BTW MODEL WITH PROBABILISTICALLY NONUNIFORM DISTRIBUTION OF PARTICLES COMING FROM THE UNSTABLE SITES

Ajanta Bhowal Acharyya

Department of Physics, Lady Brabourne College
P-1/2 Suhrawardy Avenue, Calcutta — 700017, India
ajanta.bhowal@gmail.com

(Received September 29, 2016; accepted January 5, 2017)

The two-dimensional BTW model of SOC, with probabilistically nonuniform distribution of particles among the (nearest) neighbouring sites, is studied by the computer simulation. When the value of height variable of a particular site reaches the critical value ($z_{\rm c} = 4$), the value of height variable of that site is reduced by four units by distributing four particles among the four nearest neighbouring sites. In this paper, it is considered that two particles are distributed equally among the two nearest neighbouring sites along x-axis. However, the distribution of other two particles along y-axis, is probabilistically nonuniform. The variation of spatial average of the height variable with time is studied. In the SOC state, the distributions of avalanche sizes and durations are obtained. The total number of topplings occurred during the stipulated time of evolution is also calculated.

DOI:10.5506/APhysPolB.48.207

1. Introduction

There exists some extended driven systems in nature, such as forest fire, earthquake, growth of sandpile, which can be explained by using self-organised criticality (SOC). This phenomena of SOC is characterised by a spontaneous evolution into a steady state which shows long-range spatial and temporal correlations. The concept of SOC was introduced by Bak, Tang and Wiesenfeld in terms of a simple cellular automata model [1, 2]. The steady state dynamics of the model shows a power-law behaviour in the probability distributions for the occurrence of the relaxation (avalanches) clusters of a certain size, area, lifetime, *etc.* The BTW model has been solved exactly using the commutative property of the particle addition operator [3]. Several properties of this critical state, *e.g.*, entropy, height correlation, height probabilities, cluster statistics, *etc.* have been studied in

Refs. [4–7]. An extensive numerical efforts have been also performed to study the properties of the model in the SOC state and to estimate various critical exponents [8–14]. The avalanche exponents were estimated using the renormalization scheme [15, 16]. The BTW model in a dilute lattice has been also studied recently [17, 18].

The BTW model is a so-called 'sandpile' model with deterministic and isotropic toppling rule. Later the sandpile model with different modifications have been studied. Haw and Kardar determined the critical exponents of the anisotropic sandpile model using dynamical renormalisation method [19]. The effect of anisotropy in a continuous version of sandpile model (Zhang model) has been also studied and critical exponents have been calculated [20, 21]. In [22], the authors have shown both by theoretical and numerical simulation that continuous Abelian sandpile model with anisotropies in toppling rule belongs to the same universality class of continuous sandpile model.

Dhar and Ramaswamy obtained an exact solution of the Abelian deterministic directed sandpile model [23] which has been studied in [24] both with deterministic and stochastic toppling rules.

After the introduction of stochastic sandpile model by Manna [25], various studies have been performed on stochastic sandpile model [10, 26, 27]. The critical behaviour of sandpile model with stochasticity in toppling is different from that of deterministic toppling rules [28]. Recently, the continuous transformation of BTW model to Manna model has been studied [29] by introducing an 'intermediate model'.

In this paper, we have studied the BTW model with probabilistically anisotropic toppling rule. Here, we have considered that two particles are distributed uniformly along one direction (say along x-axis) *i.e.*, one particle moves along positive x-axis and the other moves along negative x-axis from the unstable lattice site. However, along other (y) direction, there is a probability (P_r) that the remaining two particles are distributed nonuniformly (*i.e.*, both particles will go towards positive y-axis with probability P_r). Consequently, two particles are distributed uniformy with probability $(1 - P_r)$, restoring the original BTW model.

In the original BTW model (with uniform distribution of particles after topplings), in two dimensions, the distribution may be visualized as the sandpile formed on a horizontal plane. Whereas a probabilistically nonuniform distribution of particles (considered in the present study) is a manifestation of formation of sandpile on an inclined plane.

In this work, we have first studied the time evolution of the spatial average of the height variable, $\bar{z}$, for a particular probability, P_r, of nonuniform distribution of particles along y-axis. The distribution of size and duration of the avalanches in the critical state has also been obtained. We have also studied the effect of probabilistically nonuniform distribution of particles along y-axis through the spatial varaition of the total number of topplings, $N_{\mathrm{t}}(i,j)$, occurred at a particular lattice site (i,j), during the stipulated time of evolution of the system.

This paper is organised as follows. In Section 2, the model and simulation is discussed. In Section 3, the results are described. The paper ends with the conclusion, in Section 4.

2. The model and simulation

The BTW model is a lattice automata model of sandpile growth which evolves spontaneously into a critical state. We consider a two-dimensional square lattice of size $L \times L$. Each site (i,j) of the lattice is associated with a variable (so-called height) $z(i,j)$ which can take positive integer values varying from 0 to z_{c}. In every time step, one particle is added to a randomly selected site which increases the value of the height of that site, according to

$$z(i,j) = z(i,j) + 1\,. \tag{1}$$

If, at any site, the height variable exceeds a critical value z_{c} (*i.e.*, if $z(i,j) \geq z_{\mathrm{c}}$) then that site becomes unstable and it relaxes by a toppling. When an unstable site topples, the value of the height variable of that site is reduced by 4 units and that of each of the four of its neighbouring sites increased by unity (local conservation), *i.e.*,

$$z(i,j) = z(i,j) - 4\,, \tag{2}$$

$$z(i \pm 1, j) = z(i \pm 1, j) + 1 \qquad \text{and} \qquad z(i, j \pm 1) = z(i, j \pm 1) + 1 \tag{3}$$

for $z(i,j) \geq z_{\mathrm{c}}$. Each boundary site is attached to an additional site which acts as a sink. We use here the open boundary conditions so that the system can dissipate through the boundary. In our simulation, we have taken $z_{\mathrm{c}} = 4$.

In the original BTW model, when the unstable site topples, four particles from the unstable site are distributed uniformly among its four nearest neighbours. In this paper, we consider the distribution of two particles from the unstable site equally among the two nearest neighbouring sites along the directions of x-axis. But in the case of distribution of particles along y-axis, the distribution is probabilistically nonuniform. Thus, there is a probability (P_r) that two particles from the unstable site will move to the nearest neighbouring site along the positive direction of y-axis, *i.e.*, from (i,j) to $(i,j+1)$. With probability $(1 - P_r)$, two particles will be distributed equally among the two nearest neighbouring sites along both the directions of y-axis. Thus, in this work, as the unstable site topples, the particles from the unstable site will move towards its nearest neighbouring sites as follows:

$$z(i,j) = z(i,j) - 4\,, \tag{4}$$
$$z(i \pm 1, j) = z(i \pm 1, j) + 1\,. \tag{5}$$

There is a probability (P_r) that two particles from the unstable site will move along the direction of y-axis according to

$$z(i, j+1) = z(i, j+1) + 2\,, \tag{6}$$
$$z(i, j-1) = z(i, j-1) \tag{7}$$

and with probability $(1 - P_r)$, two particles will move along the two direction of y-axis according to

$$z(i, j \pm 1) = z(i, j \pm 1) + 1\,. \tag{8}$$

In this work, the system is evolved according to the dynamics (following Eqs. (4)–(8)) starting from an initial condition with all the sites having $z = 0$. With the evolution of time, the value of height variables $z(i,j)$ of different sites first increases due to a random addition of particles. As soon as the value of height variable of any site reaches (or exceeds) the critical value ($z_{\mathrm{c}} = 4$), that site topples.

Here, we have studied the following observations in BTW model with probabilistically nonuniform distribution of particles:

1. The time evolution of the average (spatial) value of z, *i.e.*,

$$\bar{z} = (1/N) \sum_{i=1}^{N} z_i\,, \qquad \left(N = L^2\right)\,.$$

2. The fraction of sites, f_z, having the height variable $z = 0, 1, 2, 3$ in the critical state.

3. The distribution of the avalanche size ($D(s)$), and avalanche time τ ($D(\tau)$).

4. The spatial variation of the total number of topplings occurred at a site (i,j) during total time (t), $N_{\mathrm{t}}(i,j)$.

3. Results

In this paper, we have studied the two-dimensional BTW model when the two particles are distributed uniformly along x-axis. But in the case of the distribution of particles along y-axis, there is a probability, $P_r = 0.4$, that two particles are distributed nonuniformly (*i.e.*, two particles move along the positive direction of y-axis). Here, we have considered a square lattice of size ($L = 200$). We first studied the time evolution of the average (spatial) value of z, (*i.e.*, $\bar{z}$), which is plotted in figure 1 (a). It shows that the value of $\bar{z}$ in critical state is 1.98. It is to be noted that this value is less than the value observed, 2.12 [2] in the case of BTW model.

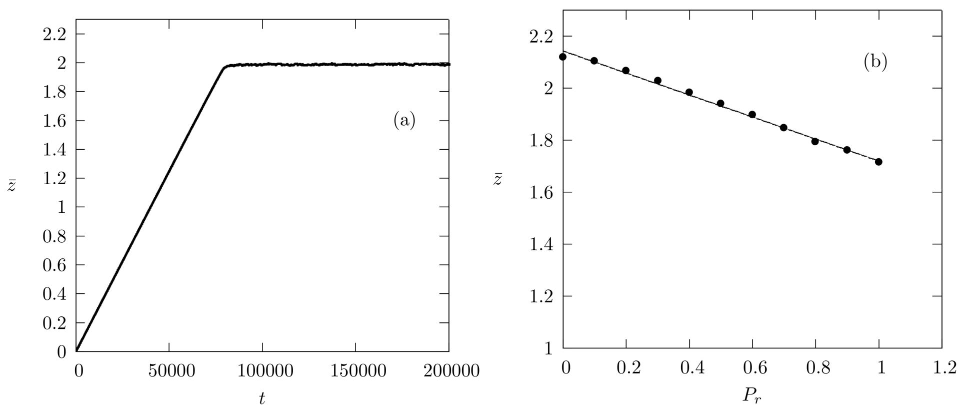

Fig. 1. (a) The plot of time variation of the spatial average of height variable, ($\bar{z}$), for the probabilistically nonuniform distribution of particles along y-axis from the unstable site ($P_r = 0.4$). (b) The plot of $\bar{z}$ against P_r ($L = 200$).

We have also studied how the value of $\bar{z}$ changes as the tendency of two particles to move along a particular direction of y-axis increases. We have calculated the value of $\bar{z}$ at critical state for different values of the probability of occuring nonuniform distribution of particles, (P_r). In figure 1 (b), the variation of the value of $\bar{z}$ at critical state is plotted for different value of P_r. Figure 1 (b) shows that the value of $\bar{z}$, decreases linearly as P_r increases. The value of $\bar{z}$ becomes 1.7, when both the particles move along a particular direction of y-axis from the unstable site, as it topples.

Various studies related to the structure of the lattice at the critical state, such as the fractal dimension and the fraction of the sites having different height variable has been studied in the case of original BTW model [9]. Here, for the BTW model with probabilistically nonuniform distribution, we have also calculated the fraction of sites (f_z) for different height variables, $z = 0, 1, 2, 3$. We have calculated f_z for different probability P_r and plotted in figure 2. Interestingly, it is observed here that f_zs (except $z = 0$) approach the value 0.28, where in the BTW model, f_zs are distinctly different for all z. The calculation of f_z is done for a lattice of size, $L = 200$, and in the critical state reached $t = 4 \times L^2$.

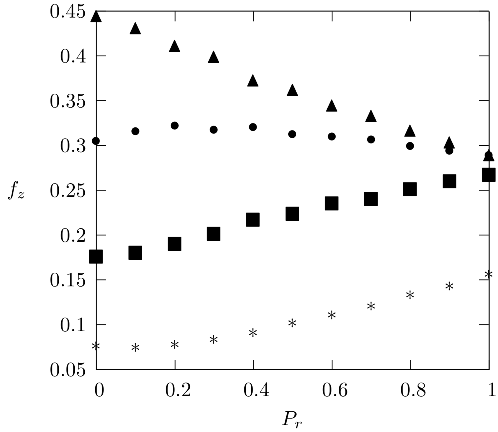

Fig. 2. Plot of f_z against P_r for $z = 0(*)$, $z = 1(\blacksquare)$, $z = 2(\bullet)$ and $z = 3(\blacktriangle)$; $(L = 200)$.

We have also calculated the distributions of duration (τ) and size (s) of the avalanches at the critical state for three different values of the $P_r = 0.1$, 0.4 and 0.8. The distribution is obtained for 80 000 number of avalanches $(L = 400)$. The distribution of avalanche size, $D(s)$, is plotted, on a doubly logarithmic scale, in figure 3 (a). Similarly, the distribution of avalanche time, $D(\tau)$, is plotted, on a doubly logarithmic scale, in figure 3 (b). We have estimated the value of the exponents within limited accuracy and given by, $D(s) \propto s^{-1.22}$ and $D(\tau) \propto \tau^{-1.55}$. The power law variations of the distribution of avalanche size (s) and avalanche time (τ) given by: $D(s) \propto s^{-1.22}$ and $D(\tau) \propto \tau^{-1.55}$, indicates that the steady state is a critical state, however, the exponents are different from the BTW model.

Here, we have calculated the total number of topplings, $N_{\mathrm{t}}(i, j)$, occurred at a site (i, j) during the total time of evolution of the system and observed its spatial variation on the square lattice. In figure 4 (a), the value of total number of topplings occurred at any site, $N_{\mathrm{t}}(i, j)$, is plotted for different lattice sites when the four particles are distributed equally among the four nearest neighbouring sites from the unstable site as it topples, *i.e.*, for the BTW model. Similarly, figure 4 (b) plots the spatial variation of $N_{\mathrm{t}}(i, j)$ when there is a probability, $P_r = 0.4$, that two particles will be nonuniformly distributed along the y-axis. Here, we have plotted the spatial variation of the total number of topplings occurred at a site, $N_{\mathrm{t}}(i, j)$, for a square lattice of size $L = 50$.

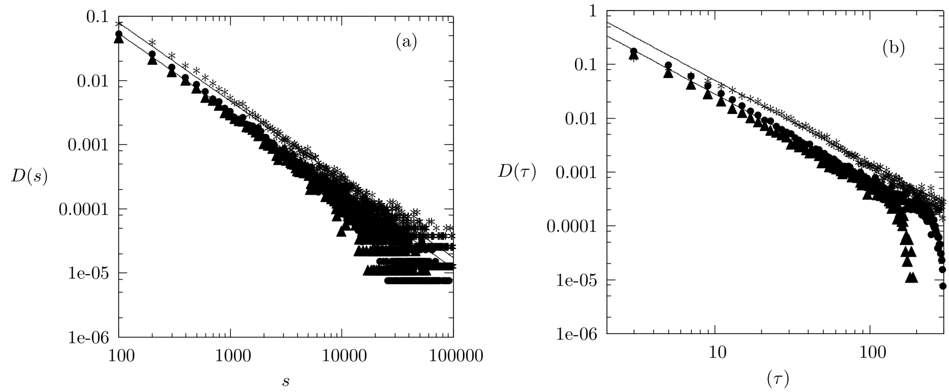

Fig. 3. (a) Log–log plot of distribution of the avalanche size (s) for the probabilistically nonuniform distribution of particles along y-axis for three different probabilities, $P_r = 0.1(*)$, $P_r = 0.4(\bullet)$ and $P_r = 0.8(\blacktriangle)$. Both the solid lines represent $y \sim x^{-1.22}$. (b) Log–log plot of distribution of the avalanche time (τ) for the probabilistically nonuniform distribution of particles along y-axis for three different probabilities, $P_r = 0.1(*)$, $P_r = 0.4(\bullet)$ and $P_r = 0.8(\blacktriangle)$. Both the solid lines represent $y \sim x^{-1.55}$.

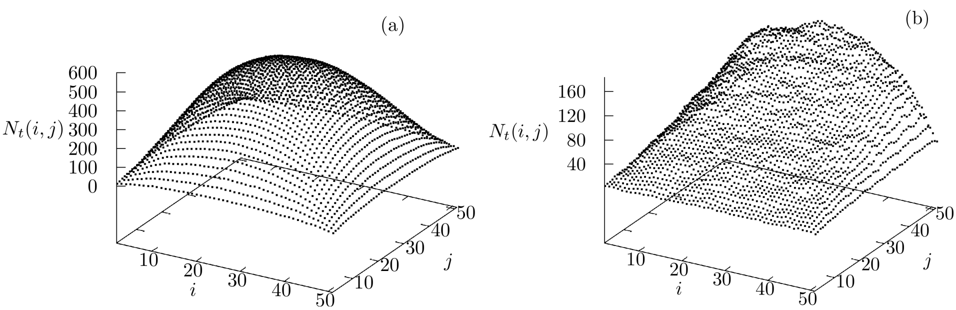

Fig. 4. Plots the variation of $N_{\rm t}(i,j)$ for different sites of the lattice for (a) uniform distribution of the particles among the nearest neighbouring sites and (b) for probabilistically nonuniform distribution of particles with a particular probability, $P_r = 0.4$.

It has been observed, as expected, that the spatial variation of the number of topplings, $N_{\rm t}(i,j)$ is symmetric for the uniform distribution of particles from the unstable site. It becomes asymmetric as the distribution of particles become probabilistically nonuniform along one direction from the unstable site as it topples.

4. Summary

We studied here the BTW model with probabilistically nonuniform distribution of two particles, from the unstable sites as it topples among its nearest neighbouring sites along a particular direction. It is observed that in the case of nonuniform distribution, the spatial average value of the height variable, $\bar{z}$, reaches a steady value, which is less than the value obtained in the BTW model with the uniform distribution of particles. The exponents of the power law obeyed by the distribution of size (s) and duration (τ) of the avalanches are calculated as $D(s) \propto s^{-1.22}$ and $D(\tau) \propto \tau^{-1.55}$. The fractions of the lattice sites, f_z, having different height variables, $z = 0, 1, 2, 3$ at the critical state have been calculated. The variations of f_z with the probability P_r have also been studied. The total number of topplings, $N_{\mathrm{t}}(i, j)$, occurred at any site, (i, j), is calculated and plotted for different lattice sites for both the cases, uniform and probabilistically nonuniform distribution of particles among the four nearest neighbouring sites from the unstable site. The total number of topplings occurred at the central site is maximal and in the case of uniform distribution of particles it is symmetric with respect to x- and y-axis. However, in the case of probabilistically nonuniform distribution of particles from the unstable site the symmetry observed in the spatial variation of the quantity, which is the total number of topplings that occurred at a given site, is broken.

The author would like to thank the anonymous referee for bringing the references [22] and [29] into her notice.

REFERENCES

[1] P. Bak, C. Tang, K. Wiesenfeld, *Phys. Rev. Lett.* **59**, 381 (1987).

[2] P. Bak, C. Tang, K. Wiesenfeld, *Phys. Rev. A* **38**, 364 (1988).

[3] D. Dhar, *Phys. Rev. Lett.* **64**, 1613 (1990).

[4] S.N. Majumdar, D. Dhar, *J. Phys. A* **24**, L357 (1991).

[5] E.V. Ivashkevich, *J. Phys. A* **27**, 3643 (1994).

[6] V.B. Priezzhev, *J. Stat. Phys.* **74**, 955 (1994).

[7] A.B. Acharyya, *Acta Phys. Pol. B* **42**, 19 (2011).

[8] S.S. Manna, *J. Stat. Phys.* **59**, 509 (1990).

[9] P. Grassberger, S.S. Manna, *J. Phys.* **51**, 1077 (1990).

[10] S.S. Manna, *Physica A* **179**, 249 (1991).

[11] V.B. Priezzhev, D.V. Ktitarev, E.V. Ivashkevich, *Phys. Rev. Lett.* **76**, 2093 (1996).

[12] A. Benhur, O. Biham, *Phys. Rev. E* **53**, R1317 (1996).

[13] S. Lubek, K.D. Usadel, *Phys. Rev. E* **55**, 4095 (1997).

[14] E.V. Ivashkevich, *J. Phys. A* **27**, L585 (1994).

[15] L. Pietronero, A. Vespignani, S. Zapperi, *Phys. Rev. Lett.* **72**, 1690 (1994).

[16] E.V. Ivashkevich, *Phys. Rev. Lett.* **76**, 3368 (1996).

[17] A.B. Acharyya, *J. Mod. Phys.* **5**, 1958 (2014).

[18] M.N. Najafi, *J. Phys. A: Math. Theor.* **49**, 335003 (2016).

[19] T. Haw, M. Kardar, *Phys. Rev. Lett.* **62**, 1813 (1989).

[20] I.M. Janosi, *Phys. Rev. A* **42**, 769 (1990).

[21] N. Azimi-Tafreshi, H. Dashti-Naserabadi, S. Moghimi-Araghi, *J. Phys. A: Math. Theor.* **41**, 435002 (2008).

[22] N. Azimi-Tafreshi, S. Moghimi-Araghi, *Phys. Rev. E* **80**, 046115 (2009).

[23] D. Dhar, R. Ramaswamy, *Phys. Rev. Lett.* **63**, 1659 (1989).

[24] R. Pastor-Satorras, A. Vespignani, *Phys. Rev. E* **62**, 6195 (2000).

[25] S.S. Manna, *J. Phys. A* **24**, L363 (1991).

[26] K. Christensen *et al.*, *Phys. Rev. Lett.* **77**, 107 (1996).

[27] D. Dhar, *Physica A* **270**, 69 (1999).

[28] A. Chessa, A. Vespignani, S. Zapperi, *Comput. Phys. Commun.* **121**, 299 (1999).

[29] H. Asasi, S. Moghimi-Araghi, M.N. Najafi, *Physica A* **419**, 196 (2015).