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NEW SOLVABLE POTENTIALS WITH BOUND STATE SPECTRUM

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A new family of solvable potentials related to the Schrödinger–Riccati equation has been investigated. This one-dimensional potential family depends on parameters and is restricted to the real interval. It is shown that this potential class, which is a rather general class of solvable potentials related to the hypergeometric functions, can be generalized to even wider classes of solvable potentials. As a consequence, the non-linear Schrödinger-type equation has been obtained.

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1. Introduction

Though, in the present paper we deal with real equations, the obtained results can be developed to the complex domain following the approaches [1]. Solvable problems of non-relativistic quantum mechanics have always attracted much attention [2, 3]. The analytical methods to resolve the Schrödinger equation are very well-known [4]. A further remarkable development in solving the Schrödinger equation was the introduction of the concept of shape invariance [3]. Many of the potentials related by supersymmetry [2, 5, 6] were found to have similar shapes (*i.e.* to depend on the coordinate in similar way), only the parameters appearing in them were different. Although the number of potentials satisfying the shape invariance condition is limited, it turned out that the energy spectrum and the wave functions can be determined by elementary calculations in this case.

2. Schrödinger equation

In the present paper, we consider the Schrödinger equation in one dimension, setting $\hbar = 2m = 1$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_n(x) + (E_n - V(x))\psi_n(x) = 0\,, \tag{1}$$

then the function

$$W_n(x) = -\frac{\psi_n'(x)}{\psi_n(x)}\,, \tag{2}$$

where prime denotes differentiation with respect to x, satisfies the corresponding Riccati equation

$$W_n'(x) - W_n^2(x) = E_n - V(x)\,. \tag{3}$$

Assuming that the function $W_0(x)$ has a zero inside interval I and

$$W_0'(x) > 0\,, \qquad \forall x \in I \subset \mathbb{R}\,, \tag{4}$$

which is associated with normalization of the basic function ψ_0, we get

$$W_0'(x) = F(W_0)\,, \tag{5}$$

where F is an arbitrary function satisfying Eq. (4). The last equation is obtained from reversibility of the function $W_0(x)$ on interval I. Taking Eq. (5) into account and comparing it with Eq. (3), we get the following result:

$$W_0'(x) = W_0^2 + f(W_0)\,, \tag{6}$$

where

$$E_0 - V(x) = f(W_0)\,. \tag{7}$$

Now, we can express the potential $V(x)$ in terms of W_0 and we can use Eq. (6) to generate potentials by choosing $f(W_0)$.

A simplest and most obvious choice seems to be a second order polynomial

$$W_0'(x) = AW_0^2 + BW_0 + C\,, \tag{8}$$

where A, B, C are parameters. This differential equation is a first-order one and it can be solved in a straightforward way. The solution of Eq. (8) has the form of

$$W_0(x) = -\frac{B}{2A} + \frac{\sqrt{-B^2+4AC}}{2A}\tan\left(\frac{1}{2}\sqrt{-B^2+4AC}(x-x_0)\right)\,, \tag{9}$$

where

$$x_0 - \frac{\pi}{\sqrt{-B^2+4AC}} \le x \le x_0 + \frac{\pi}{\sqrt{-B^2+4AC}} \,. \tag{10}$$

Thus,

$$\psi_0(x) = e^{\frac{B(x-x_0)}{2A}} \left(\cos\left(\frac{1}{2}\sqrt{-B^2+4AC}(x-x_0)\right)\right)^{\frac{1}{A}} \tag{11}$$

is the unnormalized form of the ground state.

3. Cascade of equations

In order to have the same potential function in the Riccati equation for $n = 1$, we introduce an expression for W_1:

$$W_1 = W_0 - \frac{a_1}{b_1 W_0 - c_1} \,. \tag{12}$$

Now, let us turn our attention to the explicit determination of the coefficients a_1, b_1, c_1 in terms of the A, B, C. From Eq. (3) and Eq. (8), we get

$$a_1 = (A+2)C - \frac{(A+2)^2 B^2}{4(A+1)^2} \,, \tag{13}$$

$$b_1 = A+2 \,, \tag{14}$$

$$c_1 = -\frac{(A+2)B}{2(A+1)} \,. \tag{15}$$

Strightforward calculations lead us to the form of the first excited state wave function

$$\psi_1(x) = e^{\alpha_{01} x} (\cos\theta(x-x_0))^{\gamma_{01}} (\alpha_1 \cos\theta(x-x_0) + \beta_1 \sin\theta(x-x_0))^{\gamma_1} \,, \tag{16}$$

where all coefficients, denoted in the Greek letters, depend on the parameters A, B, C. Although this relationship is rather complex but, by the use of equations (2), (9) and (12), easy to achieve.

Considerations presented above can be generalized if we take the explicit form of the function W_n in terms of W_0:

$$W_n = W_0 - \cfrac{a_n}{b_n W_0 - \cfrac{c_n}{b_{n-1} W_0 - \cdots}} \,. \tag{17}$$

This function preserves the expression of equation

$$W_n'(x) - W_n^2(x) = W_0'(x) - W_0^2(x) + E_n - E_0 \tag{18}$$

for suitable values of coefficients which are involved in a system of non-linear equations (too complicated to be presented here). We should select the appropriate values of A, B, C parameters to simplify calculations. It will be done in the next chapter.

Equations (2), (3) and (8) enable us to obtain every wave functions ψ_n and energies E_n. It is easy to prove [6] that the function W_0 fulfil the shape invariance condition, so this potential family, resulting from Eq. (8), is an example for shape-invariant solvable potentials. Using Eq. (17), the wave functions can be written, without normalization, as

$$\psi_n(x) = e^{\alpha_{0n}x}(\cos\theta(x-x_0))^{\gamma_{0n}}\prod_{i=1}^{n}(\alpha_i\cos\theta(x-x_0)+\beta_i\sin\theta(x-x_0))^{\gamma_i}\,, \quad (19)$$

where, as in the previous case, all coefficients written in Greek depend on the A, B, C parameters.

4. The classic potentials

Equation (8) offers a convenient way to link this simple method with the well-known solutions of the Schrödinger equation. For instance, choosing $B = 0$, $C = 1$, we get the following results:

$$W_0' = AW_0^2 + 1 \quad (20)$$

and the first three wave functions

$$\psi_0(x) = \left(\cos\left(\sqrt{A}x\right)\right)^{\frac{1}{A}}, \quad (21)$$

$$\psi_1(x) = \frac{\left(\cos\left(\sqrt{A}x\right)\right)^{\frac{1}{A}}\sin\left(\sqrt{A}x\right)}{\sqrt{A}}, \quad (22)$$

$$\psi_2(x) = \frac{\left(\cos\left(\sqrt{A}x\right)\right)^{\frac{1}{A}}\left(-1+(1+A)\cos\left(2\sqrt{A}x\right)\right)}{A}, \quad (23)$$

which are orthoghonal on domain I (Eq. (10)), tend to very well-known solutions of the quantum oscillator $\psi_n(x) \to H_n(x)e^{-\frac{x^2}{2}}$ for $A \to 0$. It can be seen from figure 1 that the wave functions have the same characteristic shapes but they differ in domains.

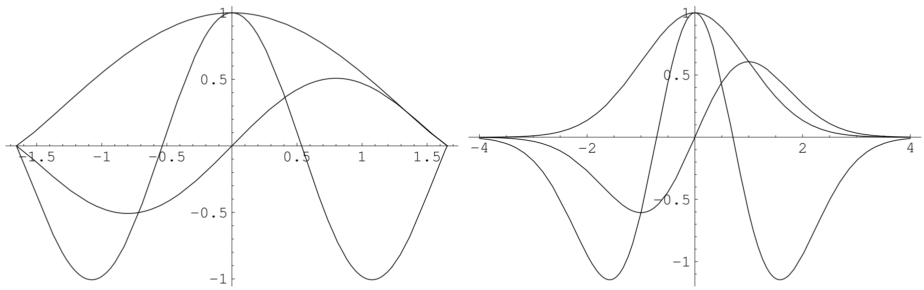

Fig. 1. Example of the wave functions with different values of the parameter A, where x is on the horizontal axis and $\psi_n(x)$ on the vertical one. Left: The first three wave functions for $A > 0 \quad (A = 0.9)$. Right: The first three wave functions of the quantum harmonic oscillator $(A \to 0)$.

Basing on the procedure described above, we are able to get solutions of the Schrödinger equation with the radial Coulomb potential (angular momentum is equal to zero). In this case,

$$W_0' = AW_0^2 - BW_0 + \frac{B^2}{4} \tag{24}$$

whose basic solution is

$$\psi_0(x) = \frac{\left(\sin\left(\frac{1}{2}\sqrt{A-1}Bx\right)\right)^{\frac{1}{A}}}{\sqrt{A-1}} e^{-\frac{B}{2A}x} \tag{25}$$

which tends to the radial part of the ground state eigenfunction of the Schrödinger equation for one-electron atom, $\psi_0 \to \frac{1}{2}Bxe^{-\frac{1}{2}Bx}$ for $A \to 1$. With the help of Eq. (17), we are able to get the wave functions for the excited states.

Another example of Eq. (8) which leads us to the very well-known solution is

$$W_0' = -AW_0^2 - W_0 + C\,, \tag{26}$$

where parameters $A > 0$, and $C > 0$. Thus,

$$W_0(x) = -\frac{1}{2A} + \frac{\sqrt{1+4AC}}{2A}\coth\left(\frac{1}{2}\sqrt{1+4AC}(x-x_0)\right)\,, \tag{27}$$

where the integration constant

$$x_0 = \frac{1}{\sqrt{1+4AC}}(\ln A + \imath\pi) \tag{28}$$

is the complex number. In this case, we have

$$W_0(x) = -\frac{1}{2A} + \frac{\sqrt{1+4AC}}{2A} \tanh\left(\frac{1}{2}\left(\sqrt{1+4AC}x - \ln A\right)\right), \tag{29}$$

and

$$\psi_0(x) = e^{\frac{x}{2A}} \cosh\left(\frac{1}{2}\left(\sqrt{1+4AC}x - \ln A\right)\right)^{-\frac{1}{A}} \tag{30}$$

is the basic wave function without normalization constant. From Eq. (29), we have $W_0 = C - e^{-x}$ for $A \to 0$ what is the standard expression for the Morse potential [5].

All potentials resulting from Eq. (8) have a trigonometric form. It means that they are expressed in terms of the tangent function. If we wish to obtain potentials interesting from the physical point of view, like the Coulomb potential, or the Morse potential, we should follow the procedure outlined above or choose the proper, initial value of the parameters in Eq. (8). It should be emphasized that every solution of the Schrödinger equation related to the orthogonal polynomials can be obtained by this method.

5. The new Hamiltonian

The results can be generalized to the form for which this method works [7]. If we take Eq. (8) in the form of

$$W_0' = AW_0^2 + \frac{P_{l+1}(W_0)}{Q_l(W_0)} = R_{l+2,l}(W_0)\,, \tag{31}$$

where $P_{l+1}(W_0)$ is a polynomial in W_0 with degree no greater than $l+1$, $Q_l(W_0)$ is a polynomial in W_0 with degree equal to l. $R_{l+2,l}(W_0)$ is a rational function such that both the numerator and the denominator are polynomials with degree $l+2$ and l respectively. Substituting

$$W_0(x) = R_{l+1,l}(\tan(\phi\cdot(x-x_0))) \tag{32}$$

into Eq. (31) and adjusting the indices of the sums to get the same powers of $\tan(\phi\cdot(x-x_0))$, we get the explicit form of W_0. It should be emphasized that the condition of Eq. (4) must be satisfied. The procedure outlined above can be applied to the function

$$W_n(x) = R_{l+n+1,l+n}(\tan(\phi\cdot(x-x_0))) \tag{33}$$

and thus the excited state wave functions $\psi_n(x)$ can be obtained.

Let us consider the simple equation, being the example of generalized Eq. (31)

$$W_0' = W_0^2 + \frac{3W_0 - 1}{W_0 - 3} = \frac{(W_0 - 1)^3}{W_0 - 3}\,, \tag{34}$$

where all coefficients has been chosen to simplify calculations.

Hence,

$$W_0(x) = \frac{2\sqrt{x + \frac{1}{4}} - 3}{2\sqrt{x + \frac{1}{4}} - 1}\,, \tag{35}$$

which gives the correspondig eigenvalue $E_0 = -1$ and the completly new potential is discovered

$$V(x) = -\frac{2}{\sqrt{x + \frac{1}{4}}} \qquad \text{for} \quad x \geq 0\,. \tag{36}$$

Thus, the ground state function, without normalization constant, has the form of

$$\psi_0(x) = e^{-x + 2\sqrt{x + \frac{1}{4}}} \left(2\sqrt{x + \tfrac{1}{4}} - 1\right)\,. \tag{37}$$

Substituting

$$W_1 = \frac{P_2(W_0)}{Q_1(W_0)} \tag{38}$$

into Eq. (3) and taking into account Eq. (36), we obtain the unnormalized wave function

$$\psi_1(x) = e^{-0.79x + 2.52\sqrt{x + \frac{1}{4}}} \left(2\sqrt{x + \tfrac{1}{4}} - 1\right)\left(2\sqrt{x + \tfrac{1}{4}} - 3.74\right)\,, \tag{39}$$

where all decimal numbers are approximated and $E_1 \approx -0.63$. It is easy to show that the latest potential does not fulfil the shape invariance condition [6], so this new potential family is an example for non-shape-invariant solvable potentials.

Let us now discuss the question of the explicit form of the Schrödinger equation. Treating Eq. (31) not as a condition but rather as the transformed Schrödinger equation and substituting Eq. (2) (for $n = 0$) into Eq. (31), we get

$$-\psi_0''(x;\alpha)\psi_0(x;\alpha) - \alpha(\psi_0'(x;\alpha))^2 = \left[E_0 - V\left(-\frac{\psi_0'(x;\alpha)}{\psi_0(x;\alpha)}\right)\right]\psi_0^2(x;\alpha)\,, \tag{40}$$

where the parameter α is usually related to the parameter A in Eq. (31) and the potential V has the form of

$$V\left(-\frac{\psi_0'(x;\alpha)}{\psi_0(x;\alpha)}\right) = R_{l+2,l}\left(-\frac{\psi_0'(x;\alpha)}{\psi_0(x;\alpha)}\right). \tag{41}$$

As we see, Eq. (40) is a non-linear differential equation. Taking into account the previous considerations regarding the quantum oscillator, the Coulomb potential and the Morse potential, we get the following form of Eq. (40) in the $\alpha \to 0$ limit:

$$-\psi_0''(x;0) = \left[E_0 - V\left(-\frac{\psi_0'(x;0)}{\psi_0(x;0)}\right)\right]\psi_0(x;0) \tag{42}$$

what is the familiar form of the Schrödinger equation, and where non-linearity is hidden in the form of the potential function.

6. Conclusions

The new method of obtaining solvable potentials has been reviewed in this paper. The main role in this method plays the Riccati equation which is a result of the transformed, one-dimensional, stationary Schrödinger equation. It allows us to emphasize the importance of function W_0 known in a literature as a "superpotential" [2, 5]. By the use of its features, we can show that the potential is not an arbitrary function of x but rather its form depends on the function W_0. As a consequence, we can find not only very well-known solutions of the Schrödinger equation but also a new class of the solvable potentials. These considerations may help us to identify new classes of the solvable potentials and may serve as an aid for further investigations concerning the relationship between solvability of the Schrödinger equation and the form of the potential.

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