A RANDOM MATRIX MODEL WHOSE EIGENVALUE SPACINGS ARE CLOSELY DESCRIBED BY THE BRODY DISTRIBUTION

John M. Nieminen

Christie Digital Systems Canada Inc.
809 Wellington Street North, Kitchener, Ontario, N2G 4Y7, Canada

Lutz Muche

Technical University Dresden, Institute of Mathematical Stochastics
01062 Dresden, Germany

(Received March 9, 2017; accepted April 24, 2017)

We introduce a primitive 2×2 random matrix model with one parameter, $1 \leq d \leq 2$. It is shown that an ensemble of such matrices has eigenvalue spacings that transition from near-Poisson statistics, when $d = 2$, to GOE statistics for 2×2 matrices, when $d = 1$. This transition can be very closely modelled by the Brody distribution, where the Brody parameter is $q = (2 - d)/d$. Exact integral forms for the complete transition are given.

DOI:10.5506/APhysPolB.48.765

The Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensembles are cornerstones of random matrix theory (RMT) — they are often referred to as the classical ensembles of RMT. As such, the statistical properties of the eigenvalues from these ensembles have been extensively studied (see Ref. [1]) and are well-known to practitioners of RMT. Long-range correlations of eigenvalues are quantified using statistics like the number variance or the spectral rigidity, whereas short-range correlations are studied via the nearest-neighbour spacing distribution (NNSD).

Of further interest to researchers are transitional (or intermediate) systems. In RMT, matrix models that transition between the classical ensembles, *e.g.* a GOE-to-GUE transition, or matrix models that transition between randomly spaced eigenvalues and the classical ensembles, *e.g.* a Poisson-to-GOE transition, have been investigated. An expert summary of these types of transitions is given in chapter 3 of Ref. [2] — we refer the reader to that book and references therein (Refs. [3–5] present other examples). Intermediate eigenvalue (or level) statistics can be indicative of some underlying symmetry breaking or a mixing of eigenvalues from different subspaces that have good symmetries, particularly for physical systems. By using insights gained from transitional RMT models, one can begin to quantify, and better understand, these intermediate cases.

In the present work, we focus our attention on the Poisson-to-GOE transition. In particular, we introduce a new 2×2 random matrix model, with one parameter ($1 \le d \le 2$), that smoothly transitions from near-Poisson statistics to (2×2) GOE statistics. An integral form for the NNSD of eigenvalues from our model is presented and shows a surprising resemblance to the Brody distribution [6].

The random matrix of interest is as follows:

$$H = \begin{pmatrix} |a|^d & \frac{1}{2}|b|^d \\ \frac{1}{2}|b|^d & 0 \end{pmatrix} , \tag{1}$$

where a and b are real numbers chosen from a Gaussian distribution, with zero mean and a variance of σ^2, and $d \ge 0$ (later, we will restrict this to $1 \le d \le 2$, but that is not necessary at this stage). The eigenvalues of H are

$$E_\pm = \tfrac{1}{2}\left[|a|^d \pm \sqrt{(|a|^d)^2 + 4\left(\tfrac{1}{2}|b|^d\right)^2}\,\right] , \tag{2}$$

from which we get the spacing of

$$s = E_+ - E_- = \sqrt{|a|^{2d} + |b|^{2d}}\,. \tag{3}$$

In order to calculate the NNSD for an ensemble of H matrices, we first start with a statement of the geometrical probability $F(s) = \mathrm{P}(|a|^{2d} + |b|^{2d} \le s^2)$

$$F(s) = \frac{1}{2\pi\sigma^2} \underbrace{\iint}_{|a|^{2d}+|b|^{2d} \le s^2} \exp\left(-\frac{a^2+b^2}{2\sigma^2}\right) \mathrm{d}a\, \mathrm{d}b\,, \tag{4}$$

where $-\infty < a, b < +\infty$. Through symmetry arguments, it is then easy to see that

$$F(s) = \frac{2}{\pi\sigma^2} \underbrace{\iint}_{a^{2d}+b^{2d} \le s^2} \exp\left(-\frac{a^2+b^2}{2\sigma^2}\right) \mathrm{d}a\, \mathrm{d}b\,, \tag{5}$$

where $0 \le a, b < \infty$. We now introduce the following transformation:

$$b = a \tan^{1/d} \varphi\,, \qquad 0 \le \varphi \le \frac{\pi}{2}\,, \tag{6}$$

which allows us to rewrite Eq. (5) as

$$F(s) = \frac{2}{\pi\sigma^2}\int_0^{\pi/2}\int_0^{s^{1/d}\cos^{1/d}\varphi} \frac{a}{d}\frac{\sin^{\frac{1}{d}-1}\varphi}{\cos^{\frac{1}{d}+1}\varphi}\exp\left(-\frac{a^2\left(1+\tan^{2/d}\varphi\right)}{2\sigma^2}\right)\mathrm{d}a\,\mathrm{d}\varphi$$

$$= \frac{2}{\pi d}\int_0^{\pi/2}\frac{\cos^{\frac{1}{d}-1}\varphi\sin^{\frac{1}{d}-1}\varphi}{\cos^{2/d}\varphi+\sin^{2/d}\varphi}\left(1-\exp\left(-\frac{s^{2/d}\left(\cos^{2/d}\varphi+\sin^{2/d}\varphi\right)}{2\sigma^2}\right)\right)\mathrm{d}\varphi\,. \tag{7}$$

Differentiation with respect to s leads us to the probability density function for the spacings

$$\begin{aligned} P(s) \;=\; & \frac{2}{\pi d^2\sigma^2}\int_0^{\pi/2} s^{\frac{2}{d}-1}\cos^{\frac{1}{d}-1}\varphi\,\sin^{\frac{1}{d}-1}\varphi \\ & \times\exp\left(-\frac{s^{2/d}\left(\cos^{2/d}\varphi+\sin^{2/d}\varphi\right)}{2\sigma^2}\right)\mathrm{d}\varphi\,. \end{aligned} \tag{8}$$

The mean value of s can be shown to be

$$\bar{s} = \int_0^{\infty} sP(s)\mathrm{d}s = \frac{2^{d/2}\Gamma(d/2)\sigma^d}{\pi}\int_0^{\pi/2}\frac{\cos^{\frac{1}{d}-1}\varphi\sin^{\frac{1}{d}-1}\varphi}{\left(\cos^{2/d}\varphi+\sin^{2/d}\varphi\right)^{d/2+1}}\mathrm{d}\varphi\,, \tag{9}$$

which allows us to define rescaled spacings as $S = s/\bar{s}$, and the probability density function for the rescaled spacings (*i.e.* the NNSD) can finally be derived

$$\begin{aligned} P(S) \;=\; & \frac{2}{\pi d^2\sigma^2}\int_0^{\pi/2}\bar{s}^{\frac{2}{d}}S^{\frac{2}{d}-1}\cos^{\frac{1}{d}-1}\varphi\,\sin^{\frac{1}{d}-1}\varphi \\ & \times\exp\left(-\frac{(\bar{s}S)^{2/d}\left(\cos^{2/d}\varphi+\sin^{2/d}\varphi\right)}{2\sigma^2}\right)\mathrm{d}\varphi\,. \end{aligned} \tag{10}$$

We next study a couple of special cases. For $d = 1$, it is easy to see that

$$\bar{s} = \sqrt{\frac{\pi}{2}}\sigma \tag{11}$$

and

$$P(S) = \frac{\pi}{2} S \exp\left(-\frac{\pi}{4} S^2\right) , \tag{12}$$

which, unsurprisingly, is the NNSD of eigenvalues from the GOE for 2×2 random matrices (also known as the Wigner surmise for the GOE [the Wigner–GOE for short] or the Wigner distribution). Shown in Fig. 1 is $P(S)$ as given by Eq. (12), as well as a numerical study of an ensemble of 10^7 random H matrices for the case of $d = 1$. (Note that for all of our numerical studies, we have set $\sigma = 1$, without loss of generality.) For $d = 2$, we get the more complicated forms of

$$\bar{s} = \frac{2\sigma^2}{\pi^{3/2}} \left(\Gamma^2\left(\frac{3}{4}\right) + 4\Gamma^2\left(\frac{5}{4}\right) \right) \tag{13}$$

and

$$P(S) = \frac{\bar{s}}{2\pi\sigma^2} \int_0^{\pi/2} \frac{1}{\sqrt{\cos\varphi \sin\varphi}} \exp\left(-\frac{\bar{s} S \left(\cos\varphi + \sin\varphi\right)}{2\sigma^2} \right) \mathrm{d}\varphi \, . \tag{14}$$

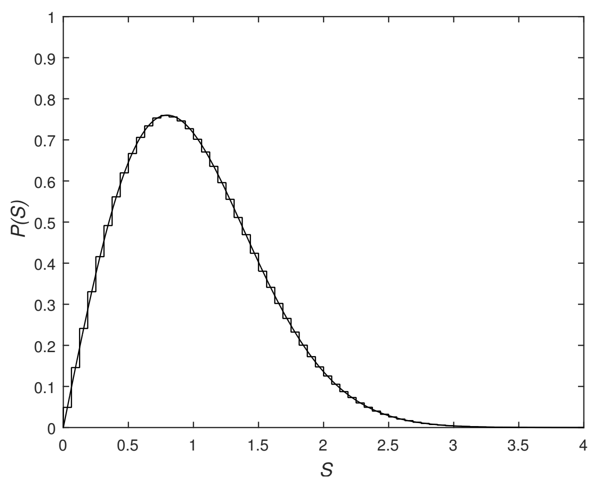

Fig. 1. The histogram shows the NNSD from a numerical study of 10^7 random H matrices (see Eq. (1)) with $d = 1$. The solid curve is the analytical solution given by Eq. (12).

Shown in Fig. 2 is $P(S)$ as given by Eq. (14), as well as a numerical study of an ensemble of 10^7 random H matrices for the case of $d = 2$. It is quite interesting how similar Eq. (14) is to the Poisson distribution

$$P(S) = \exp\left(-S\right), \tag{15}$$

which describes the NNSD of randomly spaced eigenvalues — if the Poisson distribution were plotted in Fig. 2, it would essentially be indistinguishable from the solid curve in that figure.

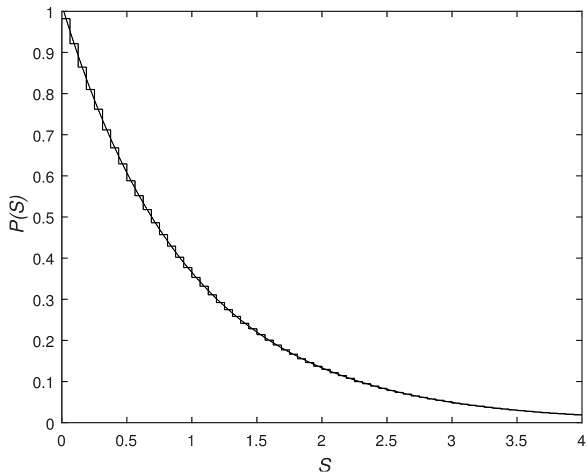

Fig. 2. The histogram shows the NNSD from a numerical study of 10^7 random H matrices (see Eq. (1)) with $d = 2$. The solid curve is the analytical solution given by Eq. (14), where we solve the integral numerically.

For values of d that are between 1 and 2, we work directly with Eq. (10) — shown in Fig. 3 is $P(S)$ for the case of $d = \pi/2$, as well as a numerical study of an ensemble of 10^7 random H matrices (for $d = \pi/2$), and shown in Fig. 4 is a near-Poisson-to-Wigner–GOE transition which was achieved by varying d from 2 to 1 in steps of 0.1. One immediately recognizes that $P(S)$ closely resembles the Brody distribution [6]

$$P_{\mathrm{B}}(S) = \alpha(q+1)S^q \exp\left(-\alpha S^{q+1}\right) , \tag{16a}$$

where

$$\alpha = \left[\Gamma\left(\frac{q+2}{q+1}\right)\right]^{q+1} \tag{16b}$$

and q is called the Brody parameter. Note that $P_{\mathrm{B}}(S)$ reduces to the Poisson distribution [Eq. (15)] when $q = 0$ and to the Wigner distribution [Eq. (12)] when $q = 1$. The Brody distribution has been successfully used, as a phenomenological distribution, to model Poisson-to-GOE transitions in many fields of study. For example, it was used to characterize the transition between regularity and irregularity in a Hamiltonian system [7] and to describe energy level statistics in a study of quantum chaos in a ripple billiard [8] and of random networks [9]. The cumulative form of the Brody distribution was

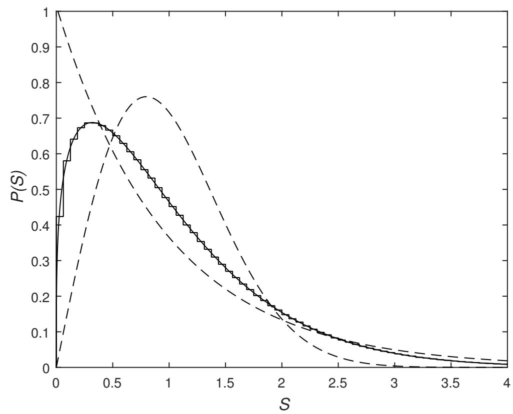

Fig. 3. The histogram shows the NNSD from a numerical study of 10^7 random H matrices (see Eq. (1)) with $d = \pi/2$. The solid curve is the analytical solution given by Eq. (10), where we solve the integral numerically. The dashed curves are the solid curves from Figs. 1 and 2.

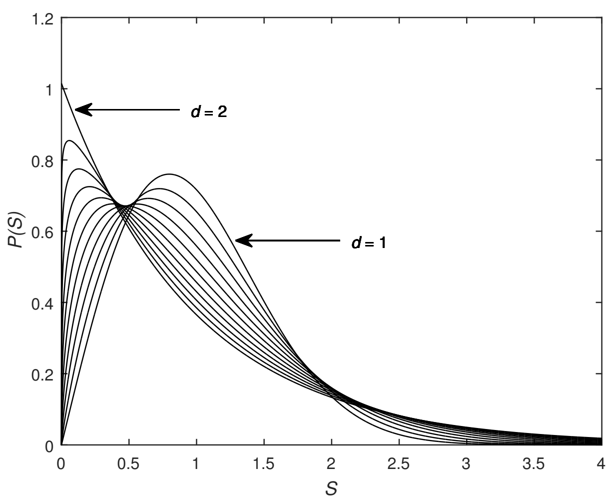

Fig. 4. $P(S)$ as obtained from numerically solving Eq. (10) for values of d ranging from 2 to 1 in steps of 0.1. Note the smooth transition from near-Poisson statistics to Wigner–GOE statistics.

successfully employed in the study of quantum chaos of the hydrogen atom in a generalized van der Waals potential [10]. In a study of random points uniformly distributed on a self-similar fractal, the NNSD of the points was, in fact, shown to be the Brody distribution [11]. Note that other 2×2 matrix models whose eigenvalue spacings transition between Poisson and GOE statistics exist in the literature [12–14], however their NNSDs are not described by $P_{\rm B}(S)$. In fact, as far as we know, a random matrix model whose eigenvalues have a NNSD that is exactly described by $P_{\rm B}(S)$ does not exist. If we now set

$$q = \frac{2-d}{d}\,, \tag{17}$$

we can study the difference between $P(S)$ and $P_{\rm B}(S)$, as shown in Fig. 5 — it is fascinating how close these functions are to one another.

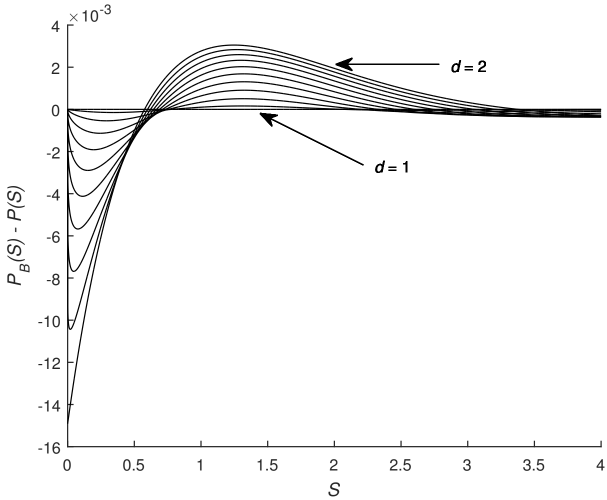

Fig. 5. Shown is $P_{\rm B}(S) - P(S)$ for values of d ranging from 2 to 1 in steps of 0.1. Once again, the integral in Eq. (10) was solved numerically.

To summarize, we introduced a new 2×2 random matrix model H, with one parameter ($1 \le d \le 2$), that smoothly transitions from near-Poisson statistics to Wigner–GOE statistics. The NNSD of eigenvalues for an ensemble of H matrices was derived and integral forms were presented. It was then noted that the NNSD for this model can be closely modelled by the Brody distribution, where the Brody parameter is $q = (2-d)/d$.

REFERENCES

[1] M.L. Mehta, *Random Matrices*, 3rd ed., Elsevier, San Diego 2004.

[2] V.K.B. Kota, *Embedded Random Matrix Ensembles in Quantum Physics*, *Lect. Notes Phys.* **884**, Springer, Heidelberg 2014.

[3] J.M. Nieminen, *J. Phys. A: Math. Theor.* **42**, 035001 (2009).

[4] F. Zypman, *J. Phys. A: Math. Theor.* **43**, 225203 (2010).

[5] J.M. Nieminen, *J. Math. Phys.* **51**, 053510 (2010).

[6] T.A. Brody, *Lett. Nuovo Cimento Soc. Ital. Fis.* **7**, 482 (1973).

[7] D. Wintgen, H. Friedrich, *Phys. Rev. A* **35**, 1464 (1987).

[8] W. Li, L.E. Reichl, B. Wu, *Phys. Rev. E* **65**, 056220 (2002).

[9] J.A. Méndez-Bermúdez *et al.*, *Phys. Rev. E* **91**, 032122 (2015).

[10] K. Ganesan, M. Lakshmanan, *J. Phys. B: At. Mol. Opt. Phys.* **27**, 2809 (1994).

[11] J. Sakhr, J.M. Nieminen, *Phys. Rev. E* **72**, 045204(R) (2005).

[12] G. Lenz, F. Haake, *Phys. Rev. Lett.* **67**, 1 (1991).

[13] V.K.B. Kota, S. Sumedha, *Phys. Rev. E* **60**, 3405 (1999).

[14] S. Schierenberg, F. Bruckmann, T. Wettig, *Phys. Rev. E* **85**, 061130 (2012).