THE GENERALIZED UNCERTAINTY PRINCIPLE AND THE STARK EFFECT

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In a series of papers, Kempf and his collaborators Mangano and Mann presented a D-dimensional (β, β')-two-parameter deformed Heisenberg algebra which leads to a non-zero minimal length. In this work, according to generalized uncertainty principle (GUP), the electrostatic field in the presence of a minimal length scale based on the Kempf algebra is studied in the special case of $\beta' = 2\beta$ up to the first order over the deformation parameter β. We use modified electrostatic field and then we find the Stark potential in the presence of a minimal length. An upper limit for the polarizability in the presence of a measurable length is obtained. Also, the modified energy shifts in the ground state and excited states of hydrogen atom is found. We estimate the isotropic minimal length. The estimation is close to the minimal observable distance which was proposed by Heisenberg ($\ell_0 \sim 10^{-13}$ cm).

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1. Introduction

In the last decade, the unification between the general theory of relativity and the Standard Model of particle physics has been one of the most important problems in theoretical physics [1]. Today, we know that this unification predicts the existence of a minimal length of the order of the Planck length. The suggestion of modifying the usual Heisenberg uncertainty principle in such a way that it includes a minimal length has been first proposed in the context of quantum gravity and string theory [2, 3]. At the present time, we know that the existence of a minimal length scale leads to an extended relation of Heisenberg uncertainty. If we keep only the first two terms on the right-hand of extended uncertainty principle, we will obtain the usual generalized uncertainty relation (GUP) as follows:

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left[1 + a_1 \left(\frac{l_{\mathrm{P}}}{\hbar}\right)^2 (\Delta P)^2\right], \tag{1}$$

where l_{P} is the Planck length and a_1 is positive numerical constant [4, 5]. It is obvious that in Eq. (1), ΔX is always larger than $(\Delta X)_{\min} = \sqrt{a_1}\, l_{\mathrm{P}}$. In recent years, many authors have studied reformulation of the quantum mechanics, gravity and quantum field theory in the presence of a minimal length scale [4–22]. There are also some researches on electrodynamics in the frame work of the generalized uncertainty relation, for instance, Refs. [13–17, 20, 22]. In the present work, we study the generalized uncertainty principle and the Stark potential. This paper is organized as follows. In Sec. 2, we give a brief review of the D-dimensional (β, β')-two-parameter deformed Heisenberg algebra that was introduced by Kempf *et al.* and it is shown that the Kempf algebra leads to a minimal measurable length [23–25]. In Sec. 3, we study the electrostatic field in the presence of a minimal length scale and then we find the modified Stark potential in the presence of a minimal measurable length. We find the modified polarizability of the ground state of a hydrogen atom and the modified Stark effect in the excited states of the hydrogen atom. We also estimate the isotropic minimal length in modified Stark effect. The estimation on the isotropic minimal length is close to the observable minimal distance, which was proposed by Heisenberg ($\ell_0 \sim 10^{-13}$ cm). Our conclusions are presented in Sec. 4. We use SI units throughout this paper.

2. A brief review of modified commutation relations with a minimal length scale

Kempf and collaborators have presented the modified Heisenberg algebra. It should be noted that modified algebra describes a D-dimensional quantized space [23–25]. The Kempf algebra which leads to the existence of a minimal length scale in a D-dimensional case takes the following form:

$$\left[X^i, P^j\right] = i\hbar\left[\left(1 + \beta \boldsymbol{P}^2\right)\delta^{ij} + \beta' P^i P^j\right], \tag{2}$$

$$\left[X^i, X^j\right] = i\hbar\, \frac{(2\beta - \beta') + (2\beta + \beta')\,\beta \boldsymbol{P}^2}{1 + \beta \boldsymbol{P}^2}\left(P^i X^j - P^j X^i\right), \tag{3}$$

$$\left[P^i, P^j\right] = 0\,, \tag{4}$$

where $\beta,\ \beta'$ are two non-negative deformation parameters $(\beta, \beta' \geq 0)$ and $i,\ j, = 0, 1, 2, \ldots, D$. In Eqs. (2)–(4), X^i and P^i are position and momentum operators in the deformed space. From Eq. (2), it is easy to show that an isotropic minimal length scale equals [26]

$$\left(\Delta X^i\right)_{\min} = \hbar\sqrt{(D\beta + \beta')}\,, \qquad \forall i \in \{1, 2, \cdots, D\}\,. \tag{5}$$

The representation of operators satisfying modified Heisenberg algebra is essential for minimal length investigations. According to Ref. [27], Stetsko and Tkachuk used the approximate representation fulfilling the modified Heisenberg algebra in the first order over the deformation parameters β and β'

$$X^i = x^i + \frac{2\beta - \beta'}{4}\left(\boldsymbol{p}^2 x^i + x^i \boldsymbol{p}^2\right)\,, \tag{6}$$

$$P^i = p^i\left(1 + \frac{\beta'}{2}\boldsymbol{p}^2\right)\,, \tag{7}$$

where $\boldsymbol{p}^2 = \sum_{i=1}^{D} p^i p^i$ and the operators x^i and $p^i = i\hbar\frac{\partial}{\partial x_i}$ are position and momentum operators in ordinary quantum mechanics. In this paper, we consider the special case of $\beta' = 2\beta$, in which the position operators commute to the first order in deformation parameter β, *i.e.* $[X^i, X^j] = 0$. Considering this linear approximation, the modified Heisenberg algebra (2)–(4) can be written as follows:

$$\left[X^i, P^j\right] = i\hbar\left[\left(1 + \beta\boldsymbol{P}^2\right)\delta^{ij} + 2\beta P^i P^j\right]\,, \tag{8}$$

$$\left[X^i, X^j\right] = 0\,, \tag{9}$$

$$\left[P^i, P^j\right] = 0\,. \tag{10}$$

In Ref. [28], Brau showed that the following representations satisfy (8)–(10), in the first order in β:

$$X^i = x^i\,, \tag{11}$$

$$P^i = p^i\left(1 + \beta\boldsymbol{p}^2\right)\,. \tag{12}$$

We note that representations (6), (7) and (11), (12) coincide when $\beta' = 2\beta$.

3. The Stark effect in the presence of a minimal length scale based on the Kempf algebra

To study the Stark effect in the presence of a minimal length, we need to find modified electrostatic field and then obtain the modified Stark potential.

3.1. Electrostatic field in the presence of a minimal length

The Lagrangian density for an electrostatic field with a charge density $\rho(x)$ in a 3-dimensions space is [15, 29]

$$\mathcal{L} = \tfrac{1}{2}\epsilon_0 E(x)^2 - \rho(x)\phi(x)$$

or

$$\mathcal{L} = \tfrac{1}{2}\epsilon_0 (\partial^i\phi(x))(\partial^i\phi(x)) - \rho(x)\phi(x)\,, \tag{13}$$

where $\phi(x)$ is the electrostatic potential.

Now, we want to find the Lagrangian density for an electrostatic field in the presence of a minimal length based on the Kempf algebra. We must replace the usual position and derivative operators with the modified position and derivative operators according to Eqs. (11) and (12), that is

$$x^i \longrightarrow X^i \;=\; x^i\,, \tag{14}$$

$$\partial^i \longrightarrow D^i \;:=\; \left(1 - \beta\hbar^2\nabla^2\right)\partial^i\,, \tag{15}$$

where $\nabla^2 := \partial^i\partial^i$ is the Laplace operator. Let us write the Lagrangian density (13) by using Eqs. (14) and (15), that is

$$\begin{aligned}
\mathcal{L}_{\mathrm{ML}} &= \tfrac{1}{2}\epsilon_0 \left(D^i\phi(x)\right)\left(D^i\phi(x)\right) - \rho(x)\phi(x) \\
&= \tfrac{1}{2}\epsilon_0 \left[\left(1 - \beta\hbar^2\nabla^2\right)\partial^i\phi(x)\right]\left[\left(1 - \beta\hbar^2\nabla^2\right)\partial^i\phi(x)\right] - \rho(x)\phi(x) \\
&= \tfrac{1}{2}\epsilon_0 \left(\partial^i\phi(x)\right)\left(\partial^i\phi(x)\right) - \left(\epsilon_0\beta\hbar^2\nabla^2\right)\left(\partial^i\phi(x)\right)\left(\partial^i\phi(x)\right) - \rho(x)\phi(x) \\
&\quad -o\left(\beta^2\hbar^2\right)\,.
\end{aligned} \tag{16}$$

If we neglect terms of the order of $(\beta^2\hbar^2)$ and higher in Eq. (16), we will obtain the Lagrangian density for an electrostatic field in the presence of a minimal length scale as follows:

$$\mathcal{L}_{\mathrm{ML}} = \tfrac{1}{2}\epsilon_0 \left(1 - 2\beta\hbar^2\nabla^2\right)\left(\partial^i\phi(x)\right)\left(\partial^i\phi(x)\right) - \rho(x)\phi(x)$$

or

$$\mathcal{L}_{\mathrm{ML}} = \tfrac{1}{2}\epsilon_0 \left(1 - 2\beta\hbar^2\nabla^2\right)E(x)^2 - \rho(x)\phi(x)\,. \tag{17}$$

A comparison between ordinary Lagrangian density in Eq. (13) and Lagrangian density in Eq. (17) clearly shows that there is an equivalence between the electrostatic field in the presence of a minimal length and the usual electrostatic field as follows:

$$\boldsymbol{e(x)}_{\mathrm{ML}} = (1 - \beta\hbar^2\nabla^2)\boldsymbol{E(x)}\,. \tag{18}$$

In the above equation, the term $-\beta\hbar^2\nabla^2\boldsymbol{E(x)}$ can be considered as a minimal length effect.

3.2. Stark effect in the presence of a minimal length

In this section, let us obtain the Stark potential in the presence of a minimal length scale which can safely be used as usual perturbation theory. When an electron in the hydrogen atom is subjected to a uniform electrostatic field in the positive z-direction [30], the Hamiltonian H is split into two parts

$$H_0 = \frac{p^2}{2m} + V_0(x) \quad \text{and} \quad V = -eEz \quad (e < 0 \quad \text{for electron}) . \tag{19}$$

The part term $-eEz$ is the ordinary Stark potential. We can substitute electrostatic field in the presence of a minimal length, from Eq. (18) into usual electrostatic field in Eq. (19). We can also substitute momentum in the deformed space from Eq. (12) into usual momentum in Eq. (19), therefore, we will obtain the following Hamiltonian in the presence of a minimal length:

$$H_{\mathrm{ML}} = \frac{p^2}{2m} + V_0(x) + \frac{\beta}{m}\left(p^4\right) - \left(1 - \beta\hbar^2\nabla^2\right) eEz + O\left(\beta^2\right) . \tag{20}$$

After neglecting terms of the order of β^2 and higher in Eq. (20), the modified Stark potential can be obtained as follows:

$$V_{\mathrm{ML}} = \frac{\beta}{m}\left(p^4\right) - \left(1 - \beta\hbar^2\nabla^2\right) eEz\,. \tag{21}$$

We must use the perturbation theory to calculate the shift of energy. We assume that the energy eigenkets and the energy spectrum for the unperturbed problem are not only known, but also no energy level is degenerate. It should be mentioned that this assumption does not hold for $n \neq 1$ levels of the hydrogen atoms, where V_0 is the pure Coulomb potential (we will discuss such cases later). The energy shift can be written as

$$\begin{aligned} \mathit{\Delta}_k^{\mathrm{ML}} &= \frac{\hbar^4\beta}{m}\left[\left|\nabla^4\right|_{kk}\right] - eE\left[|z|_{kk} - \beta\hbar^2\left|\nabla^2 z\right|_{kk}\right] \\ &\quad + \sum_{j\neq k} \frac{\left|\frac{\hbar^4\beta}{m}\nabla^4 - \left(1-\beta\hbar^2\nabla^2\right) eEz\right|_{kj}^2}{E_k^{(0)} - E_j^{(0)}} + \ldots \end{aligned} \tag{22}$$

As we know that with no degeneracy, $|k\rangle$ is expected to be a parity eigenstate, so we have

$$|z|_{kk} = \left|\nabla^2 z\right|_{kk} = \left|z\nabla^2\right|_{kk} = 0\,. \tag{23}$$

In the above equation, the position operator and the Laplace operator commute together, *i.e.* $[z, \nabla^2] = 0$. According to Eq. (22), we find the linear Stark effect in the presence of a minimal length as follows:

$$(E_{100})_{\rm ML} = \frac{\hbar^4\beta}{m}\left|\nabla^4\right|_{kk} = \hbar^2\frac{\left(\hbar\sqrt{2\beta}\,\right)^2}{2m}\frac{8-a_0^3}{a_0^4}\,, \tag{24}$$

where a_0 stands for the Bohr radius. It should be noted that no linear usual Stark effect were found. From Eq. (24), there is no term in the energy shift proportional to E, so the modified energy shift is quadratic in E if terms of the order of e^3E^3 or higher are vanished. Now, let us look at $|\frac{\hbar^4\beta}{m}\nabla^4 - (1-\beta\hbar^2\nabla^2)eEz|^2$ in Eq. (22), in which k (or j) is the collective index that stands for (n,l,m) and (n',l',m').

The polarizability α of an atom which is defined in terms of the energy shift of the atomic state will be found as follows [30]:

$$\Delta = -\tfrac{1}{2}\alpha E^2\,. \tag{25}$$

Now, we want to obtain the polarizability α for the hydrogen atom in the presence of a minimal length. Let us consider the ground state of the hydrogen atom. We know that the ground state is non-degenerate, so the formula of non-degenerate perturbation theory can be applied. To this aim, according to Eq. (22), we obtain

$$\begin{aligned}\sum_{k\neq0}|\langle k^{(0)}|V_{\rm ML}|1,0,0\rangle|^2 &= \sum_{\text{all }k}\langle k^{(0)}|V_{\rm ML}|1,0,0\rangle\langle k^{(0)}|V_{ML}|1,0,0\rangle\\ &= \langle 1,0,0|(V_{\rm ML})^2|1,0,0\rangle\\ &= \langle 1,0,0|\left(\frac{\hbar^4\beta}{m}\nabla^4 - \left(1-\beta\hbar^2\nabla^2\right)eEz\right)^2|1,0,0\rangle\\ &= \langle 1,0,0|\left(\frac{\hbar^4\beta}{m}\nabla^4\right)^2 + V_{\rm ML}^{'2} - 2\frac{\hbar^4\beta}{m}\nabla^4 V_{\rm ML}'|1,0,0\rangle\,,\end{aligned} \tag{26}$$

where$V_{\rm ML}' = [z-\beta\hbar^4(\nabla^4 z)]$ is the quadratic in E for the modified Stark potential. If we neglect terms of the order of β^2 in Eq. (26), we will find the following result:

$$\begin{aligned}&\sum_{k\neq0}|\langle k^{(0)}|V_{\rm ML}|1,0,0\rangle|^2 = \langle 1,0,0|\left(V_{\rm ML}'\right)^2 - 2\frac{\hbar^4\beta}{m}eE\nabla^4 z|1,0,0\rangle\\ &= \langle 1,0,0|\left(V_{\rm ML}'\right)^2|1,0,0\rangle - 2\frac{\hbar^4\beta}{m}eE\langle 1,0,0|\nabla^4 z|1,0,0\rangle\,.\end{aligned} \tag{27}$$

It is easily shown that $\langle 1,0,0|\nabla^4 z|1,0,0\rangle = 0$. The polarizability α for the hydrogen atom in the presence of a minimal length scale is given by

$$\alpha_{\mathrm{ML}} = -2e^2 \sum_{k\neq 0}^{\infty} \frac{|\langle k^{(0)}|V'_{\mathrm{ML}}|1,0,0\rangle|^2}{\left[E_0^{(0)} - E_k^{(0)}\right]} \,. \tag{28}$$

We assume that the denominator in Eq. (28) is constant, then we can obtain the sum as follows:

$$\begin{aligned} \sum_{k\neq 0} |\langle k^{(0)}|V'_{\mathrm{ML}}|1,0,0\rangle|^2 &= \sum_{\text{all }\ k} \langle k^{(0)}|V'_{\mathrm{ML}}|1,0,0\rangle\langle k^{(0)}|V'_{\mathrm{ML}}|1,0,0\rangle \\ &= \langle 1,0,0|\left(V'_{\mathrm{ML}}\right)^2|1,0,0\rangle \\ &= \langle 1,0,0|\left[z - \beta\hbar^2\left(\nabla^2 z\right)\right]^2|1,0,0\rangle \,. \end{aligned} \tag{29}$$

After neglecting terms of the order of (β^2) and higher in Eq. (29), we obtain

$$\sum_{k\neq 0} |\langle k^{(0)}|V_{\mathrm{ML}}|1,0,0\rangle|^2 = \langle 1,0,0|z^2|1,0,0\rangle - 2\beta\hbar^2\langle 1,0,0|\nabla^2 z^2|1,0,0\rangle \,. \tag{30}$$

The term $-2\beta\hbar^2\langle 1,0,0|\nabla^2 z^2|1,0,0\rangle$ in Eq. (30) shows the effect of minimal length corrections. With considering the wave function for ground state of hydrogen atom, z^2 and $\nabla^2 z^2$ operators, we can evaluate $\langle z^2\rangle$ and $\langle\nabla^2 z^2\rangle$ as follows:

$$\begin{aligned} \left\langle z^2\right\rangle &= a_0^2 \,, \\ \left\langle\nabla^2 z^2\right\rangle &= -\left\langle\left(\frac{-r+2a_0}{ra_0^2}\right)\right\rangle = -\left(-1+\frac{1}{2}\right) = \frac{1}{2} \,. \end{aligned} \tag{31}$$

If we substitute Eq. (31) into Eq. (30), we will obtain the following evaluate modified Stark potential:

$$|\langle V_{\mathrm{ML}}\rangle|^2 = a_0^2 - \tfrac{1}{2}\left(\hbar\sqrt{2\beta}\right)^2 \,. \tag{32}$$

Now, let us find the modified polarizability α. According to Eq. (28), if we use the inequality

$$-E_0^{(0)} + E_k^{(0)} \geq -E_0^{(0)} + E_1^{(0)} = \frac{e^2}{2a_0}\left(\frac{3}{4}\right) \tag{33}$$

together with Eq. (33), an upper limit for the modified polarizability of the ground state of the hydrogen atom is

$$\begin{aligned}\alpha_{\mathrm{ML}} &< +2e^2\left[\frac{\left(a_0^2-\frac{\left(\hbar\sqrt{2\beta}\right)^2}{2}\right)}{\frac{e^2}{2a_0}\left(\frac{3}{4}\right)}\right],\\ \alpha_{\mathrm{ML}} &< 4.5\left[a_0^3-\frac{a_0}{2}\left(\hbar\sqrt{2\beta}\right)^2\right].\end{aligned} \tag{34}$$

Inequality (34) is an upper limit for the polarizability in the presence of a minimal length scale. It is clear that for $\hbar\sqrt{2\beta}\to 0$, the modified polarizability in above inequality (34) becomes the usual upper limit for the polarizability of the ground state of the hydrogen atom.

3.3. Linear modified Stark potential in the excited states of hydrogen

As it is well-known, the excited states of hydrogen atom have a degeneracy between states because in the Schrödinger theory with a pure Coulomb potential with no spin dependence, the bound-state energy of the hydrogen atom depends only on the principal quantum number n [30]. Now, let us investigate the effect of a modified uniform electric field on the excited states of the hydrogen atom. In the special case of $n = 2$, there are four degenerate states that include an $l = 0$ state called $2s$ and three $l = 1(m = 0, \pm 1)$ states called $2p$. Now, when we apply a uniform electric field in the presence of a minimal length, the appropriate perturbation operator is given by

$$V_{\mathrm{ML}} = \frac{\hbar^4\beta}{m}\nabla^4 - \left(1-\beta\hbar^2\nabla^2\right)eEz\,. \tag{35}$$

Before we obtain the matrix elements in detail using the ordinary (n, l, m) basis, it should be noted that the modified perturbation in Eq. (35) has non-vanishing matrix elements only between states with the same m-values because the component angular momentum, L_z, commutes with z, ∇^2 and $\nabla^2 z$ operators, *i.e.* $[L_z, z] = [L_z, \nabla^2 z] = [L_z, \nabla^2] = 0$.

Thus, matrix elements for $m \neq m'$ are zero, then we have

$$\langle V_{\mathrm{ML}}\rangle = \begin{pmatrix}\langle 2,0,0|V_{\mathrm{ML}}|2,0,0\rangle & 0 & \langle 2,0,0|V_{\mathrm{ML}}|2,1,0\rangle & 0\\ 0 & \langle 2,1,-1|V_{\mathrm{ML}}|2,1,-1\rangle & 0 & 0\\ \langle 2,1,0|V_{\mathrm{ML}}|2,0,0\rangle & 0 & \langle 2,1,0|V_{\mathrm{ML}}|2,1,0\rangle & 0\\ 0 & 0 & 0 & \langle 2,1,1|V_{\mathrm{ML}}|2,1,1\rangle\end{pmatrix}. \tag{36}$$

If we consider the $V_{\rm ML}$ in two terms $V'_{\rm ML} = -(1-\beta\hbar^2\nabla^2)eEZ$ and $V''_{\rm ML} = \frac{\hbar^4\beta}{m}\nabla^4$, we can find the following results:

$$\begin{aligned}
\langle 2,1,0|V''_{\rm ML}|2,0,0\rangle &= \frac{\hbar^4\beta}{m}\langle 2,1,0|\nabla^4|2,0,0\rangle = \left(\frac{\hbar}{a_0}\right)^4\frac{\beta}{m}\left[\frac{19}{32}\right],\\
\langle 2,1,-1|V''_{\rm ML}|2,1,-1\rangle &= \langle 2,1,1|V''_{\rm ML}|2,1,1\rangle = \langle 2,1,0|V''_{\rm ML}|2,1,0\rangle\\
&= \left(\frac{\hbar}{a_0}\right)^4\frac{\beta}{8m},\\
\langle 2,0,0|V''_{\rm ML}|2,1,0\rangle &= \langle 2,1,0|V''_{\rm ML}|2,0,0\rangle = \delta_{ll'} = 0\,.
\end{aligned}\tag{37}$$

In Eq. (37), the ∇^4 operator is defined as follows:

$$\nabla^4 = \left(\nabla^2\right)^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\boldsymbol{L}^2}{\hbar^2 r^2}\right)^2.\tag{38}$$

After calculating for term $V'_{\rm ML}$, we have

$$\begin{aligned}
\langle 2,0,0|V'_{\rm ML}|2,0,0\rangle &= \langle 2,1,-1|V'_{\rm ML}|2,1,-1\rangle = 0\,,\\
\langle 2,1,0|V'_{\rm ML}|2,1,0\rangle &= \langle 2,1,1|V'_{\rm ML}|2,1,1\rangle = 0\,,\\
\langle 2,0,0|V'_{\rm ML}|2,1,0\rangle &= 3eEa_0 - eE\beta\hbar^2\frac{(7.14)}{a_0}\,,\\
\langle 2,1,0|V'_{\rm ML}|2,0,0\rangle &= 3eEa_0 - eE\beta\hbar^2\frac{(7.14)}{a_0}\,.
\end{aligned}\tag{39}$$

If we substitute the evaluates of Eqs. (37) and (39) into Eq. (36), we will obtain the modified matrix as follows:

$$\langle V_{\rm ML}\rangle = \begin{pmatrix}
\left(\frac{\hbar}{a_0}\right)^4\frac{\beta}{m}[\frac{19}{32}] & 0 & 3eEa_0 - eE\beta\hbar^2\frac{(7.14)}{a_0} & 0\\
0 & \left(\frac{\hbar}{a_0}\right)^4\frac{\beta}{8m} & 0 & 0\\
3eEa_0 - eE\beta\hbar^2\frac{(7.14)}{a_0} & 0 & \left(\frac{\hbar}{a_0}\right)^4\frac{\beta}{8m} & 0\\
0 & 0 & 0 & \left(\frac{\hbar}{a_0}\right)^4\frac{\beta}{8m}
\end{pmatrix}.\tag{40}$$

Now, we want to discuss how to find the eigenvalues of modified matrix in Eq. (40). For this aim, by setting the determinant of matrix in Eq. (40)

equal to zero, we have

$$\left(\frac{\hbar}{a_0}\right)^4\left(\frac{3\beta}{8m}\right)\lambda^3+(eE)^2\left(9a_0^2-42.8\hbar^2\beta\right)\lambda^2-(eE)^2\left(\frac{9\beta}{4m}\right)\left(\frac{\hbar^4}{a_0^2}\right)\lambda+O\left(\beta^2\right)=0\,. \tag{41}$$

After neglecting terms of the order of β^2 and higher in Eq. (41), we can obtain the following equation:

$$\lambda\left[\left(\frac{\hbar}{a_0}\right)^4\left(\frac{3\beta}{8m}\right)\lambda^2+(eE)^2\left(9a_0^2-21.4\left(\hbar\sqrt{2\beta}\,\right)^2\right)\lambda-(eE)^2\left(\frac{9\beta}{4m}\right)\left(\frac{\hbar^4}{a_0^2}\right)\right]=0\,. \tag{42}$$

It is clearly showed that one of the roots of the cubic equation in Eq. (42) is $\lambda=0$. Also, we can find the roots of the second equation which is remained as Eq. (42). If we neglect terms of the order of β^2 and higher, the modified energy shift can be obtained as follows:

$$(\varDelta)_{1\mathrm{ML}} = 0\,, \tag{43}$$

$$(\varDelta)_{2\mathrm{ML}} = \frac{-2(eE)^2\left(9a_0^2-21.4\left(\hbar\sqrt{2\beta}\,\right)^2\right)}{3\frac{\beta}{m}\left(\frac{\hbar}{a_0}\right)^4}\,.$$

3.4. An upper bound on the minimal length in the modified Stark effect

In this section, we want to estimate an upper bound on the isotropic minimal length in the modified Stark effect. Substituting $\beta'=2\beta$ into Eq. (5), the isotropic minimal length becomes

$$\left(\Delta X^i\right)_{\mathrm{min}}=\sqrt{\frac{D+2}{2}}\hbar\sqrt{2\beta}\,,\qquad \forall i\in\{1,2,\cdots,D\}\,. \tag{44}$$

The isotropic minimal length in three spatial dimensions is given by

$$\left(\Delta X^i\right)_{\mathrm{min}}=\sqrt{\tfrac{5}{2}}\hbar\sqrt{2\beta}\,. \tag{45}$$

Now, if we consider the condition of Eq. (43) non-zero, we can obtain the following equation:

$$\left(\hbar\sqrt{2\beta}\right)\le\frac{9}{21.4}a_0\,. \tag{46}$$

If we insert Eq. (46) into Eq. (45), we can obtain the following upper bound on the minimal length scale:

$$\left(\Delta X^i\right)_{\mathrm{min}}=5.9\times10^{-10}\,\mathrm{cm}\,. \tag{47}$$

4. Conclusions

We know that recent investigation in perturbative string theory and quantum gravity suggests that there is a measurable minimal length in nature [1, 31]. According to the GUP, an immediate consequence is that the ordinary position and derivative operators must be replaced by modified position and derivative operators. We have obtained the electrostatic field in the presence of a minimal length scale based on the Kempf algebra. We have used modified electrostatic field and we have found the Stark potential in the presence of a minimal length. It was shown that the linear Stark effect in the presence of a minimal length scale was non-zero, while there can be non-linear Stark effect in ordinary form. An upper limit for the polarizability in the presence of a measurable length has been obtained. Moreover, the modified energy shifts in the excited states of hydrogen atom has been found. In the limit of $\hbar\sqrt{2\beta} \to 0$, the modified polarizability becomes the ordinary polarizability. Finally, we have found the upper bound on the isotropic minimal length scale in Eq. (47). It is interesting to note that the estimation on the isotropic minimal length is close to the observable minimal distance which was proposed by Heisenberg ($\ell_0 \sim 10^{-13}$ cm) [32–35].

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