

DIMENSION 4: QUANTUM ORIGINS OF SPACETIME  
SMOOTHNESS\*

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We further develop recently proposed cosmological model based on exotic smoothness structures in dimension 4 and Boolean-valued models of Zermelo–Fraenkel set theory. The approach indicates quantum origins of large-scale smoothness and justifies the dimension 4 as the unique dimension for a spacetime. Of particular importance is the hyperbolic geometry of exotic  $R^4$  submanifolds of codimensions 1 and 0. It is argued that the global 4-dimensional manifold representing the Universe beyond the present observational scope is the direct sum of complex surfaces  $K3\#\overline{CP}(2)$ .

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**1. Introduction**

In current mainstream cosmological models, some fundamental phenomena are taken for granted or introduced as free parameters, *e.g.* smoothness structure of spacetime. In the standard  $\Lambda$ CDM cosmological model, the evolution of the Universe begins with the initial singularity which, from the point of view of general relativity (GR) and the Friedmann–Robertson–Walker (FRW) geometry, is a strong singularity [1]. Therefore, it is not evident that the large scale smoothness structure and 4-dimensionality of spacetime are emergent from and compatible with the quantum nature of the initial singularity. The cosmologically important examples are  $\mathbb{R}^4$  or

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$S^3 \times \mathbb{R}$ , and it is crucial that these manifolds admit infinitely continuum many different nondiffeomorphic smoothness structures [2, 3]. Each of them is a perfectly smooth manifold which topologically is  $\mathbb{R}^4$  or  $S^3 \times \mathbb{R}$  (similarly, there exist plethora of smoothness structures on other smoothable open topological 4-manifolds). Observe that  $S^3 \times \mathbb{R}$  is directly realized in FRW cosmological models (see *e.g.* [4]) and  $\mathbb{R}^4$  serves as underlying differentiable structure for spacetime. Thus, the possibility that the smoothness of the Universe is not the standard but some exotic one definitely cannot be ruled out at the moment (see *e.g.* [5, 6]). Dealing appropriately with these *exotic* structures in physics can be very beneficial (*e.g.* [7, 8]). Impressively, one can derive the value of cosmological constant (CC) which reasonably well matches the Planck experimental data [7–10]. Consequently, we are interested whether it is possible to *formally* justify the choice of exotic smoothness structure of spacetime. In particular, it is the question whether initial quantum state bears information about differentiability and dimensionality of spacetime at cosmological scales. This is expected to occur, since all matter and energy along with spacetime were initially confined to the region within quantum singularity and, in fact, emerge from the singularity itself. It was shown in [10] that the answer is *affirmative*. Indeed, the smooth large scale structure of spacetime which agrees with the initial quantum state of the cosmological evolution has to be chosen as exotic  $\mathbb{R}^n$ . A direct conclusion is that such a model can be realized exclusively in dimension 4 since exotic  $\mathbb{R}^n$ s exist *only* for  $n = 4$ . This line of reasoning is presented shortly in Section 2. Then in Section 3, we explain how hyperbolic geometry of certain 3- and 4-manifolds considered as submanifolds of the exotic  $\mathbb{R}^4$  helps to understand the value of CC as a topological invariant [7]. Moreover, the scenario predicts the large scale global structure of our Universe as a direct sum of two complex surfaces, namely  $K3\#\overline{CP(2)}$  [7]. We will discuss this important issue briefly.

## 2. From quantum mechanics to 4-dimensionality of spacetime

We assume that the quantum state of the initial singularity is described by quantum mechanics (QM). Let  $\mathcal{H}$  be a separable complex Hilbert space. The basic object of our interest is the orthomodular lattice of projections  $\mathbb{L} \equiv (\mathbb{L}(\mathcal{H}), \wedge, \vee, 0, 1)$  defined on  $\mathcal{H}$  which is non-distributive if  $\dim \mathcal{H} \geq 2$  [11]. The strategy is to look for (complete) Boolean algebras in  $\mathbb{L}$  as local frames for non-Boolean logic of QM (see *e.g.* [12]). This strategy can be pushed even further — for each complete Boolean algebra  $B \subset \mathbb{L}$ , one builds the Boolean-valued model  $V^B$  of Zermelo–Fraenkel set theory (ZFC) [13]. Every ZFC model carries its own object of real numbers  $\mathbb{R}^B$  among other model-dependent notions. Surprisingly, given a family of commuting self-

adjoint operators  $\{A_\alpha\}$  on  $\mathcal{H}$ , there is a complete Boolean algebra  $B$  containing spectral resolutions of  $\{A_\alpha\}$  such that the real numbers in  $V^B$  are in 1–1 correspondence with all spectral families in  $B$  and hence with self-adjoint operators built of the projections from  $B$  [13]. The Boolean valued model  $V^B$  determines its 2-valued classical model  $V^B/U$ , where  $U$  is an (sometimes *generic*) ultrafilter in  $V^B$ . The relation between internal (to the model  $V^B/U$ ) 1<sup>st</sup> order line of real numbers  $R$  and external 2<sup>nd</sup> order  $\mathbb{R}$  is crucial for correct understanding of the large scale structure of the Universe (as emerging from quantum regime). The large scale structure is described by some differentiable (smooth) manifold  $M^n$  with local coordinate patches  $\mathbb{R}^n$  glued together into the smooth structure. Quantum mechanical counterparts of these patches are isomorphic to  $R^n$  (internal to the model  $V^B/U$ ). We take this relation between  $R^n$  and  $\mathbb{R}^n$  as strict and rigid, *i.e.* every macroscopic local patch  $A_{(n)} \simeq \mathbb{R}^n$  emerges from a quantum patch  $a_{(n)} \simeq R^n$  such that if  $A_{(n)} \cap C_{(n)} = \emptyset$  then  $a_{(n)} \in V^{B_1}/U_1, c_{(n)} \in V^{B_2}/U_2$  and  $B_1 \neq B_2$ , where  $B_1, B_2$  are maximal complete Boolean algebras contained in  $\mathbb{L}$  and  $C_{(n)}, c_{(n)}$  are different macroscopic and quantum patches, respectively. Then one can prove the theorem stating that if  $M^n = \mathbb{R}^n$  and  $M^n$  is to be covered with local neighbourhoods  $\mathbb{R}^n$  indexed by maximal complete Boolean algebras from  $\mathbb{L}$  (as above) then  $M^n$  has to be diffeomorphic to some exotic smooth  $\mathbb{R}^n$  [10].  $M^n$  in this case has to be an exotic smooth  $R^4$  since all dimensions  $n \neq 4$  exclude the existence of exotic  $\mathbb{R}^n$ .

To summarize, accepting QM lattice  $\mathbb{L}$  of initial quantum singularity as the driving force for spacetime smoothness structure simultaneously points at both exotic smoothness of spacetime and its four-dimensional nature.

### 3. Exotic smoothness of the Universe and hyperbolic geometry

The possibility that the value of CC can be understood as a topological invariant of some smooth 3- and 4-manifolds has been recently under intense study [7, 8]. Such analysis is entirely new among all approaches to CC [14]. If the CC was indeed realized as a topological invariant, this would serve as an extremely natural explanation of the observed non-zero tiny CC energy density value  $\sim 10^{-29} \text{g/cm}^3$  ( $\sim 10^{-47} \text{GeV}^4$ ). Consequently, it would also solve the big part of the ‘old’ CC problem relying on the unbelievable fine-tuning of various theoretical terms derived from the Standard Model of particles and quantum zero-modes of fields which have to contribute to the CC density [15]. The typical outmatch of terms derived theoretically and the observed value of the energy density is referred to as much as 40–70 orders of magnitude [15]. Their fine-tuning in all orders of perturbation theory such that  $\rho_{\text{eff}} \sim 10^{-47} \text{GeV}^4$  would be the result seems to be unlike scenario. Nevertheless, it has to occur somehow since the tiny value of the

energy density is the one that is observed. That is why the possibility to calculate the observed CC value would solve the puzzle. Being topologically protected as a topological invariant, CC cannot be manipulated as long as the underlying differential and topological structures stay unaltered. We would like to pursue the analysis and focus on main ingredients which will bring us closer to the invariant [7]. The complete and detailed exposition of this invariant will appear elsewhere.

Given exotic  $R^4$  obtained from the above quantum initial state, its construction determines variety of topological data. Firstly, there is a hyperbolic geometry defined on special submanifolds of  $R^4$  embedded in  $K3\#\overline{CP}(2)$ . Namely, the initial quantum state is represented by the (widely embedded) sphere  $S^3 \subset R^4$  with the radius of the order of the Planck length. This sphere is embedded in the boundary  $\partial K$  of the so-called Akbulut cork  $K$  for the 4-manifold  $K3\#\overline{CP}(2)$ , *i.e.*  $S^3 \subset \partial K \subset R^4$ . More precisely, there exists the homology 3-sphere (Brieskorn sphere)  $\Sigma(2, 5, 7)$  embedded in  $\partial K$  such that  $S^3 \subset \Sigma(2, 5, 7) \subset \partial K \subset R^4$ . Since  $\Sigma(2, 5, 7)$  has constant negative curvature, it is a hyperbolic 3-submanifold of  $R^4$ . Now, each small exotic  $R^4$  can be embedded in the standard  $\mathbb{R}^4$  by  $E : R^4 \rightarrow \mathbb{R}^4$  such that  $E(R^4)$  has negative scalar curvature and further it can be given the 4-hyperbolic structure [7]. The end of  $R^4$  is represented as the infinite chain of 4-cobordisms

$$\text{End}(R^4) = W(Y_1, Y_2) \cup_{Y_2} W(Y_2, Y_3) \cup_{Y_3} \dots$$

such that  $R^4 = K \cup_{Y_1} \text{End}(R^4)$  and  $\partial K = Y_1$ . The important property of the infinite chain of 3-manifolds  $Y_i, i = 1, 2, \dots$  and the 4-cobordisms  $W(Y_i, Y_{i+1}), i = 1, 2, \dots$  is that they are hyperbolic submanifolds of  $R^4$ . Furthermore, the hyperbolic geometry of the cobordism is best expressed by the FRW metric [7]

$$ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k. \quad (1)$$

In the case of the embedding  $R^4 \hookrightarrow \mathbb{R}^4 \hookrightarrow K3\#\overline{CP}(2)$  starting with  $S^3$  of the Planck length ( $\ell_P$ ) radius, we have two subsequent topology changes in dimension 3

$$S^3 \rightarrow \Sigma(2, 5, 7) \rightarrow P\#P, \quad (2)$$

where  $P\#P$  is the connected sum of two copies of the Poincaré homology 3-sphere  $P$ .

The cosmological constant  $\Lambda$  is defined as  $R_{\mu\nu} = -\Lambda g_{\mu\nu}$ , where  $R_{\mu\nu}$ ,  $g_{\mu\nu}$  are the Ricci and metric tensors, respectively. Then in terms of the scaling function  $a(t)$  from (1), we have the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2}. \quad (3)$$

This is where the hyperbolic geometry enters the stage. The Mostow rigidity gives  $\dot{a} = 0$  and, finally,

$$\Lambda = \frac{3k}{a^2}. \quad (4)$$

It is the relation of 4-dimensional CC with the 3-dimensional scalar curvature. Applying (4) to sequence (2) one obtains, after rather tedious manipulations [7], the value of cosmological constant

$$\Lambda \ell_{\text{P}}^2 = \exp\left(-\frac{3}{\text{CS}(\Sigma(2, 5, 7))} - \frac{3}{\text{CS}(P\#P)} - \frac{\chi(A_{\text{cork}})}{4}\right).$$

Here,  $\text{CS}(\Sigma(2, 5, 7))$  and  $\text{CS}(P\#P)$  are the Chern–Simons 3-dimensional topological invariants of  $\Sigma(2, 5, 7)$  and  $P\#P$ , respectively, while  $A_{\text{cork}}$  is the Akbulut cork of  $R^4$  and  $\chi(A_{\text{cork}})$  is its Euler characteristic. The term  $-\frac{\chi(A_{\text{cork}})}{4}$  represents quantum corrections to  $\Lambda$ . Note that calculations of the Chern–Simons and Euler invariants can be performed explicitly. Introducing their values and physical parameters like the Planck length for the radius of the 3-sphere and referring the obtained value to the Hubble constant  $H_0$ , we get the result for  $\Omega_\Lambda$  as the fraction of the critical density [7]

$$\Omega_\Lambda \approx 0.6869$$

which is in a good agreement with the measured value in the Planck mission [9].

Let us repeat that this approach works for the embedding of exotic  $R^4$  into  $\text{K3}\#\overline{\text{CP}(2)}$ . Conversely, it can be shown that the embedding of  $R^4$  into the standard  $\mathbb{R}^4$  gives rise to the vanishing CC. The canonical embeddings  $R^4 \hookrightarrow \mathbb{R}^4 \hookrightarrow \text{K3}\#\overline{\text{CP}(2)}$  and the above vanishing of CC indicate that it is indeed essential to consider enlarged model for the Universe, *i.e.*  $\text{K3}\#\overline{\text{CP}(2)}$ . This should be thought as representing the global structure of our Universe which extends the observed local one which, in turn, is given by the patch  $\mathbb{R}^4$ . If the entire Universe was represented by  $\mathbb{R}^4$ , the topological explanation of the CC value given above would be impossible. Thus, the model predicts the nontrivial global 4-dimensional structure of the Universe supported by the tiny value of the vacuum energy density.

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