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THE FIVE-LOOP BETA FUNCTION FOR A GENERAL GAUGE GROUP*

Thomas Luthe

Faculty of Physics, University of Bielefeld, 33501 Bielefeld, Germany

Andreas Maier

Institute for Particle Physics Phenomenology, Durham University
Durham, DH1 3LE, United Kingdom

Peter Marquard

Deutsches Elektronen Synchrotron (DESY)
Platanenallee 6, 15738 Zeuthen, Germany

York Schröder

Grupo de Cosmología y Partículas Elementales, Universidad del Bío-Bío
Casilla 447, Chillán, Chile

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We present results for the gluon field anomalous dimension in perturbative QCD and derive the corresponding Beta function at five-loop order. All given results are valid for a general gauge group.

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1. Introduction

In modern high-energy physics experiments, in order to closely scrutinize (and eventually go beyond) our established particle physics models such as the Standard Model (SM), it is important to push the precision of theoretical predictions that follow from these models to the highest possible level. All parameters that appear in these quantum field theories such as the SM

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change as functions of the energy scale, in a well-defined way that is governed by so-called renormalization group equations. These, in turn, depend on a number of renormalization group parameters that can be deduced from the underlying quantum field theory.

Perhaps the most fundamental of such renormalization group parameters is the Beta function, governing the running of the gauge coupling constant and, consequently, much effort has been invested into precision determinations of this coefficient. After seminal work at one-loop order [1, 2], demonstrating the asymptotically free nature of the strong coupling constant and, therefore, establishing Quantum Chromodynamics (QCD) as a central part of the Standard Model, perturbative corrections have been pushed to 2-loop [3, 4], 3-loop [5, 6] and 4-loop [7, 8] level. Five-loop results have appeared over the last ten years or so, first for the case of Quantum Electrodynamics (QED) [9–11], then for physical QCD with gauge group SU(3) [12, 13], and finally for QCD with more general gauge groups [14, 15, 18, 19].

Of course, the (gauge-invariant) Beta function is not the only fundamental parameter governing renormalization of the gauge theory. All fields and parameters of the theory need to be renormalized, giving rise to a set of renormalization constants (RCs) that can be evaluated order-by-order in perturbation theory. Perhaps the second most important representative of this set is the (gauge-invariant) renormalization constant for the quark mass, needed for a precise evolution of measured low-energy quark masses to current and future high-energy collider experiments. It has been known at two [20] and three loops [21, 22] for a long time already; at four loops, complete results for SU(N) and QED as well as general Lie groups are available [23, 24]; at five loops, mass renormalization is known for SU(3) as well as general Lie groups [16, 25, 26].

The remaining members of the set of RCs depend on the gauge parameter. At four loops, these are known for more than a decade for SU(N) and Lie groups, see [8, 27] and references therein. Full gauge dependence for the case of Lie groups has been added only recently [16, 17]. At five loops and for a general Lie group, all of them are presently known in Feynman gauge from [16, 17]. The linear dependence on the gauge parameter has been calculated in [18] and the full gauge dependence in [19].

In this work, we present our results for the gluon field anomalous dimension obtained in [18], which together with the results obtained in [17] allows for the extraction of the Beta function [18]. Our result for the Beta function confirms the result as given in [15] by an independent approach, which is mandatory given the complexity of the five-loop calculation.

In the following, we define the set of renormalization constants and anomalous dimensions we are after, and introduce the set of group invariants that are needed to express the higher order results. Finally, we present our results for the five-loop gluon field anomalous dimension (in the Feynman gauge), from which we extract the (gauge-invariant) Beta function. We refrain from presenting the details of the calculation but refer the reader to Ref. [18] for an extended description of the procedure. Throughout the paper, we work in dimensional regularization around $d = 4 - 2\varepsilon$ space-time dimensions and in the $\overline{\mathrm{MS}}$ scheme.

2. Renormalization constants and anomalous dimensions

The fermion-, gauge- and ghost fields as well as fermion mass, gauge coupling and gauge-fixing parameter of the gauge theory are renormalized multiplicatively via

$$\psi_{\mathrm{b}} = \sqrt{Z_2}\psi_{\mathrm{r}}\,, \qquad A_{\mathrm{b}} = \sqrt{Z_3}A_{\mathrm{r}}\,, \qquad c_{\mathrm{b}} = \sqrt{Z_3^c}c_{\mathrm{r}}\,, \tag{1}$$

$$m_{\mathrm{b}} = Z_m m_{\mathrm{r}}\,, \qquad g_{\mathrm{b}} = \mu^{\varepsilon} Z_g g_{\mathrm{r}}\,, \qquad \xi_{L,\mathrm{b}} = Z_{\xi}\xi_{L,\mathrm{r}}\,. \tag{2}$$

We have used the subscript 'b' and 'r' for bare and renormalized quantities, respectively. All renormalization constants (RCs) have the form of $Z_i = 1 + \mathcal{O}(g_{\mathrm{r}}^2)$. There actually is no need to renormalize the gauge-fixing term $\sim (\partial A)^2/\xi_L$, such that setting $Z_\xi = Z_3$ leaves us with five independent renormalization constants only. A very economic way of recording the various renormalization constants Z_i is to merely list the corresponding anomalous dimensions, defined by

$$\gamma_i = -\partial_{\ln\mu^2} \ln Z_i\,. \tag{3}$$

Following usual conventions, instead of considering Z_g, one renormalizes the gauge coupling squared (which in our notation is

$$a \equiv \frac{C_{\mathrm{A}}\, g_{\mathrm{r}}^2(\mu)}{16\pi^2} \tag{4}$$

with C_{A} the quadratic Casimir operator of the adjoint representation of the gauge group) with the factor $Z_a \equiv Z_g^2$ and calls the corresponding anomalous dimension $\gamma_a = 2\gamma_g \equiv \beta$ the Beta function. Note that, due to the renormalization scale independence of the bare gauge coupling, using Eqs. (2) and (3), this immediately implies

$$\beta = \varepsilon + \partial_{\ln\mu^2} \ln a \quad \Leftrightarrow \quad \partial_{\ln\mu^2} a = -a\,[\varepsilon - \beta]\,. \tag{5}$$

The Beta function is a gauge invariant object. The second gauge invariant anomalous dimension is γ_m, corresponding to the renormalization of the quark mass.

To complete the renormalization program, we are left with choosing (besides the gauge invariants β and γ_m) three further RCs. These three coefficients will necessarily be gauge-parameter-dependent. In practical calculations, it can sometimes be convenient to consider 'vertex RCs' which are products of the Z_i, such as those that multiply the 3-gluon, 4-gluon, ghost-gluon and quark-gluon vertex. These vertex RCs are usually denoted as Z_1^j (where $j \in \{3g, 4g, ccg, \psi\psi g\}$). Out of this set, we found it convenient to evaluate the combination $Z_1^{ccg} = \sqrt{Z_3}\, Z_3^c\, Z_g$, giving us the anomalous dimension γ_1^{ccg}. For the remaining two of the minimal set of five RCs, we simply pick Z_2 and Z_3^c, encoded in the respective anomalous dimensions γ_2 and γ_3^c.

Once the minimal set of renormalization constants (chosen here to be γ_m, β, γ_3^c, γ_1^{ccg} and γ_2, as explained above) is known, all other anomalous dimensions can be reconstructed from simple linear relations, since they are related via gauge invariance of the QCD action (see *e.g.* [27])

$$\gamma_3 \;=\; 2\,(\gamma_1^{ccg} - \gamma_3^c) - \beta\,, \qquad \gamma_1^{3g} = 3\,(\gamma_1^{ccg} - \gamma_3^c) - \beta\,, \tag{6}$$

$$\gamma_1^{4g} \;=\; 4\,(\gamma_1^{ccg} - \gamma_3^c) - \beta\,, \qquad \gamma_1^{\psi\psi g} = \gamma_1^{ccg} - \gamma_3^c + \gamma_2\,. \tag{7}$$

In order to be able to present our results for γ_3 and β, we first need to define our notation concerning group invariants. To this end, we reiterate notation that we had already utilized in previous works [14, 16, 17]. We focus on a Yang–Mills theory coupled to N_f fermions in the fundamental representation. It is straightforward to generalize our results to fermions in a (single) arbitrary representation R by substituting all generators of the fundamental representation with generators of R.

The real and antisymmetric structure constants f^{abc} are defined by the commutation relations $T^aT^b - T^bT^a = if^{abc}T^c$ between Hermitian generators T^a of a semi-simple Lie algebra, with trace normalization $\mathrm{Tr}(T^aT^b) = T_\mathrm{F}\delta^{ab}$. The quadratic Casimir operators of the fundamental and adjoint representations (of dimensions N_F and N_A, respectively) are then defined in the usual way, as $T^aT^a = C_\mathrm{F}\mathbb{1}$ and $f^{acd}f^{bcd} = C_\mathrm{A}\delta^{ab}$. To facilitate compact representations of our results, we find it convenient to use the following normalized combinations of group invariants:

$$n_f = \frac{N_\mathrm{f}\, T_\mathrm{F}}{C_\mathrm{A}}\,, \qquad c_f = \frac{C_\mathrm{F}}{C_\mathrm{A}}\,. \tag{8}$$

In loop diagrams, one typically encounters traces of more than two group generators, giving rise to higher order group invariants. These higher order traces can be systematically classified in terms of combinations of symmetric tensors [29]. Rewriting the generators of the adjoint representation as

$[F^a]_{bc} = -if^{abc}$, we need the following three combinations (again, we normalize conveniently):

$$d_1 = \frac{\left[\mathrm{sTr}\left(T^aT^bT^cT^d\right)\right]^2}{N_\mathrm{A}T_\mathrm{F}^2C_\mathrm{A}^2}\,, \tag{9}$$

$$d_2 = \frac{\mathrm{sTr}\left(T^aT^bT^cT^d\right)\,\mathrm{sTr}\left(F^aF^bF^cF^d\right)}{N_\mathrm{A}T_\mathrm{F}C_\mathrm{A}^3}\,, \tag{10}$$

$$d_3 = \frac{\left[\mathrm{sTr}\left(F^aF^bF^cF^d\right)\right]^2}{N_\mathrm{A}C_\mathrm{A}^4}\,. \tag{11}$$

Here, $\mathrm{sTr}(ABCD) = \frac{1}{6}\mathrm{Tr}(ABCD + ABDC + ACBD + ACDB + ADBC + ADCB)$ is a fully symmetrized trace.

Taking the gauge group to be $\mathrm{SU}(N)$ and setting $T_\mathrm{F} = \frac{1}{2}$ and $C_\mathrm{A} = N$, our set of normalized invariants then reads [29]

$$n_f = \frac{N_\mathrm{f}}{2N}\,, \qquad c_f = \frac{N^2-1}{2N^2}\,, \tag{12}$$

$$d_1 = \frac{N^4 - 6N^2 + 18}{24N^4}\,, \qquad d_2 = \frac{N^2+6}{24N^2}\,, \qquad d_3 = \frac{N^2+36}{24N^2}\,. \tag{13}$$

From here, one can, for example, easily obtain the SU(3) coefficients, corresponding to physical QCD.

3. Results

In the following, we present our results for the gauge field anomalous dimension γ_3 and the Beta function. The results for the remaining anomalous dimension have been presented in [16, 17].

In terms of the renormalized gauge coupling a as defined in Eq. (4), we have obtained

$$\gamma_3 = -a\left[\frac{8n_f - (13 - 3\xi_L)}{6} + \gamma_{31}a + \gamma_{32}a^2 + \gamma_{33}a^3 + \gamma_{34}a^4 + \ldots\right]. \tag{14}$$

The coefficients γ_{3n} are functions of the group invariants and the gauge parameter. At five loops and in Feynman gauge $\xi_L = 1$, we have obtained [18]

$$\begin{aligned} 2^{13}3^5\,\gamma_{34} &= \gamma_{344}\,[16n_f]^4 + \gamma_{343}\,[16n_f]^3 + \gamma_{342}\,[16n_f]^2 + \gamma_{341}\,[16n_f] + \gamma_{340}\,, \\ \gamma_{344} &= \{c_f, 1\}\cdot\{107 + 144\zeta_3, -619/2 + 432\zeta_4\}\,, \end{aligned} \tag{15}$$

$$\begin{aligned}
\gamma_{343} \;=\;& \left\{c_f^2, c_f, d_1, 1\right\} \cdot \{576(4961/48 - 238\zeta_3 + 99\zeta_4), \\
& 576(16973/288 + 221\zeta_3 - 198\zeta_4 + 72\zeta_5), \\
& -10368(55/3 - 41\zeta_3 + 12\zeta_4 + 20\zeta_5), \\
& 144(14843/36 + 722\zeta_3 + 165\zeta_4 - 816\zeta_5)\}\,, \qquad (16) \\
\gamma_{342} \;=\;& \left\{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\right\} \cdot \{82944\,(2509/48 + 67\zeta_3 - 145\zeta_5)\,, \\
& -1152(135571/16 + 4225\zeta_3 - 3024\zeta_3^2 - 99\zeta_4 - 18900\zeta_5 + \\
& 5400\zeta_6), -6635520\,(13/8 + 2\zeta_3 - 5\zeta_5)\,, 288(476417/72 \\
& -23035\zeta_3 - 25056\zeta_3^2 + 34929\zeta_4 - 44640\zeta_5 + 10800\zeta_6), \\
& 13824\left(230 - 2354\zeta_3 + 54\zeta_3^2 + 360\zeta_4 - 295\zeta_5 + 225\zeta_6\right), \\
& 6912\left(2373 - 4715\zeta_3 + 288\zeta_3^2 + 900\zeta_4 - 820\zeta_5\right), \\
& -72(1524019/8 - 33931\zeta_3 - 47808\zeta_3^2 \\
& +108225\zeta_4 - 73572\zeta_5 - 39600\zeta_6)\}\,, \qquad (17) \\
\gamma_{341} \;=\;& \left\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_3, d_2, 1\right\} \cdot \{20736\,(4157 + 768\zeta_3)\,, \\
& -165888\,(11277/4 + 1541\zeta_3 + 335\zeta_5 - 2520\zeta_7)\,, \\
& 1152(2208371/3 + 396403\zeta_3 + 91800\zeta_3^2 - 65115\zeta_4 \\
& -647460\zeta_5 + 229500\zeta_6 - 362880\zeta_7), \\
& 165888\left(236 - 386\zeta_3 - 216\zeta_3^2 - 895\zeta_5 - 357\zeta_7\right), \\
& -5184(1139437/9 - 29587\zeta_3 + 18744\zeta_3^2 + 42880\zeta_4 \\
& -124360\zeta_5 + 25500\zeta_6 - 33362\zeta_7), \\
& -1728(11659/2 - 116251\zeta_3 + 8880\zeta_3^2 + 171\zeta_4 \\
& +59980\zeta_5 + 40200\zeta_6 - 99099\zeta_7), \\
& -1728(77920 - 735952\zeta_3 - 61272\zeta_3^2 + 150480\zeta_4 \\
& +249580\zeta_5 + 76500\zeta_6 + 52479\zeta_7), \\
& 72(124662829/18 - 4899045\zeta_3 - 63192\zeta_3^2 + 3669873\zeta_4 \\
& +4836692\zeta_5 - 2278200\zeta_6 - 4098024\zeta_7)\}\,, \qquad (18) \\
\gamma_{340} \;=\;& \{d_3, 1\} \cdot \{6912(47317 - 814000\zeta_3 + 15294\zeta_3^2 + 42300\zeta_4 \\
& +61390\zeta_5 + 427125\zeta_6 + 358848\zeta_7), \\
& -144(112182361/9 - 12985044\zeta_3 - 2403444\zeta_3^2 + 6431460\zeta_4 \\
& +53855480\zeta_5 - 12870750\zeta_6 \\
& -30266775\zeta_7)\}\,, \qquad (19)
\end{aligned}$$

where we used a scalar-product-like notation (*e.g.* $\{c_f^2, c_f, 1\} \cdot \{a, b, c\} = c_f^2 a + c_f b + c$) to clearly expose the group structure. Our result has been confirmed in [19].

If one needs to reconstruct renormalization constants Z_i from the anomalous dimensions γ_i, one can start from Eq. (3), recalling that $Z_i(a, \xi_L)$ depends on the renormalization scale through both of its variables. Using the d-dimensional Beta function of Eq. (5); remembering that the gauge parameter renormalizes as the gluon field $\xi_{L,b} = Z_3 \xi_{L,r}$; expressing the gauge parameter as $\xi_L = 1 - \xi$, where $\xi = 0$ now corresponds to Feynman gauge; and converting all anomalous dimensions to our preferred minimal set, one obtains the relation

$$\gamma_i \;=\; -a(\beta - \varepsilon)(\partial_a \ln Z_i) - \left(2\gamma_1^{ccg} - 2\gamma_3^c - \beta\right)(\xi - 1)(\partial_\xi \ln Z_i)\,. \quad (20)$$

The coefficients $z_i^{(n)}$ of the RCs $Z_i = 1 + \sum_{n>0} z_i^{(n)}/\varepsilon^n$ finally follow from solving Eq. (20), requiring γ_1^{ccg}, γ_3^c and β at one loop lower only. In turn, once the RCs Z_i are available, the corresponding anomalous dimensions can be extracted from the single poles, $\gamma_i = a\partial_a z_i^{(1)}$.

From the first of Eq. (6), using the relation $\beta = 2(\gamma_1^{ccg} - \gamma_3^c) - \gamma_3$, this enables us to obtain the corresponding terms of the Beta function, whose coefficients we define as

$$\partial_{\ln\mu^2}\, a = -a\,[\varepsilon - \beta] = -a\left[\varepsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \ldots\right]. \quad (21)$$

We refrain from listing the 2–4 loop results and only show the one-loop result for normalization and the result at five loops [18]

$$\begin{aligned}
3^1\, b_0 \;&=\; [-4]n_f + 11\,, && (22)\\
3^5\, b_4 \;&=\; b_{44}\, n_f^4 + b_{43}\, n_f^3 + b_{42}\, n_f^2 + b_{41}\, n_f + b_{40}\,, \\
b_{44} \;&=\; \{c_f, 1\} \cdot \{-8(107 + 144\zeta_3), 4(229 - 480\zeta_3)\}\,, && (23)\\
b_{43} \;&=\; \left\{c_f^2, c_f, d_1, 1\right\} \cdot \{-6(4961 - 11424\zeta_3 + 4752\zeta_4), \\
&\quad -48(46 + 1065\zeta_3 - 378\zeta_4), 1728(55 - 123\zeta_3 + 36\zeta_4 + 60\zeta_5), \\
&\quad -3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)\}\,, && (24)\\
b_{42} \;&=\; \left\{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\right\} \cdot \{-54(2509 + 3216\zeta_3 - 6960\zeta_5), \\
&\quad 9(94749/2 - 28628\zeta_3 + 10296\zeta_4 - 39600\zeta_5), \\
&\quad 25920(13 + 16\zeta_3 - 40\zeta_5), 3(5701/2 + 79356\zeta_3 - 25488\zeta_4 + 43200\zeta_5), \\
&\quad -864(115 - 1255\zeta_3 + 234\zeta_4 + 40\zeta_5), \\
&\quad -432(1347 - 2521\zeta_3 + 396\zeta_4 - 140\zeta_5), \\
&\quad 843067/2 + 166014\zeta_3 - 8424\zeta_4 - 178200\zeta_5\}\,, && (25)
\end{aligned}$$

$$\begin{aligned}
b_{41} \;=\; & \{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_3, d_2, 1\} \cdot \{ -81(4157/2 + 384\zeta_3), \\
& 81(11151 + 5696\zeta_3 - 7480\zeta_5), \\
& -3(548732 + 151743\zeta_3 + 13068\zeta_4 - 346140\zeta_5), \\
& -25920(3 - 4\zeta_3 - 20\zeta_5), \\
& 8141995/8 + 35478\zeta_3 + 73062\zeta_4 - 706320\zeta_5, \\
& 216(113 - 2594\zeta_3 + 396\zeta_4 + 500\zeta_5), \\
& 216(1414 - 15967\zeta_3 + 2574\zeta_4 + 8440\zeta_5), \\
& -5048959/4 + 31515\zeta_3 - 47223\zeta_4 + 298890\zeta_5\}\,, \qquad (26) \\
b_{40} \;=\; & \{d_3, 1\} \cdot \{ -162(257 - 9358\zeta_3 + 1452\zeta_4 + 7700\zeta_5), \\
& 8296235/16 - 4890\zeta_3 + 9801\zeta_4/2 - 28215\zeta_5\}\,. \qquad (27)
\end{aligned}$$

Out of these 5-loop coefficients, b_{44} has, in fact, been known already for quite some time from a large-N_{f} analysis [30, 31], while b_{43} was given in [14], as a proof-of-concept of our setup that we have used in this and earlier works [16, 17]. The three coefficients b_{42}, b_{41} and b_{40} have first been computed by an independent group [15], using the background field method, infrared rearrangement [32] and the so-called R^* operation [28] in order to map UV divergences onto the class of massless four-loop two-point functions which were evaluated via their code FORCER [33–35]. Equations (25)–(27) fully coincide with the results of [15]. As a further check of the 5-loop expressions given above, all coefficients reduce to the results given in [13] when setting the group invariants to their SU(3) values (*cf.* Eq. (12)).

4. Conclusions

We presented results for the gauge field anomalous dimension and the QCD Beta function at five-loop orders. All presented results are in agreement with results obtained by other groups using different means. At five loops for a general gauge group, there are now three independent calculations for the Beta function and two for the remaining renormalization constants.

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