CLASSICAL MECHANICS AND QUANTUM FIELDS IN NONCOMMUTATIVE PHASE SPACES

Everton M.C. Abreu[a,b,†], Mario J. Neves[a,‡], Vahid Nikoofard[c,§]

[a]Grupo de Física Teórica e Matemática Física, Departamento de Física
Universidade Federal Rural do Rio de Janeiro
465-07, 23890-971, Seropédica, RJ, Brazil
[b]Departamento de Física, ICE, Universidade Federal de Juiz de Fora
36036-330, Juiz de Fora, MG, Brazil
[c]Faculdade de Tecnologia, Universidade do Estado do Rio de Janeiro (UERJ)
Rodovia Presidente Dutra, Km 298, Pólo Industrial
27537-000, Resende, RJ, Brazil

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In this paper, we have analyzed noncommutative (NC) structures in the case when the NC parameter is not constant. Firstly, it is a variable in the configuration space making part of the dynamics of the relativistic particle action. Secondly, through a straightforward approach, we have constructed quantum scalar fields and its algebra in an NC space-time.

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1. Introduction

The search for the holy grail in theoretical physics is composed of the main challenges that have dwelt among us since the last century. One of these challenges is to unify in a single and consistent framework both theories of quantum mechanics and general relativity. The combination of special relativity and quantum field theory has already been accomplished through the Klein–Gordon and Dirac approaches. However, the path to reconcile the general relativity with the quantum field theory is still a mystery.

This so-called quantization procedure of general relativity has stumbled onto another theoretical physics challenge, *i.e.*, the infinities (divergences) that appear in some specific calculations during the quantization process.

† evertonabreu@ufrrj.br
‡ mariojr@ufrrj.br
§ vahid@fat.uerj.br

This issue is directly connected to the understanding of the behavior of quantum fields at the high-energy scale which is also connected to the structure of space-time at (or near) the Planck scale. The understanding of the structure of the space-time at this scale is necessary to construct the Hilbert space inner product, essential to the definition of the particle states. There are several formalisms that deal with these questions and one of those is the noncommutative (NC) geometry, which can, for these reasons, be considered as a theoretical laboratory for quantum gravity.

For example, one attempt to free us from the infinities that appear in quantum field theory was made by Snyder [1] when he constructed a five-dimensional NC algebra in order to define a minimum length for space-time structure. Unfortunately, a little time after the Snyder's effort, Yang [2] demonstrated that even in this Snyder's NC algebra, the divergences are still there.

This result condemned Snyder noncommutativity (NCY) to be outcast for more than fifty years until Seiberg and Witten [3] demonstrated that the algebra resulting from string theory embedded into a magnetic field has shown itself to have an NC algebra. Since then, we have seen a massive research production concerning several NC formulations that deserves our attention and investigation.

In the 1980s, NC geometry was considered as a way to extend the Standard Model in a number of different ways [4–8]. In condensed matter physics, NCY appears naturally. For example, NC geometry describes the dynamics of electrons in a magnetic field at the lowest energy level which is related to the quantum Hall effect [9, 10]. In matrix models of M-theory, for example, compactification leads to NC tori. As we have mentioned above, the so-called SW map [3] between commutative and NC gauge theories explained that gauge symmetries, including diffeomorphisms, can be realized by standard commutative transformations on commutative fields.

This theoretical framework is called NC field theory and it may be a relevant physical model at scales in between ℓ_{P} ($\simeq 1.6 \times 10^{-33}$cm) and ℓ_{LHC} ($\simeq 2 \times 10^{-18}$cm). In fact, one of the main threads of research in this field has been related to studies of energetic cosmic rays.

It was expected that some quantum field theories would be better behaved on NC space-time than on ordinary space-time. The interested reader can find some interesting and encouraging results in [11]. In this manner, space-time NCY presents an alternative to supersymmetry or string theory. Besides, it is a useful arena for studying physics beyond the Standard Model, and also for standard physics in strong external fields. Finally, it sheds light on alternative lines of attack to address various fundamental issues in QFT. For instance, it naturally relates field theory to gravity. Since the field theory may be easier to quantize, this may provide significant insights into the problem of quantizing gravity. However, in his approach, Snyder postulated an identity between coordinates and generators of the SO(4,1) algebra. Hence, he promoted the space-time coordinates to the Hermitian operators.

Mathematically speaking, the late 80s and the 90s have shown considerable progress in the NC geometry [12]. The introduction of these new techniques in QFT was considered in [13], where these approaches, in general, use a generalization of the canonical commutation relations [14].

In this paper, we will attack the quantization approaches of theories described in NC phase spaces by investigating its dynamics features. Firstly, we will disclose new characteristics about the NC relativistic particle which were not analyzed in [15], where the author presented the Dirac's constraint point of view. We have used the symplectic structure to show new results such as Newton's second law and actual role played by the Lagrangian multipliers used in [15]. After that, we have constructed straightforwardly the scalar field and its algebra within Doplicher–Fredenhagen–Roberts (DFR) quantum gravity scenario [16, 17] in a Lorentz invariant space-time.

To accomplish these tasks, we have organized the paper in the following way: In Section 2, we have discussed the extended analysis of the NC relativistic particle in extra dimensions. This model brought interesting results besides the ones connected directly to what we want to show. In Section 3, we have computed the basic commutation relations for the scalar field in the DFR phase space. Consequently, we believe that we have filled the gap that exists in DFR literature which does not see the necessity of an associated momentum relative to the θ-coordinate. Finally, in Section 4, the conclusions and perspectives were depicted. We have constructed an appendix with some explanations about the Moyal–Weyl product.

2. The NC relativistic particle

In [15], the author proposed that the cure for the lack of relativistic invariance for NC models is to modify the constant feature of the NC parameter, *i.e.*, the NC parameter would be a dynamical configuration space variable $\theta^{\mu\nu}(\tau)$. Consequently, he has analyzed the NC version for the D-dimensional relativistic particle with a θ-variable phase space and a π-momentum.

Here, we are interested in the dynamics of a particle in this NC phase space. On the other hand, in [15], the author was interested in analyzing mainly the constraint algebra and quantization via the Dirac bracket approach. We have calculated the equations of motion and the NC relativistic acceleration in order to discuss the $\theta_{\text{constant}} \Longrightarrow \theta_{\text{variable}}$ duality and its consequence. We will see through the symplectic structure that, although $p_\theta^{\mu\nu} = 0$, $\dot{p}_\theta^{\mu\nu}$ is not zero, which is an interesting and unusual result.

2.1. Noncommutative relativistic free particle

In this section, since we are interested in the DFR features that exist in the analyzed model, we will mention only the relative points of [15] where more details can be found.

The action of the free relativistic particle in this NC configuration space is

$$S(x,\theta) = \int \mathrm{d}\tau \left[\dot{x}^{\mu} v_{\mu} - \frac{e}{2}\left(v^2 - m^2\right) + \frac{1}{\theta^2}\,\dot{v}_{\mu}\theta^{\mu\nu}v_{\nu}\right] , \tag{1}$$

where $\theta^2 \equiv \theta^{\mu\nu}\theta_{\mu\nu}$, $\eta = \mathrm{diag}(+,-,\ldots,-)$. p^{μ}, π^{μ}, p_e, $p_{\theta}^{\mu\nu}$ are the conjugate momenta associated to $x^{\mu}(\tau)$, $v^{\mu}(\tau)$, $e(\tau)$ and $\theta^{\mu\nu}(\tau)$, respectively. As we have said just above, it is a D-dimensional space-time and μ, ν are the space-time indices. To clarify in detail, the "e" index in p_e means that p_e is the conjugate momenta associated with the $e(\tau)$ phase-space coordinate. An analogous meaning has the index θ in $p_{\theta}^{\mu\nu}$.

We will use the fundamental algebra [15] defined by

$$\begin{aligned}
\{x^{\mu}, x^{\nu}\} &= -\frac{2}{\theta^2}\,\theta^{\mu\nu} , \qquad \{x^{\mu}, p^{\nu}\} = \eta^{\mu\nu} , \qquad \{v^{\mu}, \pi^{\nu}\} = \eta^{\mu\nu} , && (2)\\
\{x^{\mu}, v^{\nu}\} &= \eta^{\mu\nu} , \qquad \{x^{\mu}, \pi^{\nu}\} = -\frac{1}{\theta^2}\,\theta^{\mu\nu} , && (3)\\
\left\{\theta_{\mu\nu}, p_{\theta}^{\rho\sigma}\right\} &= -\delta_{\mu}^{[\rho}\,\delta_{\nu}^{\sigma]} , && (4)\\
\left\{x^{\mu}, p_{\theta}^{\rho\sigma}\right\} &= -\left\{\pi^{\mu}, p_{\theta}^{\rho\sigma}\right\} = \frac{1}{\theta^2}\,\eta^{\nu[\rho}v^{\sigma]} - \frac{4}{\theta^4}(\theta v)^{\mu}\theta^{\rho\sigma} . && (5)
\end{aligned}$$

This system is singular and has the following primary constraints

$$\begin{aligned}
G^{\mu} &= p^{\mu} - v^{\mu} , && (6)\\
T^{\mu} &= \pi^{\mu} - \frac{1}{\theta^2}\,\theta^{\mu\nu}v_{\nu} , && (7)\\
p_{\theta}^{\mu\nu} &= 0 , && (8)\\
p_e &= 0 && (9)
\end{aligned}$$

and we can write the total Hamiltonian as

$$H = \frac{e}{2}\left(v^2 - m^2\right) + \lambda_{1\mu}G^{\mu} + \lambda_{2\mu}T^{\mu} + \lambda_e p_e + \lambda_{\theta\mu\nu}p_{\theta}^{\mu\nu} , \tag{10}$$

where the λs are the Lagrangian multipliers. Using the time consistency condition (*i.e.*, $\dot{\chi} = \{\chi, H\} = 0$, where χ is a constraint), we can obtain the secondary constraint

$$K \equiv v^2 - m^2 = 0 \tag{11}$$

and other relations that allow us to determine the Lagrangian multipliers

$$\dot{G}^{\mu} = \{G^{\mu}, H\} = 0 \implies \lambda_2^{\mu} = 0\,, \tag{12}$$

$$\dot{T}^{\mu} = \{T^{\mu}, H\} = 0 \implies \lambda_1^{\mu} = ev^{\mu} + \frac{2}{\theta^2}\left(\lambda_\theta v\right)^{\mu} - \frac{4}{\theta^4}\left(\theta\lambda_\theta\right)\left(\theta v\right)^{\mu}\,. \tag{13}$$

If we substitute the fixed Lagrangian multipliers into the Hamiltonian, we have that

$$\begin{aligned} H &= \frac{e}{2}\left(p^2 - m^2\right) + \left(ev_\mu + \frac{2}{\theta^2}\left(\lambda_\theta v\right)_\mu - \frac{4}{\theta^4}\left(\theta\lambda_\theta\right)\left(\theta v\right)_\mu\right) \\ &\quad \times \left(p^\mu - v^\mu\right) + \lambda_e p_e + \lambda_{\theta\mu\nu} p_\theta^{\mu\nu} \end{aligned} \tag{14}$$

and it can be seen that we were left with two undetermined Lagrangian multipliers.

Let us define the following symplectic variables α^i as $(x^i,\ p_i,\ \theta^{ij},\ \pi_{ij})$. We can write the generalized Poisson bracket for this system in a compact and symplectic form as

$$\{F, G\} = \left\{\alpha^i, \alpha^j\right\} \frac{\partial F}{\partial \alpha^i}\frac{\partial G}{\partial \alpha^j}\,, \tag{15}$$

where we are using the sum rule for repeated indices. In the same way, we will define the following symplectic variables

$$\begin{aligned} \xi^\mu &\to \left(x^\mu, p_\mu\right)\,, \\ \zeta^\mu &\to \left(v^\mu, \pi_\mu\right)\,, \\ \chi^\mu &\to \left(e, p_e\right)\,, \\ \Omega^{\mu\nu} &\to \left(\theta^{\mu\nu}, p_{\theta\mu\nu}\right)\,. \end{aligned} \tag{16}$$

We can write the generalized Poisson brackets for this system in a compact and symplectic form as

$$\begin{aligned} \{F, G\} &= \{\xi^\mu, \xi^\nu\}\frac{\partial F}{\partial \xi^\mu}\frac{\partial G}{\partial \xi^\nu} + \{\zeta^\mu, \zeta^\nu\}\frac{\partial F}{\partial \zeta^\mu}\frac{\partial G}{\partial \zeta^\nu} \\ &\quad + \{\chi^\mu, \chi^\nu\}\frac{\partial F}{\partial \chi^\mu}\frac{\partial G}{\partial \chi^\nu} + \{\Omega^{\mu\nu}, \Omega^{\rho\sigma}\}\frac{\partial F}{\partial \Omega^{\mu\nu}}\frac{\partial G}{\partial \Omega^{\rho\sigma}} \\ &\quad + \{\xi^\mu, \zeta^\nu\}\frac{\partial F}{\partial \xi^\mu}\frac{\partial G}{\partial \zeta^\nu} + \{\xi^\mu, \chi^\nu\}\frac{\partial F}{\partial \xi^\mu}\frac{\partial G}{\partial \chi^\nu} \\ &\quad + \{\xi^\mu, \Omega^\nu\}\frac{\partial F}{\partial \xi^\mu}\frac{\partial G}{\partial \Omega^\nu} + \{\chi^\mu, \zeta^\nu\}\frac{\partial F}{\partial \chi^\mu}\frac{\partial G}{\partial \zeta^\nu} \\ &\quad + \{\chi^\mu, \Omega^\nu\}\frac{\partial F}{\partial \chi^\mu}\frac{\partial G}{\partial \Omega^\nu} + \{\zeta^\mu, \Omega^\nu\}\frac{\partial F}{\partial \zeta^\mu}\frac{\partial G}{\partial \Omega^\nu}\,. \end{aligned} \tag{17}$$

According to Eq. (17), we can obtain the following equation of motion for x^μ as

$$\begin{aligned}
\dot{x}^\mu &= \{x^\mu, H\} \\
&= \{x^\alpha, p_\beta\} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial H}{\partial p_\beta} + \{p_\beta, x^\alpha\} \frac{\partial x^\mu}{\partial p_\beta} \frac{\partial H}{\partial x^\alpha} + \left\{x^\alpha, x^\beta\right\} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial H}{\partial x^\beta} \\
&\quad + \{x^\alpha, p_{\theta\rho\sigma}\} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial H}{\partial p_{\theta\rho\sigma}} + \{p_{\theta\rho\sigma}, x^\alpha\} \frac{\partial x^\mu}{\partial p_{\theta\rho\sigma}} \frac{\partial H}{\partial x^\alpha} \qquad (18)
\end{aligned}$$

$$\Rightarrow \dot{x}^\mu = e p^\mu + \frac{2}{\theta^2} (\lambda_\theta v)^\mu - \frac{4}{\theta^4} (\theta\lambda_\theta) (\theta v)^\mu \,, \qquad (19)$$

where we have shown in (18) the nonzero terms of (17). For p_μ, we have that

$$\begin{aligned}
\dot{p}^\mu &= \{p^\mu, H\} \\
&= \{x^\alpha, p_\beta\} \frac{\partial p^\mu}{\partial x^\alpha} \frac{\partial H}{\partial p_\beta} + \{p_\beta, x^\alpha\} \frac{\partial p^\mu}{\partial p_\beta} \frac{\partial H}{\partial x^\alpha} \Rightarrow \dot{p}^\mu = 0 \,. \qquad (20)
\end{aligned}$$

Analogously, we can compute the equations of motion for the other variables, namely,

$$\begin{aligned}
\dot{\theta}^{\mu\nu} &= -2\,\lambda_\theta^{\mu\nu} \,, && (21)\\
\dot{v}_\mu &= 0 \,, && (22)\\
\dot{e} &= \lambda_e \,, && (23)\\
\dot{p}_e &= -v \cdot p + \frac{1}{2}\left(v^2 + m^2\right) \,, && (24)\\
\dot{\pi}^\mu &= \frac{4}{\theta^4} (\theta\lambda_\theta) (\theta\, v)^\mu - \frac{1}{\theta^2} \eta^{\mu[\rho} v^{\sigma]} \lambda_{\theta\rho\sigma} \,, && (25)\\
\dot{p}_\theta^{\mu\nu} &= \frac{8}{\theta^4} \left[\frac{\theta^{\mu\nu}}{\theta^2} (\theta\lambda_\theta)(\theta v)^\sigma p_\sigma - \lambda_\theta^{\mu\nu} (\theta v)^\sigma p_\sigma - \theta^{\mu\nu} (\lambda_\theta v)^\sigma p_\sigma \right. \\
&\quad \left. + \frac{1}{2} (\theta\lambda_\theta) v^{[\mu} p^{\nu]} \right] . && (26)
\end{aligned}$$

Finally, in the same way, we can calculate the acceleration in this NC phase space, namely, $\ddot{x}^\mu = \{\dot{x}^\mu, H\}$, which brings us the result

$$\ddot{x}^\mu = \frac{8}{\theta^4} \left[(\theta\lambda_\theta) (\lambda_\theta v)^\mu - \frac{4}{\theta^2} (\theta\lambda_\theta)^2 (\theta v)^\mu + \lambda^2 (\theta v)^\mu - (\theta\lambda_\theta)(\lambda_\theta v)^\mu \right] , \qquad (27)$$

where $\lambda^2 = \lambda_{\theta\mu\nu} \lambda_\theta^{\mu\nu}$. This last result is very interesting since the equation of motion (21) shows us that if we have that $\theta =$ constant, we have $\lambda_\theta = 0$. In this way, we will not have p_θ in the Hamiltonian written in (14). However, we can easily see from Eq. (26) that we have that $\lambda_\theta = 0 \Longrightarrow \dot{p}_\theta = 0 \Longrightarrow p_\theta =$

constant, but the important fact is that the phase space for the Hamiltonian in Eq. (14) will not have p_θ as a dynamical variable. If θ is not constant, the NC phase space contains p_θ. Consequently, if θ is constant, we do not have p_θ within the phase space. Notice that although the λs are auxiliary variables in order to construct the total Hamiltonian, they are connected to the momenta by construction of the constraints formalism.

Otherwise, if $\theta =$ constant in Eq. (27), the acceleration is zero. This is an interesting result since we do not have any time derivative of θ in Eq. (27) but this result is a consequence of the nulification of λ_θ. So, we can say that, although it is not part of the phase-space structure, the Lagrangian multiplier is underlying for the system's dynamics. It was not explored in [15]. However, the time derivative of x^μ in Eq. (19) is not zero when $\lambda_\theta = 0$ neither it is constant since $e(\tau)$ is variable (p_μ is constant since $\dot{p}_\mu$ is identically zero). Although it was not explored in [15], the author has claimed that θ can actually be made constant by exploiting the additional local symmetry of the action in (1). So, as we explained just above, the corrections in Newton's second law in (27) can be gauged away, since on the condition $\theta =$ constant $\Longrightarrow \lambda_\theta = 0$ scenario, only the λ^2-term survives.

3. Quantum NC scalar field theory

Motivated by the dynamical analysis of a system in NC phase space, let us analyze from now on the scalar fields in a θ-variable NC approach, as we have mentioned above, the so-called DFR formalism [16, 17]. To clarify, in their original papers, DFR have stressed that their formalism is based only on the classical gravitational collapse in general relativity and on standard quantum principles. Namely, it does not require any quantum gravity scenario. The approach here is very different from the ones investigated in [18–20]. Here, we will show a direct approach to construct the scalar field. The objective here is to construct auxiliary functions which will help us to build basic fields in DFR formalism.

3.1. Preliminaries

Let us begin this section by defining the manifold Σ used in [16, 17]. Let us follow [16, 17] in order to define the manifold Σ where the so-called joint eigenvalues of the commuting selfadjoint operators $\theta^{\mu\nu}$ dwell. In [16, 17], the manifold Σ can be decomposed so that $\Sigma = \Sigma_+ \cup \Sigma_-$, where

$$\Sigma_\pm = \left\{\sigma | \sigma^{\mu\nu} = -\sigma^{\nu\mu}\,,\, \sigma = (\vec{e}, \vec{m})\,;\, e^2 = m^2\,,\, \vec{e}\cdot\vec{m} = \pm 1\right\}\,,$$

where $\vec{e}$ and $\vec{m}$ are the "electric" and "magnetic" parts of $\theta^{\mu\nu}$. In this way, we can write that

$$\tfrac{1}{2}\,\theta_{\mu\nu}\,\theta^{\mu\nu} = m^2 - e^2 \qquad \text{and} \qquad \tfrac{1}{4}\,\theta^{\mu\nu}(*\theta^{\mu\nu}) = \vec{e}\cdot\vec{m}\,,$$

where $\theta_{\mu\nu}(*\theta^{\mu\nu}) = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\theta^{\mu\nu}\theta^{\lambda\rho}$, the relations given by $e^2 = m^2$ and $(\vec{e}\cdot\vec{m})^2 = \mathbb{I}$, and $\{\sigma^{\mu\nu}\} = (\vec{e},\vec{m})$ is the spectrum of $\theta^{\mu\nu}$. Σ_+ and Σ_- are connected manifolds [16, 17] and they are topologically equivalent to the tangent bundle TS^2 of the unit sphere S^2 in $\mathbb{R}^3$. If $\vec{e} = \pm\,\vec{m}$, they must be of length one, and span the base $\Sigma^{(1)}$ of Σ. So, Σ can be understood as $T\Sigma^{(1)}$ [16, 17].

In DFR algebra, the simplest condition says that x^μ and $\theta^{\mu\nu}$ commute. Besides, we have the quantum conditions given by [16, 17]

$$\theta_{\mu\nu}\,\theta^{\mu\nu} = 0\,, \qquad \left[\tfrac{1}{4}\theta_{\mu\nu}\,(*\theta^{\mu\nu})\right]^2 = \mathbb{I}\,, \qquad [\hat{x}_\mu,[\hat{x}_\nu,\hat{x}_\lambda]] = 0\,, \tag{28}$$

where, in generic units, the identity operator above would be multiplied by λ_{P}^8.

Let us consider that (28) is valid in the more restrictive Weyl form [16, 17]

$$e^{i\alpha_\mu\hat{x}^\mu}\,e^{i\beta_\mu\hat{x}^\mu} \;=\; e^{\frac{1}{2}\alpha_\mu\theta^{\mu\nu}\beta_\nu}\,e^{i(\alpha+\beta)_\mu\hat{x}^\mu}\,, \tag{29}$$

where $e^{i\alpha_\mu\hat{x}^\mu}$ are unitary and continuous operators in the real four-vector α.

Considering our manifold Σ, we can say that the representations that satisfy (29) correspond to the representations κ of a C^*-algebra $\mathcal{C}$ generated by F functions, which are continuous and vanish at infinity from Σ to $L^1(\mathbb{R}^4)$. We can construct a correspondence equation given by

$$\hat{\kappa}(F) = g(\theta)\int f(\alpha)\,e^{i\alpha_\mu\hat{x}^\mu}\,\mathrm{d}^4x\,,$$

where $F : \sigma\in\Sigma \to g(\sigma)\,f$, with $g\in\mathcal{C}_0(\Sigma)$ and $f\in L^1(\mathbb{R}^4)$. Notice that $\mathcal{C}$ depicts the quantum space-time and substitutes the commutative C^*-algebra $\mathcal{C}_0(\mathbb{R}^4)$ [16, 17]. It is important to explain that the C^*-algebra $\mathcal{C}$ can be connected to the algebra of all continuous functions that vanish at infinity from Σ to the compact operators algebra over a fixed separable infinite dimensional Hilbert space.

Using the von Neumann–Wigner–Moyal equation, the Weyl relations permit us to compute the functions of the quantum position operator $\hat{x}^\mu$

$$\hat{f}(\hat{x}) \;=\; \int \tilde{f}(\alpha)\,e^{i\alpha_\mu\hat{x}^\mu}\,\mathrm{d}^4\alpha\,,$$

where

$$\tilde{f}(\alpha) \;=\; \frac{1}{(2\pi)^2}\int f(x)\,e^{-i\alpha_\mu x^\mu}\,\mathrm{d}^4x$$

is an L^1 function of α and these operators form a linear subspace which is not stable under multiplication. $\mathcal{C}$ can be spanned, as a normed vector space, by elements of the form of $g(\theta)\hat{f}(\hat{x})$, where $g \in \mathcal{C}_0(\Sigma)$ and $f \in L^1(\mathbb{R}^4)$. In a moment, we will use these definitions in Σ in order to construct scalar fields in DFR space-time. We will see that the α_μ elements are, in fact, the p_μ conjugated to x^μ and the $\pi_{\mu\nu}$ conjugated to $\theta^{\mu\nu}$.

3.2. The DFR scalar field

Using the concepts defined above, we will construct the first basic step of a QFT with the phase-space definitions established above. Since we have shown that the DFR and DFR-extended phase space are, in fact, the same, we will use the name DFR to define the formalism embedded in the complete phase space (x, p, θ, π).

In papers published by two of us [18–20], one can see that the construction of the commutation relations between the bosonic/fermionic fields with themselves and with its associated momenta are missing. It is our intention in this section to fill this gap. In other words, we will demonstrate precisely the basic commutation relations using only the DFR elements. The fermionic construction is an ongoing research that will be published in a near future.

In other papers which consider the DFR formalism or θ-variable approaches, such as [15–17, 21–26] for example, we can find this basic step in an indirect way where the associated momenta are not defined. The quantity used to construct the scalar field, which was used as being associated with the variable θ, is an ill-defined scalar quantity.

After these considerations, let us employ the von Neumann–Wigner–Moyal formula mentioned in the last subsection to construct the field operators over the full phase-space (x, p, θ, π). So, we can write a map between a member of the operator algebra and an ordinary function

$$\hat{f}\left(\hat{x}, \hat{\theta}\right) = \mathrm{Tr}\left[e^{i\left(p\cdot\hat{x}\,+\,\pi\cdot\hat{\theta}\right)}\, f(x,\theta)\right] = \int \mathrm{d}^4p\,\mathrm{d}^6\pi\; e^{i\left(p\cdot\hat{x}\,+\,\pi\cdot\hat{\theta}\right)}\,\widetilde{f}(p,\pi)\,, \quad (30)$$

where $\widetilde{f}$ can be defined by

$$\widetilde{f}(p,\pi) = \int \frac{\mathrm{d}^4x}{(2\pi)^4}\frac{\mathrm{d}^6\theta}{(2\pi)^6}\; e^{-i(p\cdot x\,+\,\pi\cdot\theta)}\, f(x,\theta)\,, \quad (31)$$

where $p\cdot\hat{x} = p_\mu\hat{x}^\mu$ and $\pi\cdot\hat{\theta} = \frac{1}{2}\,\pi_{\mu\nu}\hat{\theta}^{\mu\nu}$ (the 1/2 factor avoids the sum over repeated terms), $f(x,\theta)$ is the corresponding function to the operator

$\hat{f}(\hat{x},\hat{\theta})$ and the integration measures are

$$\begin{aligned} \mathrm{d}^6\pi &= \mathrm{d}\pi_{01}\,\mathrm{d}\pi_{02}\,\mathrm{d}\pi_{03}\,\mathrm{d}\pi_{12}\,\mathrm{d}\pi_{13}\,\mathrm{d}\pi_{23}\,, \\ \mathrm{d}^6\theta &= \mathrm{d}\theta^{01}\mathrm{d}\theta^{02}\mathrm{d}\theta^{03}\mathrm{d}\theta^{12}\mathrm{d}\theta^{13}\mathrm{d}\theta^{23}\,. \end{aligned} \tag{32}$$

The details about θ and π are described in [27] (and references therein), where the θ-variable and π are not necessarily connected as we have discussed so far.

Since the momentum $\pi_{\mu\nu}$ is an element of the NC phase space, let us construct the operator field in this DFR algebra in Weyl representation [21]

$$\hat{\phi}\left(\hat{x},\hat{\theta}\right) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{d}^6\pi}{(2\pi)^6}\,\widetilde{\phi}(p,\pi)\;e^{i\left(p\cdot\hat{x}+\pi\cdot\hat{\theta}\right)}\,, \tag{33}$$

where $\widetilde{\phi}(p,\pi)$ is the Fourier transform of $\hat{\phi}(\hat{x},\hat{\theta})$ and $\mathrm{d}^6\pi$ is a Lorentz invariant measure given in (32). Notice that the difference between the issues explored here and in [21] is that now we know that the phase space is described by (x,p,θ,π).

From the commutation relation $[\hat{x}^\mu,\,\hat{x}^\nu] = i\hat{\theta}^{\mu\nu}$, one can define the shifted coordinate (Bopp shift) operator

$$\hat{X}^\mu = \hat{x}^\mu + \frac{i}{2}\,\hat{\theta}^{\mu\nu}\hat{p}_\nu\,, \tag{34}$$

which commute, namely, $[\hat{X}^\mu,\,\hat{X}^\nu] = 0$.

The $\widehat{X}^\mu$-operator commutes with $\widehat{\theta}^{\mu\nu}$, $\widehat{X}^\nu$ and $\widehat{\pi}^{\mu\nu}$ operators, and the commutation relation for $\hat{p}^\nu$ remains zero. Since $\widehat{X}^\mu$ commutes with itself, we can define a basis $|X,\theta\rangle = |X\rangle\otimes|\theta\rangle$ in the Hilbert space, such that

$$\hat{X}^\mu|X,\theta\rangle = X^\mu|X,\theta\rangle \qquad \text{and} \qquad \hat{\theta}^{\mu\nu}|X,\theta\rangle = \theta^{\mu\nu}|X,\theta\rangle\,. \tag{35}$$

Notice that, obviously, $|X\rangle$ is an eigenvector of $\widehat{X}^\mu$ and not of $\hat{x}^\mu$. In order to obtain a Fourier representation of a scalar ϕ from the operator $\hat{\phi}$, let us make the diagonalization operation [21] using (33)

$$\widetilde{\phi}(x,\theta) = \langle X,\theta|\,\hat{\phi}\left(\hat{x},\hat{\theta}\right)\,|X,\theta\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\mathrm{d}^6\pi}{(2\pi)^6}\;\phi(p,\pi)\;e^{-i(p\cdot x+\pi\cdot\theta)}\,, \tag{36}$$

where we have used that $p\cdot\widehat{X} = p\cdot\hat{x}$.

The Lagrangian density of a real spin-0 field ϕ with mass m can be written as [27]

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\star\partial^\mu\phi + \frac{\lambda^2}{4}\,\partial_{\mu\nu}\phi\star\partial^{\mu\nu}\phi - \frac{1}{2}\,m^2\,\phi\star\phi\,, \tag{37}$$

where $\partial_{\mu\nu} = \partial/\partial\theta^{\mu\nu}$ and λ is a parameter with length dimension. It is important to remember that when Lagrangian (37) is integrated throughout the DFR space-time, the Moyal product in the quadratic terms reduces to usual product (see Appendix). Therefore, the action in (37) gives us the Klein–Gordon equation

$$\Big(\Box + \lambda^2\Box_\theta + m^2\Big)\phi = 0\,, \tag{38}$$

where $\Box = \partial_\mu\partial^\mu$ and $\Box_\theta = \frac{1}{2}\,\partial_{\mu\nu}\partial^{\mu\nu}$ are the four- and six-dimensional Laplace operators, respectively. The canonical conjugate momentum associated with ϕ is given by

$$\pi(x,\theta) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x,\theta)} = \dot{\phi}(x,\theta)\,, \tag{39}$$

which leads us to the Hamiltonian density

$$\begin{aligned}\mathcal{H} = &\frac{1}{2}\,\pi(x,\theta)\star\pi(x,\theta) + \frac{1}{2}\,\nabla\phi(x,\theta)\star\nabla\phi(x,\theta) + \frac{\lambda^2}{2}\,\nabla_\theta\phi(x,\theta)\star\nabla_\theta\phi(x,\theta)\\ &+\frac{1}{2}\,m^2\phi(x,\theta)\star\phi(x,\theta)\,,\end{aligned} \tag{40}$$

where $\nabla_\theta = \frac{1}{2}\,\partial^{ij}$. The conserved field energy is defined by the integral of the Hamiltonian density in the space $(\boldsymbol{x},\theta)$

$$H = \int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^6\theta\,\frac{1}{2}\left[\pi^2(x,\theta) + (\nabla\phi(x,\theta))^2 + \lambda^2\left(\nabla_\theta\hat{\Phi}(x,\theta)\right)^2 + m^2\,\phi^2(x,\theta)\right]. \tag{41}$$

In [21], the author has written an incomplete $\hat{\phi}(x,\theta)$ using the Weyl representation. We say incomplete because now we know that $\theta^{\mu\nu}$ has an associated momentum given by $\pi_{\mu\nu}$. In this way, we can expand the field $\phi(x,\theta)$ with respect to a basis. Let us use the set of plane waves such as

$$u_{\boldsymbol{p},\pi}(\boldsymbol{x},\theta) = N_{\boldsymbol{p},\pi}\,e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta)}\,, \tag{42}$$

which means that we can write the Fourier modes as

$$\phi(\boldsymbol{x},\theta,t) = \int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^6\pi\;N_{\boldsymbol{p},\pi}\,e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta)}\,a_{\boldsymbol{p},\pi}(t)\,, \tag{43}$$

where $N_{\boldsymbol{p},\pi}$ is a normalization constant. If we substitute Eq. (43) into (38), we will have the following equation of motion

$$\ddot{a}_{\boldsymbol{p},\pi}(t) + \omega^2_{\boldsymbol{p},\pi}\,a_{\boldsymbol{p},\pi}(t) = 0\,, \tag{44}$$

which has a general solution given by

$$a_{\boldsymbol{p},\pi}(t) = a^{(1)}_{\boldsymbol{p},\pi}\, e^{-i\omega_{\boldsymbol{p},\pi} t} + a^{(2)}_{\boldsymbol{p},\pi}\, e^{i\omega_{\boldsymbol{p},\pi} t}\,, \tag{45}$$

and the dispersion relation is

$$\omega_{\boldsymbol{p},\pi} = \sqrt{\boldsymbol{p}^{\,2} + \frac{\lambda^2}{2}\,\pi^2 + m^2}\,, \tag{46}$$

and from (45), we can easily see that $a^{(1)}_{\boldsymbol{p},\pi}$ and $a^{(2)}_{\boldsymbol{p},\pi}$ are constants in time. The real-valued feature of the classical field shows us that, of course, the operator is hermitian, hence,

$$\left(a^{(1)}_{\boldsymbol{p},\pi}\right)^{\dagger} = a^{(2)}_{-\boldsymbol{p},-\pi}\,, \tag{47}$$

which is a standard constraint. The free field can be expanded in terms of creation and annihilation operators, namely,

$$\left[\hat{a}_{\boldsymbol{p},\pi}\,,\,\hat{a}^{\dagger}_{\boldsymbol{p}',\pi'}\right] \;=\; \delta^3(\boldsymbol{p}-\boldsymbol{p}')\,\delta^6(\pi-\pi')\,, \tag{48}$$

$$\left[\hat{a}_{\boldsymbol{p},\pi}\,,\,\hat{a}_{\boldsymbol{p}',\pi'}\right] \;=\; \left[\hat{a}^{\dagger}_{\boldsymbol{p},\pi}\,,\,\hat{a}^{\dagger}_{\boldsymbol{p}',\pi'}\right] = 0\,. \tag{49}$$

One can ask if the field quanta will obey a kind of Bose–Einstein statistics in this NC phase space. For the time being, we will associate both $a_{\boldsymbol{p},\pi}$ and $a^{\dagger}_{\boldsymbol{p},\pi}$ with annihilation and creation operators, respectively, in the DFR formalism. Therefore, the field ϕ in (43) can be promoted to the field-operator $\hat{\Phi}$ expanded in this basis as

$$\hat{\Phi}(\boldsymbol{x},\theta,t) = \int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^6\pi\, N_{\boldsymbol{p},\pi}\left[\hat{a}_{\boldsymbol{p},\pi}\, e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta-\omega_{\boldsymbol{p},\pi}t)} + \hat{a}^{\dagger}_{\boldsymbol{p},\pi}\, e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta-\omega_{\boldsymbol{p},\pi}t)}\right]. \tag{50}$$

Thus, we construct the conjugate momentum operator $\hat{\Pi}$, that is, $\hat{\Pi}(\boldsymbol{x},\theta,t) = \dot{\hat{\Phi}}(\boldsymbol{x},\theta,t)$, so we have that

$$\begin{aligned}\hat{\Pi}(\boldsymbol{x},\theta,t) \;=\; & \int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^6\pi\, N_{\boldsymbol{p},\pi}\,(-i\omega_{\boldsymbol{p},\pi}) \\ & \times\left[\hat{a}_{\boldsymbol{p},\pi}\, e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta-\omega_{\boldsymbol{p},\pi}t)} - \hat{a}^{\dagger}_{\boldsymbol{p},\pi}\, e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta-\omega_{\boldsymbol{p},\pi}t)}\right].\end{aligned} \tag{51}$$

We can construct the Moyal commutation relation between two field-operators in equal times as

$$\left[\hat{\Phi}(\boldsymbol{x},\theta,t)\,,\,\hat{\Phi}(\boldsymbol{x}'\,,\,\theta',t)\right]_{\star} := \hat{\Phi}(\boldsymbol{x},\theta,t)\star\hat{\Phi}\left(\boldsymbol{x}',\theta',t\right) - \hat{\Phi}\left(\boldsymbol{x}',\theta',t\right)\star\hat{\Phi}(\boldsymbol{x},\theta,t) \tag{52}$$

and substituting Eq. (50) into Eq. (52), and using relations (48) and (49), we obtain that

$$\begin{aligned}\left[\hat{\Phi}(\boldsymbol{x},\theta,t),\hat{\Phi}\left(\boldsymbol{x}',\theta',t\right)\right]_\star &= \int \mathrm{d}^9P \int \mathrm{d}^9P'\, N_{\boldsymbol{p},\pi}\, N_{\boldsymbol{p}',\pi'}\,(-2i)\,\sin\left(\frac{p\wedge p'}{2}\right)\\ &\times\Big[\hat{a}_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\\ &-\hat{a}^\dagger_{\boldsymbol{p}',\pi'}\,\hat{a}_{\boldsymbol{p},\pi}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\ &-\hat{a}^\dagger_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\ &+\hat{a}^\dagger_{\boldsymbol{p},\pi}\,\hat{a}^\dagger_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\Big],\end{aligned}\tag{53}$$

where $\mathrm{d}^9P := \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^6\pi$, and we have defined the product $p\wedge p' = \theta^{\mu\nu}p_\mu\,p'_\nu$. Using the calculation above, the Moyal-commutation relation between the field operator $\hat{\Phi}$ and momenta $\hat{\Pi}$ is given by

$$\begin{aligned}\left[\hat{\Phi}(\boldsymbol{x},\theta,t),\hat{\Pi}\left(\boldsymbol{x}',\theta',t\right)\right]_\star &= \int \mathrm{d}^9P\,N^2_{\boldsymbol{p},\pi}\,\left(i\omega_{\boldsymbol{p},\pi}\right)\\ &\times\left[e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')+i\pi\cdot(\theta-\theta')}+e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')-i\pi\cdot(\theta-\theta')}\right]\\ &+\int \mathrm{d}^9P\int \mathrm{d}^9P'\,N_{\boldsymbol{p},\pi}\,N_{\boldsymbol{p}',\pi'}\,\left(i\omega_{\boldsymbol{p}',\pi'}\right)(2i)\,\sin\left(\frac{p\wedge p'}{2}\right)\\ &\times\Big[\hat{a}_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\\ &+\hat{a}^\dagger_{\boldsymbol{p}',\pi'}\,\hat{a}_{\boldsymbol{p},\pi}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\ &-\hat{a}^\dagger_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\ &+\hat{a}^\dagger_{\boldsymbol{p},\pi}\,\hat{a}^\dagger_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\Big].\end{aligned}\tag{54}$$

If we choose the normalization constant such as

$$N_{\boldsymbol{p},\pi} = \frac{1}{\sqrt{2\,(2\pi)^9\,\omega_{\boldsymbol{p},\pi}}}\,,\tag{55}$$

the result in (54) is simplified as

$$
\begin{aligned}
&\left[\hat{\Phi}(\boldsymbol{x},\theta,t),\hat{\Pi}\left(\boldsymbol{x}',\theta^{'},t\right)\right]_{\star}=i\,\delta^{3}\left(\boldsymbol{x}-\boldsymbol{x}'\right)\,\delta^{6}\left(\theta-\theta'\right)\\
&-\int\frac{\mathrm{d}^{9}P\,\mathrm{d}^{9}P'}{(2\pi)^{9}}\sqrt{\frac{\omega_{\boldsymbol{p}',\pi'}}{\omega_{\boldsymbol{p},\pi}}}\,\sin\left(\frac{p\wedge p'}{2}\right)\\
&\times\left[\hat{a}_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\right.\\
&+\hat{a}^{\dagger}_{\boldsymbol{p}',\pi'}\,\hat{a}_{\boldsymbol{p},\pi}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\
&-\hat{a}^{\dagger}_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\
&\left.+\hat{a}^{\dagger}_{\boldsymbol{p},\pi}\,\hat{a}^{\dagger}_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\right],
\end{aligned}
\tag{56}
$$

which is the opposite direction followed in [27] for example, and the delta functions are assumed to have the same form as the ones used in [27]. Finally, we have the commutation relation involving the momentum operators

$$
\begin{aligned}
&\left[\hat{\Pi}(\boldsymbol{x},\theta,t),\hat{\Pi}(\boldsymbol{x}',\theta^{'},t)\right]_{\star}=\int\mathrm{d}^{9}P\int\mathrm{d}^{9}P'\,N_{\boldsymbol{p},\pi}\,N_{\boldsymbol{p}',\pi'}\,(-i\omega_{\boldsymbol{p},\pi})\left(-i\omega_{\boldsymbol{p}',\pi'}\right)\\
&\times\,(-2i)\,\sin\left(\frac{p\wedge p'}{2}\right)\left[\hat{a}_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\right.\\
&+\hat{a}^{\dagger}_{\boldsymbol{p}',\pi'}\,\hat{a}_{\boldsymbol{p},\pi}\,e^{i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\
&-\hat{a}^{\dagger}_{\boldsymbol{p},\pi}\,\hat{a}_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t-\boldsymbol{p}'\cdot\boldsymbol{x}'+\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta-\pi'\cdot\theta'\right)}\\
&\left.+\hat{a}^{\dagger}_{\boldsymbol{p},\pi}\,\hat{a}^{\dagger}_{\boldsymbol{p}',\pi'}\,e^{-i\left(\boldsymbol{p}\cdot\boldsymbol{x}-\omega_{\boldsymbol{p},\pi}\,t+\boldsymbol{p}'\cdot\boldsymbol{x}'-\omega_{\boldsymbol{p}',\pi'}\,t+\pi\cdot\theta+\pi'\cdot\theta'\right)}\right].
\end{aligned}
\tag{57}
$$

Notice that what we have done here was to demonstrate the canonical commutation relations using the field operators constructed with DFR phase-space definitions. These canonical relations involving the Moyal product are not equal to the usual case, in which we have obtained combinations between creation and annihilation operators. It is clear that in the commutative limit ($\theta=0$), these terms involving $\hat{a}$ and $\hat{a}^{\dagger}$ are zero naturally. If we use the vacuum properties of the operators $\hat{a}$ and $\hat{a}^{\dagger}$, *i.e.*, if we define a vacuum state $|0\rangle$ such that $\hat{a}_{\boldsymbol{p},\pi}|0\rangle=0$, the expected value of the previous commutator relations in the vacuum state can be given by

$$
\begin{aligned}
\langle 0|\left[\hat{\Phi}(\boldsymbol{x},\theta,t),\hat{\Phi}\left(\boldsymbol{x}',\theta',t\right)\right]_{\star}|0\rangle &= 0\,,\\
\langle 0|\left[\hat{\Phi}(\boldsymbol{x},\theta,t),\hat{\Pi}\left(\boldsymbol{x}',\theta',t\right)\right]_{\star}|0\rangle &= i\,\delta^{3}\left(\boldsymbol{x}-\boldsymbol{x}'\right)\,\delta^{6}\left(\theta-\theta'\right)\,,\\
\langle 0|\left[\hat{\Pi}(\boldsymbol{x},\theta,t),\hat{\Pi}\left(\boldsymbol{x}',\theta',t\right)\right]_{\star}|0\rangle &= 0\,.
\end{aligned}
\tag{58}
$$

We can see that the result in (58) corroborates the construction of the operator in $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ with a convenient normalization choice and obeying the commutation operators. We believe that this formalism completes the ones depicted in [21] since in it the existence of a NC six-dimensional phase space is missing. Concerning [27], the path here is different since we have demonstrated that the field operator in an NC space-time can be written in terms of plane waves as

$$u_{\boldsymbol{p},\pi}(\boldsymbol{x},\theta) = \frac{e^{-i(\boldsymbol{p}\cdot\boldsymbol{x}+\pi\cdot\theta)}}{\sqrt{2\,(2\pi)^9\,\omega_{\boldsymbol{p},\pi}}}\,. \tag{59}$$

When we substitute Eq. (59) in the Fourier expansion in Eq. (50), the same can be accomplished for $\hat{\Pi}$.

We can apply the quantization to the field energy (41), so the quantized Hamiltonian operator is given by

$$\hat{H} = \int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^6\theta\,\tfrac{1}{2}$$
$$\times\left[\hat{\Pi}^2(\boldsymbol{x},\theta,t) + \left(\nabla\hat{\Phi}(\boldsymbol{x},\theta,t)\right)^2 + \left(\lambda\nabla_\theta\hat{\Phi}(\boldsymbol{x},\theta,t)\right)^2 + m^2\,\hat{\Phi}^2(\boldsymbol{x},\theta,t)\right]. \tag{60}$$

Using the plane wave expansion of the operators $\hat{\Phi}$ and $\hat{\Pi}$, the quantized energy in terms of the creation and annihilation operators is given by

$$\hat{H} = \int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^6\pi\,\omega_{\boldsymbol{p},\pi}\left(a^\dagger_{\boldsymbol{p},\pi}\,a_{\boldsymbol{p},\pi} + \tfrac{1}{2}\right)\,, \tag{61}$$

so that we can obtain the vacuum energy E_0

$$E_0 = \langle 0|\hat{H}|0\rangle = \int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^6\pi\,\tfrac{1}{2}\,\omega_{\boldsymbol{p},\pi}\,. \tag{62}$$

We can use the Hamiltonian operator (61), and operators (50) and (51) to calculate the Hamilton's equations of motion as

$$\dot{\hat{\Phi}}(\boldsymbol{x},\theta,t) = -i\left[\hat{\Phi}(\boldsymbol{x},\theta,t),\hat{H}\right] = \hat{\Pi}(\boldsymbol{x},\theta,t) \tag{63}$$

and

$$\dot{\hat{\Pi}} = -i\left[\hat{\Pi}(\boldsymbol{x},\theta,t),\hat{H}\right] = \left(\nabla^2 + \lambda^2\nabla^2_\theta - m^2\right)\hat{\Phi}(\boldsymbol{x},\theta,t)\,, \tag{64}$$

where we have used conveniently integrations by parts. Notice that, using Eqs. (63) and (64), we can construct the NC Klein–Gordon equation

$$\ddot{\hat{\Phi}}(\boldsymbol{x},\theta,t) = \left(\nabla^2 + \lambda^2\nabla^2_\theta - m^2\right)\hat{\Phi}(\boldsymbol{x},\theta,t)\,, \tag{65}$$

which shows clearly a different path from [21] since the author did not consider the existence of a canonical momentum.

4. Conclusions

The investigation of physical theories that occur in NC space-time has brought great interest through the last years and one of the reasons for that interest is the hope to increase the comprehension about gravity at the Planck scale.

The existence of a parameter that allows the appearance of NC terms that contribute perturbatively in well-known theories acts as a theoretical laboratory to study the physics of the very early Universe. It motivates theoretical physicists to pursue this NC knowledge to search for a way to unify gravitation and quantum mechanics.

In other words, we hope to find an algebraic unified model [28] or an arguably understanding of a quantum space as the beginning of a quantum gravity theory which avoids current problems and is free of singularities, for instance. The seminal objective would be that the deformation of space-time would act as a regularization scheme which would keep the algebraic properties of the theory.

It is well-known that basically, we can classify the NC theories into those where the NC parameter is constant and those where it is not constant. Of course, we are talking only about the theories where the results of the position coordinates commutation rely on the NC parameter. The main motivation to have a non-constant NC parameter is to recover the Lorentz invariance which is lost in a constant parameter structure.

In order to understand the NC formulations where the θ-parameter is not constant we, in this paper, have analyzed systems where the θ-parameter is a variable of the NC phase space. The first system is the relativistic particle studied in [15]. The focus there was on the Dirac constraint analysis. Here, we have used symplectic formulation all the way to obtain all the equations of motion and to obtain also the Newton's second law in this NC structure. We have seen that the Lagrangian multipliers, having a kind of secondary role in a constrained approach, are dynamically fundamental since, in the commutative limit ($\theta = 0$), they are also zero and, consequently, the acceleration is also zero. In other words, the NC relativistic particle shows, besides the $\theta_{\text{constant}} \longrightarrow \theta_{\text{variable}}$ duality, another interesting result. Since the equations of motion have shown that for $\theta =$ constant we have the multiplier $\lambda_\theta = 0$ and this value zeroes the NC acceleration, the velocity is not constant since it has a parameter that is time-dependent. Besides, we have calculated here that $\dot{e} \neq 0$, which confirms that, following the equations of motion, the velocity $\dot{x}$ is not constant. It is important to compute $\dot{e}$ because although it is defined as $e = e(\tau)$ its calculation could result as zero, which would show a paradox, but it did not happen.

After that, we have considered the analysis of another approach where the NC parameter is not constant, the well-known DFR formalism. Using the formulation, we have constructed the scalar field algebra and the Klein–Gordon equation. This procedure is completely different from the ones in the literature. We believe that it is more natural to obtain scalar fields in this way.

As a perspective, we can analyze other θ_{variable} algebras different from DFR (of course) to verify if the behavior is the same. Another possible research is to construct the fermion DFR QFT. It is an ongoing research and it will be published elsewhere.

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Appendix

The Moyal–Weyl product

To investigate field theories defined in spaces with NC variables corresponding to deformations of flat spaces as *e.g.* the Euclidean plane or Minkowski space $\mathbb{M}^d$, one must replace the (commuting) flat space space-time coordinates by Hermitian operators $\hat{x}^\mu$ (with $\mu = 0, 1, \ldots, (d-1)$) [28]. We consider a canonical structure defined by the following algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \qquad [\theta^{\mu\nu}, \hat{x}^\rho] = 0\,. \tag{A.1}$$

The simplest case is when the $\theta^{\mu\nu}$ matrix is constant, which means that we have only the first commutation relation of (A.1). Furthermore, it is real and antisymmetric. In natural units, where $\hbar = c = 1$, it can be easily seen from (A.1) that it has squared mass dimension.

In order to construct the perturbative field theory formulation, it is more convenient to use fields $\Phi(x)$ (which are functions of ordinary commuting coordinates) instead of operator valued objects like $\hat{\Phi}(\hat{x})$. Considering the properties (A.1), one must redefine the multiplication law of functional (field) space. One, therefore, defines the linear map $\hat{f}(\hat{x}) \longmapsto S[\hat{f}](x)$, called the symbol of the operator $\hat{f}$, and can then represent the original operator multiplication in terms of the so-called star products of symbols as

$$\hat{f}\hat{g} = S^{-1}\left[S\left[\hat{f}\right] \star S[\hat{g}]\right]\,, \tag{A.2}$$

(see, for example, references [7, 8]). By using the Weyl-ordered symbol (which corresponds to the Weyl-ordering prescription of the operators), one can arrive at the following definitions, with $S[\hat{f}](x) = \Phi(x)$,

$$\begin{aligned} \hat{\Phi}(\hat{x}) &\longleftrightarrow \Phi(x), \\ \hat{\Phi}(\hat{x}) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \; \tilde{\Phi}(k) \; e^{ik\cdot\hat{x}}, \\ \tilde{\Phi}(k) &= \mathrm{Tr}\left[\hat{\Phi}(\hat{x}) \, e^{-ik\cdot\hat{x}} \right] = \int \mathrm{d}^d x \; \Phi(x) \; e^{-ik\cdot x}, \end{aligned} \tag{A.3}$$

where k is a real variable, and $\hat{x}$ is the position operator. For any two arbitrary scalar fields $\hat{\Phi}_1$ and $\hat{\Phi}_2$, one can therefore write that[1]

$$\begin{aligned} \hat{\Phi}_1(\hat{x})\,\hat{\Phi}_2(\hat{x}) &= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \; \tilde{\Phi}_1(k_1)\,\tilde{\Phi}_2(k_2) \; e^{ik_1\cdot\hat{x}} \; e^{ik_2\cdot\hat{x}} \\ &= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \; \tilde{\Phi}_1(k_1)\,\tilde{\Phi}_2(k_2) \; e^{i(k_1+k_2)\cdot\hat{x} - \frac{1}{2}\,[\hat{x}^\mu, \hat{x}^\nu]\, k_{1\mu} k_{2\nu}} \\ &= \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \; \tilde{\Phi}_1(k_1)\,\tilde{\Phi}_2(k_2) \; e^{i(k_1+k_2)\cdot\hat{x} - \frac{i}{2}\,\theta^{\mu\nu} k_{1\mu} k_{2\nu}} \, . \end{aligned} \tag{A.4}$$

Hence, one has the following Weyl–Moyal correspondence [3, 7, 8]

$$\hat{\Phi}_1(\hat{x})\,\hat{\Phi}_2(\hat{x}) \longleftrightarrow \Phi_1(x) \star \Phi_2(x)\,, \tag{A.5}$$

where, using relation (A.1) to replace the commutator in the exponent of (A.4), the generalized Moyal–Weyl star product is given by

$$\Phi_1(x) \star \Phi_2(x) \;=\; \Phi_1(x) \exp\left(\frac{i}{2}\, \overleftarrow{\partial}_{x\mu} \theta^{\mu\nu} \overrightarrow{\partial}_{y\nu} \right) \Phi_2(y) \Big|_{x=y} \, . \tag{A.6}$$

This means that we can work in the same way as in an usual commutative space for which the multiplication operation is modified by the star product (A.6). For the ordinary commuting coordinates, this implies[2] that

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}\,, \qquad [\theta^{\mu\nu}, x^\rho]_\star = 0\,. \tag{A.7}$$

At this point, one also has to mention that the commutation relations (A.1) between the coordinates explicitly break Lorentz invariance because of the fact that we assumed that θ is a constant matrix [3, 7, 8].

[1] One has to use the Baker–Campbell–Hausdorff formula, as well as the relations in (A.1).

[2] The Weyl bracket is defined as $[A, B]_\star = A \star B - B \star A$.

Some other possibilities for a non-constant θ are, for example, $\theta^{\mu\nu} = C^{\mu\nu}{}_{\rho}\, x^{\rho}$ (Lie algebra) or $\theta^{\mu\nu} = R^{\mu\nu}{}_{\rho\sigma}\, x^{\rho}\, x^{\sigma}$ (quantum space structure) — see, for instance, references [7, 8] for a detailed discussion about these two approaches.

Another solution of this problem leads us to the NC formulation of the space-time used here which was formulated by Doplicher, Fredenhagen and Roberts (DFR) [16, 17], which is based on general relativity and quantum mechanics arguments. This formalism recovers Lorentz invariance through the promotion of $\theta^{\mu\nu}$ to be a standard coordinate of this extra dimensional system. Of course, being the coordinate, the algebra turns out to be, together with Eq. (A.1),

$$[\hat{x}^{\mu}, \hat{p}^{\nu}] = i\eta^{\mu\nu}\,, \qquad \left[\hat{x}^{\mu}, \hat{\theta}^{\mu\nu}\right] = [\hat{p}^{\mu}, \hat{p}^{\nu}] = \left[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\rho\lambda}\right] = 0\,, \tag{A.8}$$

which completes the basic DFR algebra.

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