NEW DEVELOPMENTS WITH THE LOOP-TREE DUALITY*

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In this paper, we review the most recent developments of the four-dimensional unsubstraction (FDU) and loop-tree duality (LTD) methods. In particular, we make emphasis on the advantages of the LTD formalism regarding asymptotic expansions of loop integrands.

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1. Introduction

Theoretical predictions in high-energy physics are based on the Quantum Field Theory (QFT) called the Standard Model (SM) or any of its possible extensions. Although QFT is a very powerful theoretical tool, it also suffers from some unphysical weaknesses, or in other words, it is poorly defined in some aspects. Quantum corrections in QFT are described by loop Feynman diagrams in which the validity of the theory is extrapolated to arbitrary large energy scales, much above the Planck scale. In QFT, massless particles, such as gluons or photons, can be emitted with zero energy, and are represented by a quantum state which is different from the state describing the absence of real radiation. Finally, the emission of particles in parallel directions cannot be distinguished from the emission of a single particle. These unphysical aspects have a price, such as the proliferation of infinities when the theory is defined in the four dimensions of the space-time.

The traditional approach to solve this problem, and to extract physical predictions from the theory, consists in altering the dimensions of the space-time to *e.g.* $d = 4 - 2\varepsilon$. In Dimensional Regularization (DREG) [1–5], the singularities appear as explicit poles in $1/\varepsilon$ after integration of the loop

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momenta and the phase space of real radiation. After renormalisation of the ultraviolet (UV) singularities, virtual and real quantum corrections contribute with poles in $1/\varepsilon$ of the opposite sign such that the total result is finite. Although this procedure efficiently transforms the theory into a calculable and well-defined mathematical framework, a big effort needs to be invested to evaluate loop integrals in higher space-time dimensions, and to adequately subtract the singularities of the real radiation contributions, particularly at higher perturbative orders.

Therefore, perturbative calculations would be much simpler if we could keep the dimensions of the space-time to four. To this extent, and besides different variants of DREG, several groups have defined new regularisation schemes that do not alter the dimensions of the space-time or change it to an integer number, such as the four-dimensional formulation (FDF) [6] of the four-dimensional helicity scheme, the six-dimensional formalism (SDF) [7], implicit regularisation (IREG) [8], four-dimensional regularisation/renormalisation (FDR) [9], and four-dimensional unsubtraction (FDU) [10]. For a review of all these methods, see *e.g.* Ref. [11].

In this paper, we introduce FDU [10, 12–15], and review the main features of the loop-tree duality (LTD) [16–20] on which it is based. The idea behind FDU is to exploit a suitable mapping of momenta between the virtual and real kinematics in such a way that the summation over the degenerate soft and collinear states is performed locally at integrand level without the necessity to introduce infrared (IR) substractions. Suitable counter-terms are used to cancel, also locally, the UV singularities in such a way that calculations can be performed without altering the dimensions of the space-time. The method should improve the efficiency of Monte Carlo event generators because it is meant for integrating simultaneously real and virtual contributions. We also explain why LTD is advantageous for asymptotic expansions of loop integrands, due to the fact that it reduces the loop integration domain to the Euclidean space of the loop three-momentum. The LTD formalism or a similar framework has also been used to derive causality and unitarity constraints [21], or to integrate numerically subtraction terms [22], and can be related to the forward limit of scattering amplitudes [16, 23]. It has also been used in the framework of the color-kinematics duality [24].

2. Four-dimensional unsubstraction from the loop-tree duality

LTD [16–18] transforms any loop integral or loop scattering amplitude into a sum of tree-level-like objects that are constructed by setting on-shell a number of internal loop propagators equal to the number of loops. Explicitly, LTD is realised by modifying the $\imath 0$ prescription of the Feynman propagators that remain off-shell

$$G_{\rm F}(q_j) = \frac{1}{q_j^2 - m_j^2 + \imath 0} \quad \xrightarrow[\text{on-shell}]{G_{\rm F}(q_i)} \quad G_{\rm D}(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - \imath 0\, \eta k_{ji}} \,, \quad (1)$$

with $k_{ji} = q_j - q_i$, and η^μ a future-like vector. From now on $\eta^\mu = (1, \mathbf{0})$, which is equivalent to integrate out the loop energy component through the Cauchy residue theorem. The left-over integration is then restricted to the loop three-momenta. The dual prescription can hence be $+\imath 0$ for some dual propagators, and $-\imath 0$ for others, and encodes in a compact and elegant way the contribution of the multiple cuts that are introduced by the Feynman tree theorem [25]. The on-shell condition is given by $\tilde{\delta}\,(q_i) = \imath\, 2\pi\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$, and determines that the loop integration is restricted to the positive energy modes of the on-shell hyperboloids (light-cones for massless particles).

It is interesting to note that although the on-shell loop three-momenta are unrestricted, after analysing the singular behaviour of the loop integrand, one realises that thanks to a partial cancellation of singularities among different dual components, all the physical IR singularities are restricted to a compact region of the loop three-momentum [19]. This relevant fact allows to construct the mappings between the virtual and real kinematics based on the factorisation properties of QCD, as illustrated graphically in Fig. 1, and then the summation over degenerate soft and collinear states.

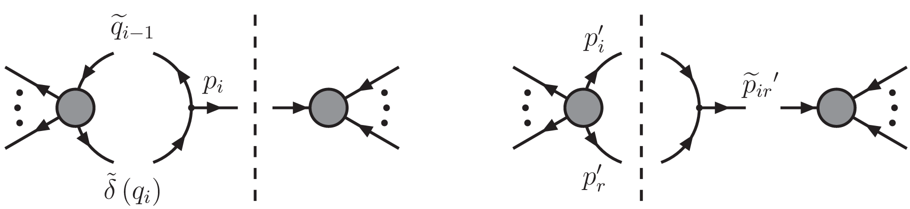

Fig. 1. Interference of the Born process with the one-loop scattering amplitude with internal momentum q_i on-shell, $\mathcal{M}_N^{(1)}(\tilde{\delta}\,(q_i)) \otimes \mathcal{M}_N^{(0)\,\dagger}$ (left), and interference of real processes with parton splitting $p'_{ir} \to p'_i + p'_r$: $\mathcal{M}_{N+1}^{(0)} \otimes \mathcal{M}_{N+1}^{(0)\,\dagger}(p'_{ir})$ (right). The dashed line represents momentum conservation. In the soft/collinear limits, the momenta $q_{i-1} = q_i - p_i$ and p'_{ir} become on-shell and the scattering amplitudes factorise.

As usual, the NLO cross section is constructed in FDU from the one-loop virtual correction with m partons in the final state and the exclusive real cross section with $m+1$ partons in the final state

$$\sigma^{\rm NLO} = \int_m {\rm d}\sigma_{\rm V}^{(1,{\rm R})} + \int_{m+1} {\rm d}\sigma_{\rm R}^{(1)} \,, \quad (2)$$

where the virtual contribution is obtained from its LTD representation

$$\mathrm{d}\sigma_{\mathrm{V}}^{(1,\mathrm{R})} = \sum_i \int_\ell 2\,\mathrm{Re}\left\langle \mathcal{M}_N^{(0)} \middle| \mathcal{M}_N^{(1,\mathrm{R})}\left(\tilde{\delta}\left(q_i\right)\right)\right\rangle \mathcal{O}_N(\{p_j\})\,. \tag{3}$$

In Eq. (3), $\mathcal{M}_N^{(0)}$ is the N-leg scattering amplitude at LO, and $\mathcal{M}_N^{(1,\mathrm{R})}$ is the renormalised one-loop scattering amplitude, which also contains integrand representations of the self-energy corrections of the external legs, even if they are massless and then ignored in the usual calculations because their integrated form vanishes. The delta function $\tilde{\delta}\left(q_i\right)$ symbolises the dual contribution with the internal momentum q_i set on-shell. The integral is weighted with the function $\mathcal{O}_N$ that defines a given observable, for example the jet cross section in the k_{T}-algorithm. The renormalised amplitude includes appropriate counter-terms that subtract the UV singularities locally, as discussed in Refs. [10, 15], including UV singularities of degree higher than logarithmic that integrate to zero.

The real phase space is rewritten in terms of the virtual phase space and the loop three-momentum

$$\int_{m+1} = \int_m \int_\ell \sum_i \mathcal{J}(q_i, \{p_j\})\,\mathcal{R}_i(q_i, \{p_j\})\,, \tag{4}$$

where $\mathcal{J}(q_i, \{p_i\})$ is the Jacobian of the transformation, and $\mathcal{R}_i(\{p'_j\}) = \mathcal{R}_i(q_i, \{p_j\})$ defines a complete partition of the real phase space

$$\sum_i \mathcal{R}_i(q_i, \{p_j\}) = \sum_i \prod_{jk \neq ir} \theta\left(y'_{jk} - y'_{ir}\right) = 1\,, \tag{5}$$

which is equivalent to split the phase space as a function of the minimal dimensionless two-body invariants $y'_{ir} = s'_{ir}/s$. In this way, the NLO cross section can be cast in the form

$$\begin{aligned}\sigma^{\mathrm{NLO}} = & \int_m \int_\ell \sum_i \Big[2\,\mathrm{Re}\left\langle \mathcal{M}_N^{(0)} \middle| \mathcal{M}_N^{(1,\mathrm{R})}\left(\tilde{\delta}\left(q_i\right)\right)\right\rangle \mathcal{O}_N\left(\{p_j\}\right) \\ & + \mathcal{J}\left(q_i, \{p_j\}\right) \mathcal{R}_i\left(\{p'_j\}\right) \left|\mathcal{M}_{N+1}^{(0)}\left(\{p'_j\}\right)\right|^2 \mathcal{O}_{N+1}\left(\{p'_j\}\right)\Big]\,,\end{aligned} \tag{6}$$

where the external momenta p'_j, the phase space and the tree-level scattering amplitude $\mathcal{M}_{N+1}^{(0)}$ are rewritten in terms of the loop three-momentum (equivalently, the internal on-shell loop momenta) and the external momenta p_i of the Born process. The cross section defined in Eq. (6) has a smooth four-dimensional limit and can be evaluated directly in four space-time dimensions. DREG is only necessary to fix the UV renormalisation counter-terms in order to define the cross section in *e.g.* the $\overline{\text{MS}}$ scheme, the rest of the calculation is stable with $d = 4$.

Analogously, at NNLO, the total cross section consists of three contributions

$$\sigma^{\text{NNLO}} = \int_m \mathrm{d}\sigma_{\text{VV}}^{(2,\text{R})} + \int_{m+1} \mathrm{d}\sigma_{\text{VR}}^{(2,\text{R})} + \int_{m+2} \mathrm{d}\sigma_{\text{RR}}^{(2)} \,, \tag{7}$$

where the double virtual cross section $\mathrm{d}\sigma_{\text{VV}}^{(2,\text{R})}$ receives contributions from the interference of the two-loop with the Born scattering amplitudes, and the square of the one-loop scattering amplitude with m final-state particles, the virtual–real cross section $\mathrm{d}\sigma_{\text{VR}}^{(2,\text{R})}$ includes the contributions from the interference of one-loop and tree-level scattering amplitudes with one extra external particle, and the double real cross section $\mathrm{d}\sigma_{\text{RR}}^{(2)}$ are tree-level contributions with emission of two extra particles. The LTD representation of the two-loop scattering amplitude is obtained by setting two internal lines on-shell [17]. It leads to the two-loop dual components $\langle \mathcal{M}_N^{(0)} | \mathcal{M}_N^{(2,\text{R})}(\tilde{\delta}\,(q_i)\,, \tilde{\delta}\,(q_j)) \rangle$, while the two-loop momenta of the squared one-loop amplitude are independent and generate dual contributions of the type $\langle \mathcal{M}_N^{(1,\text{R})}(\tilde{\delta}\,(q_i)) | \mathcal{M}_N^{(1,\text{R})}(\tilde{\delta}\,(q_j)) \rangle$. In both cases, there are two independent loop three-momenta and m final-state momenta, from where we can reconstruct the kinematics of the one-loop corrections entering $\mathrm{d}\sigma_{\text{VR}}^{(2,\text{R})}$, and the tree-level corrections in $\mathrm{d}\sigma_{\text{RR}}^{(2)}$.

3. Asymptotic expansions in the Euclidean space of the loop three-momentum

Asymptotic expansions are useful when we are interested in kinematical configurations exhibiting a hierarchy of physical scales, masses and external momenta, and the final answer can be approximated by a series in the ratio of two or several scales. Although the series expansion of integrated expressions is well-defined, it is desirable to first find a suitable expansion of the integrand which is expected to be simpler to integrate than the full expression. However, it is well-known that the naive expansion of loop integrands does not lead, in general, to the correct result. Several complementary expansions in different kinematical regions need to be considered, the so-called expansion by regions [26, 27] describing the hard and the soft regions. The reason for that is simply due to the fact that in a Minkowski space, such as the loop momentum, the square of a vector can vanish even if the components are non-zero, due to the metric. This is not the case of an Euclidean space. And this is precisely one of the advantages of LTD, because after integrating out the energy component of the loop momentum, the left over integration lies in the Euclidean space of the loop three-momentum.

Consider, for example, the following one-loop integral in the case when $0 < p^2 \ll M^2$ with $p_0 > 0$. The naive Taylor expansion of the integrand leads to

$$\int_\ell \frac{1}{(\ell^2 - M^2 + \imath 0)[(\ell + p)^2 - M^2 + \imath 0]} = \int_\ell \frac{1}{(\ell^2 - M^2 + \imath 0)^2} \left(1 + \frac{2\ell \cdot p + p^2}{\ell^2 - M^2 + \imath 0} + \cdots \right) . \tag{8}$$

This expansion, however, is not valid in the region where $\ell^2 \simeq M^2$ because in that case, p^2 is not the smallest quantity in the denominator of the second Feynman propagator. Therefore, Eq. (8) needs to be balanced with the expansion in a complementary kinematical region to obtain the full result.

The corresponding LTD representation on the opposite, and for the first of the two cuts, is given by

$$-\int_\ell \frac{\tilde{\delta}\,(\ell)}{2\ell \cdot p + p^2 - \imath 0} = -\int_\ell \frac{\tilde{\delta}\,(\ell)}{2\ell \cdot p} \sum_{n=0}^{\infty} \left(\frac{-p^2}{2\ell \cdot p} \right)^n \tag{9}$$

with $\tilde{\delta}\,(\ell) = \imath\, 2\pi\, \theta(\ell_0)\, \delta(\ell^2 - M^2)$. But since ℓ is set on-shell with mass M and has positive energy, the scalar product $2\ell \cdot p$ is positive definite and of $\mathcal{O}(M)$, and the Taylor expansion in Eq. (9) is well-defined for any value of the loop momentum. In Ref. [14], this procedure has been used for the first time as a proof of the efficiency of LTD, in particular, asymptotic expressions for the $H \to \gamma\gamma$ one-loop amplitude have been obtained from the corresponding LTD representation both in the large and the small mass limits of the internal particle running in the loop, including charged scalars, fermions (top quarks) and W bosons.

4. Conclusions

We have reviewed the FDU/LTD formalism for the calculation of physical cross sections and differential distributions. In FDU/LTD, all the IR and UV singularities are cancelled locally, and virtual and real corrections are evaluated simultaneously, which should be advantageous in Monte Carlo implementations. Also, we have shown that LTD is advantageous regarding the direct evaluation of asymptotic expansions of loop integrands because it involves an Euclidean space.

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