

where $\varphi_0(q)$ is integration "constant". It is easy to see that function (76) cannot realize a propagation of a localized in the variable ρ object. So, case (73) is not realized. The roots of equation (69) for (72), (74) are

$$B_1(q,\eta)_{1,2} = \frac{1}{2}c_1c_2\alpha_1 e^{-2\nu_0-\varphi\left(c-\frac{\chi}{2c}\right)}\left[1 \pm \sqrt{1-\left(2\frac{\varphi_{,\eta}}{\sqrt{\alpha_1\beta}c_2}\right)^2}\right]\,. \tag{77}$$

Then, using (66), we obtain

$$A_1(q,\eta)_{1,2} = \frac{1}{2}c_1c_2\beta e^{-2\nu_0-\varphi\left(c-\frac{\chi}{2c}\right)}\left[1 \mp \sqrt{1-\left(2\frac{\varphi_{,\eta}}{\sqrt{\alpha_1\beta}c_2}\right)^2}\right]\,. \tag{78}$$

The remaining three equations of system (24), (28) and (29) which are considered for (56), (77) and (78) define the conditions, connecting the functions (integration "constants") $c(q)$, $c_1(q)$, $c_2(q)$, $\alpha_1(q)$, $\nu_0(q)$, $\beta(q)$, their derivatives and the constant γ.

Thus, the difference between equations (28) and (29) for (56), (59), (63) is

$$\left(\frac{(\beta B_1)_{,q}}{\beta B_1} - \frac{(\alpha A_1)_{,q}}{\alpha A_1}\right)_{,\eta} + \frac{1}{2}\left(\frac{(\beta B_1)_{,q}}{\beta B_1} - \frac{(\alpha A_1)_{,q}}{\alpha A_1}\right)\left(\frac{(B_1)_{,\eta}}{B_1} + \frac{(A_1)_{,\eta}}{A_1}\right) = 0\,. \tag{79}$$

According to (39), (43), the function $\varphi(q,\eta)$, defining the scalar field distribution, satisfies the equations

$$\varphi_{,\eta\eta} - \frac{f_{,\eta\eta}}{f_{,\eta}}\varphi_{,\eta} - \frac{1}{\gamma}\left(\varphi_{,\eta}\right)^2 \;=\; 0\,, \tag{80}$$

$$\varphi_{,qq} - \frac{\alpha_{,qq}}{\alpha_{,q}}\varphi_{,q} - \frac{1}{\gamma}\left(\varphi_{,q}\right)^2 \;=\; 0\,, \tag{81}$$

$$\varphi_{,q\eta} - \frac{\lambda_{,q}}{\lambda(q)}\varphi_{,\eta} - \frac{1}{\gamma}\varphi_{,q}\varphi_{,\eta} \;=\; 0\,. \tag{82}$$

For (77), (78), (80), (82), equation (79) can be represented in the form of

$$\psi_0+\psi_1\varphi+\psi_2\varphi_{,q}+\psi_3\varphi_{,\eta}+\psi_4\varphi_{,\eta}\varphi+\psi_5\varphi_{,\eta}\varphi_{,q}+\psi_6\left(\varphi_{,\eta}\right)^2\varphi$$
$$+\psi_7\left(\varphi_{,\eta}\right)^3\varphi+\psi_8\left(\varphi_{,\eta}\right)^3+\psi_9\left(\varphi_{,\eta}\right)^3\varphi_{,q}+\psi_{10}\varphi_{,q\eta}=0\,, \tag{83}$$

where ψ_i, $i=0,1,\ldots,10$ are the functions, containing integration "constants", their derivatives, the constant γ, and, also, the functions, defining the scalar field distribution, for example,

$$\begin{aligned}\psi_0 &= \frac{f_{,\eta\eta}}{f_{,\eta}}\frac{\mathrm{d}}{\mathrm{d}q}\ln\left(\frac{\lambda(q)e^{2\nu_0(q)}}{\sqrt{\alpha_1(q)\beta(q)}c_2(q)}\right),\\ \psi_2 &= \left(\frac{1}{\gamma}+2c(q)\right)\frac{f_{,\eta\eta}}{f_{,\eta}},\\ \psi_{10} &= \left(\frac{1}{\gamma}+2c(q)\right).\end{aligned}$$

It can be shown that the demand

$$\left(\frac{1}{\gamma}+2c(q)\right) = 0 \quad\Rightarrow\quad c(q)=c=-\frac{1}{2\gamma}=\mathrm{const}\,, \tag{84}$$

$$\frac{\lambda(q)e^{2\nu_0(q)}}{\sqrt{\alpha_1(q)\beta(q)}c_2(q)} = c_4=\mathrm{const} \tag{85}$$

is the "trivial" solution of equation (83) (*i.e.* the solution for which $\psi_i=0$, $i=0,1,\ldots,10$). For equation (79), it corresponds to the case

$$\frac{(\beta B_1)_{,q}}{\beta B_1}-\frac{(\alpha A_1)_{,q}}{\alpha A_1}=0\,. \tag{86}$$

On the other hand, it can be shown that the demand

$$\left(\frac{1}{\gamma}+2c(q)\right)\neq 0 \tag{87}$$

which corresponds to a case

$$\frac{(\beta B_1)_{,q}}{\beta B_1}-\frac{(\alpha A_1)_{,q}}{\alpha A_1}\neq 0 \tag{88}$$

for equation (79) leads to a contradiction. Thus, for (87), equating the coefficients at the same powers of the functions ϕ and its derivatives in equations (82) and (83), we obtain the demand $\psi_2 = 0$. It leads to the equation

$$\frac{f_{,\eta\eta}}{f_{,\eta}} = 0 \quad \Rightarrow \quad f(\eta) = \tilde{c}\eta + \hat{c}\,, \tag{89}$$

where $\tilde{c}$ and $\hat{c}$ are constants. The solution of this equation, obviously, cannot satisfy conditions (46), (47). That is why the case (88) is not realized.

The sum of equations (28) and (29) for (56), (59), (63), (77) and (78), considering (84)–(86), is

$$\varphi_{,q\eta} - \frac{2}{1+2\chi\gamma^2}\left(\frac{\mathrm{d}}{\mathrm{d}q}\ln\left(\lambda(q)c_1(q)\sqrt{\alpha_1(q)\beta(q)}\right)\right)\varphi_{,\eta} - \frac{1}{\gamma}\varphi_{,q}\varphi_{,\eta} = 0\,. \tag{90}$$

Equating the coefficients at the same powers of the function φ and its derivatives in equations (82) and (90), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}q}\ln\left(\lambda(q)c_1(q)\sqrt{\alpha_1(q)\beta(q)}\right) = \frac{1+2\chi\gamma^2}{2}\frac{\mathrm{d}}{\mathrm{d}q}\ln(\lambda(q))\,. \tag{91}$$

Integrating (91), we find

$$c_1(q) = \tilde{c}_1\frac{(\lambda(q))^{\left(\chi\gamma^2-\frac{1}{2}\right)}}{\sqrt{\alpha_1(q)\beta(q)}}, \tag{92}$$

where $\tilde{c}_1$ is integration constant.

For (56), (59), (63), (77), (78), considering (84)–(86), (92), equation (24) is

$$\Psi_0(q) + \varphi_{,qq} + \Psi_1(q)\varphi_{,q} + \Psi_2(\varphi_{,q})^2 = 0\,, \tag{93}$$

where

$$\Psi_0(q) = \frac{\gamma\left(1+2\chi\gamma^2\right)}{(1-2\chi\gamma^2)}\left[\left(\frac{\lambda_{,q}}{\lambda(q)}\right)_{,q} - 2\nu_0(q)\frac{\lambda_{,q}}{\lambda(q)} + \frac{1}{4}\left(1+2\chi\gamma^2\right)\left(\frac{\lambda_{,q}}{\lambda(q)}\right)^2\right], \tag{94}$$

$$\Psi_1(q) = -2\nu_0(q) + \left[\frac{(1+2\chi\gamma^2)}{(1-2\chi\gamma^2)} + \frac{1}{2}\left(1+2\chi\gamma^2\right)\right]\frac{\lambda_{,q}}{\lambda(q)}\,, \tag{95}$$

$$\Psi_2 = \left(\frac{1}{2\gamma} - \chi\gamma\right)^{-1}\left(\frac{1}{2\gamma^2}\left(1-2\chi\gamma^2\right) + \frac{1}{8\gamma^2}\left(1+2\chi\gamma^2\right)^2 + \chi\right)\,. \tag{96}$$

Equating the coefficients at the same powers of the function φ and its derivatives in equations (81) and (93), we find the conditions for the functions Ψ_i, $i = 0, 1, 2$ of equation (93)

$$\Psi_0(q) = 0\,, \tag{97}$$

$$\Psi_1(q) = -\frac{\alpha_{,qq}}{\alpha_{,q}}\,, \tag{98}$$

$$\Psi_2 = -\frac{1}{\gamma}\,. \tag{99}$$

Applying (96) for (99), we obtain an algebraic equation

$$4\chi^2\gamma^4 + 12\chi\gamma^2 + 9 = 0 \tag{100}$$

which has single positive solution (since $\gamma > 0$)

$$\gamma = \sqrt{\frac{3}{2\chi}}\,. \tag{101}$$

For (94), (101), equality (97) takes the form of

$$2\nu_0(q) = \frac{\lambda_{,qq}}{\lambda_{,q}}\,. \tag{102}$$

Integrating (102), we find (considering the parity of the function $\nu_0(q)$)

$$e^{2\nu_0(q)} = c_5|\lambda_{,q}|\,, \tag{103}$$

where c_5 is integration constant. Let us note that for (43), (95), (101), (102), equality (98) holds identically.

Thus, using (39), (56), (59), (63), (77), (78), (84), (85), (92), (101), (103), we obtain an expression for the unknown metric functions of the quadratic form (21)

$$e^{2\nu} = c_5|\lambda_{,q}|(\alpha(q) + \lambda(q)f(\eta))\,, \tag{104}$$

$$B = \frac{3\tilde{c}_1 c_4}{\chi}\lambda^2(q)(\alpha(q) + \lambda(q)f(\eta))\frac{(f_{,\eta})^2(R_{,\theta})^2}{\left[1 \mp \sqrt{1 - \left(c_4\sqrt{\frac{6}{\chi}}f_{,\eta}\right)^2}\,\right]}\,, \tag{105}$$

$$A = \frac{3\tilde{c}_1 c_4}{\chi}\lambda^2(q)(\alpha(q) + \lambda(q)f(\eta))\frac{(f_{,\eta})^2}{\left[1 \pm \sqrt{1 - \left(c_4\sqrt{\frac{6}{\chi}}f_{,\eta}\right)^2}\,\right]}\,. \tag{106}$$

The value of the constant c_4 (normalization constant) must satisfy the condition

$$0 < \left(c_4 \sqrt{\frac{6}{\chi}} f_{,\eta} \right)^2 \leq 1$$

and, according to (49), depends on the value of the constants f_0, μ and ζ. These constants define the size ("thickness") of the "ring" inside which the scalar field is concentrated in variable ρ, specifically

$$c_4 \sim \frac{\sqrt{\chi}}{f_0 \mu \zeta^2} \,. \tag{107}$$

For determination of the value of the constant $\tilde{c}_1$, it is convenient to investigate the expression of the scalar curvature: $K = g^{\alpha\beta} g^{\mu\nu} R_{\alpha\mu\beta\nu}$. Here, $R_{\alpha\mu\beta\nu}$ is the Riemann–Christoffel tensor. The scalar curvature for the found metric functions has the form of

$$K = -\chi \left[\frac{(\varphi_{,\rho})^2}{A} + \frac{(\varphi_{,\theta})^2}{B} \right] = \frac{-\chi}{\tilde{c}_1 c_4 (\alpha(q) + \lambda(q) f(\eta))^3} \,. \tag{108}$$

Comparing (108) with the r.h.s. of equation (25), it can be seen that during the contraction of scalar field into a string, *i.e.* at $\Delta q \to 0$, $\Delta \rho \to 0$ in region III defined by (36), the following should be valid:

$$K|_{q \to 0, \eta \to 0} = -\frac{\chi}{\tilde{c}_1 c_4 F} \to 0 \,, \tag{109}$$

where, applying (43), (48), (49),

$$F = \big(\exp(-1/\epsilon) + (1 - \exp(-1/\epsilon)) e^{-\mu} \big)^3 \,. \tag{110}$$

In regions I and II defined by (34) and (35), according to (43), (45), (46), the function $(\alpha(q) + \lambda(q) f(\eta)) \to 1$. Then from equation (108), it follows that:

$$K = -\frac{\chi}{\tilde{c}_1 c_4} \,. \tag{111}$$

From equalities (107), (109), (110), (111), it can be seen that scalar curvature in regions I–III depends on the value of constants defining the size ("thickness") of the "ring" in which the scalar field is concentrated. Moreover, in a limit of contraction of the "smeared" distribution into the one-dimensional object (null string), the following condition should be satisfied in regions I and II: $K \to 0$. For example, fixing $\tilde{c}_1 = \sqrt{\chi} (c_4)^{-2} F^{-1}$ in (108) and (111), we obtain

$$K = \frac{-\chi F}{f_0 \mu \zeta^2 (\alpha(q) + \lambda(q) f(\eta))^3} \,. \tag{112}$$

Applying (50), (52), it follows that during contraction of the "smeared" distribution into the one-dimensional object (null string), equation (112) takes the form

— in regions I and II

$$K = \frac{-\chi F}{f_0 \mu \zeta^2} \to 0\,, \tag{113}$$

— in region III

$$K|_{q\to 0,\eta\to 0} = \frac{-\chi}{f_0 \mu \zeta^2} \to 0\,. \tag{114}$$

The value of the constant c_5 in expression (104), which is scale multiplier, is convenient to be chosen equal to

$$c_5 = \frac{3\tilde{c}_1 c_4}{\chi} = \frac{3 f_0 \mu \zeta^2}{F}\,. \tag{115}$$

Considering (104)–(106), (115), quadratic form (21) can be represented in the form of

$$\begin{aligned} \mathrm{d}S^2 \;&=\; W \left\{ \frac{|\lambda_{,q}|}{\lambda^2(q)} \left((\mathrm{d}t)^2 - (\mathrm{d}z)^2 \right) \right. \\ &\left. - \frac{\left(f_{,\rho}\right)^2 (\mathrm{d}\rho)^2}{\left[1 \pm \sqrt{1 - \left(\tilde{c}_4 f_{,\eta}\right)^2}\,\right]} - \frac{(f_{,\theta})^2 (\mathrm{d}\theta)^2}{\left[1 \mp \sqrt{1 - \left(\tilde{c}_4 f_{,\eta}\right)^2}\,\right]} \right\}, \end{aligned} \tag{116}$$

where $W = W(q,\eta) = \frac{3 f_0 \mu \zeta^2}{F} \lambda^2(q)[\alpha(q) + \lambda(q) f(\eta)]$, $\tilde{c}_4 = c_4 \sqrt{\frac{6}{\chi}}$.

A particular case of the motion trajectories defined by (2) is the trajectory corresponding to a demand $R(\theta) = R = \text{const}$. It describes the motion of a closed null string of constant in time radius R along the z-axis. The solution of the Einstein equations system for this case was found in [20]. It is interesting to note that the Einstein equation system (24)–(30) for an axially symmetrical null string completely coincides with the system for a null string of constant radius R in a limit case $R(\theta) = R = \text{const}$. Wherein, solution (116) in a limit $R(\theta) \to R = \text{const}$ cannot be led to the solution found in [20]. An absence of a limit transition between these two solutions at $R(\theta) \to R = \text{const}$ may indicate a stability of configuration of a closed null string in a shape of circle during its motion in an external gravitational field. Consequently, an influence of this field for this null string may be reduced to a change of its motion as a whole or to a change of its radius.

Initial analysis of the test null string motion equations (6), (7) has showed the following. A part of null string gas properties, such as an ability to form a domain structure and an existence of polarized states (multi-string systems), does not depend on a shape of a null string. However, a dynamics of a test null string in a field of such a multi-string system will certainly have peculiarities.

6. Conclusions

In this work, we have found the conditions at which the components of the energy-momentum tensor of a scalar field asymptotically coincide with the ones of a closed null string in a limit of contraction of a scalar field into one-dimensional object (null string). It has been accomplished by comparison of the Einstein equation systems of two types. The first one corresponds to a distribution of real, massless scalar field concentrated inside of a "thin region". The second one corresponds to a closed axially symmetrical null string, moving along the z-axis without a change of its shape, which in each moment of time t is completely placed in a plane orthogonal to its motion. A common form of the distribution function, describing a motion along the z-axis of a scalar field concentrated inside of a "thin region", has been suggested. An example of scalar field distribution, satisfying found conditions, has been provided. The solution of the Einstein equations for a "smeared", axially symmetrical, closed null string, moving along the z-axis, which is completely placed in a plane orthogonal to the z-axis in each moment of time t has been found. An absence of a limit transition between the Einstein equation system solution for an axially symmetrical, "smeared" null string and a solution for a closed null string of constant radius provided in [20] has been shown. It may tell about stability of a configuration of a closed null string in shape of circle during motion in an external gravitational field. An influence of this field for such null string may be reduced to a change of its motion as a whole (without a change of its shape and size) or to a change of its radius. In the work, it was noted that part of characteristics of null string gas, such as an ability to form a domain structure and an existence of polarized states (multi-string systems), does not depend on a shape of a null string. However, a dynamics of a test null string in a field of such multi-string system will have peculiarities.

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