CLASSICAL AND QUANTUM CHAPLYGIN GAS HOŘAVA–LIFSHITZ SCALAR-METRIC COSMOLOGY

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In this work, we study the Friedmann–Robertson–Walker cosmology in which a Chaplygin gas is coupled to a non-linear scalar field in the framework of the Hořava–Lifshitz theory. In writing the action of the matter part, we use the Schutzs formalism so that the only degree of freedom of the Chaplygin gas plays the role of an evolutionary parameter. In a minisuperspace perspective, we construct the Lagrangian for this model and show that in comparison with the usual Einstein–Hilbert gravity, there are some correction terms coming from the Hořava theory. In such a set-up and by using some approximations, the classical dynamics of the model is investigated and some discussions about their possible singularities are presented. We then deal with the quantization of the model in the context of the Wheeler–DeWitt approach of quantum cosmology to find the cosmological wave function. We use the resulting wave functions to investigate the possibility of the avoidance of classical singularities due to quantum effects.

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1. Introduction

Various modern cosmological theories such as grand unified theories imply the existence of the classical and semiclassical scalar fields [1]. In cosmological viewpoint, scalar–tensor models have attracted much attention in which a non-minimal coupling appears between the space-time geometry and a scalar field [2–5]. This is due to the fact that various research areas in cosmology such as spatially flat and accelerated expanding universe

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at the present time [6–8], inflation [9, 10], dark matter and dark energy [11, 12], and many other behaviors can be explained phenomenologically by the scalar fields. Cosmological models are usually described by a single scalar field with a canonical kinetic term in the form of $\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and a self-interaction potential $V(\phi)$ where the scalar field is often minimally coupled to gravity. However, in scalar–tensor theories, the scalar field is not simply added to the action. Indeed, it is added to the tensor gravitational field by a non-minimal coupling term [13].

In recent years, the so-called Hořava–Lifshitz (HL) gravity theory, presented by Hořava, is proved to be power-countable renormalizable. It is based on the anisotropic scaling of space $\boldsymbol{x}$ and time t as

$$\boldsymbol{x} \to b\boldsymbol{x}\,, \qquad t \to b^z t\,, \tag{1}$$

where b is a scaling parameter and z is the dynamical critical exponent. Notice that for $z=1$, the standard relativistic scale invariance obeying Lorentz symmetry is recovered in the IR limit. However, the UV gravitational theory implies $z=3$ [14–17]. Due to the asymmetry of space and time in HL theory, it is common to use the Arnowitt–Deser–Misner (ADM) formalism to represent the space-time metric $g_{\mu\nu}(t,\boldsymbol{x})$, in terms of three-dimensional metric $\gamma_{ab}(t,\boldsymbol{x})$, shift vector $N_a(t,\boldsymbol{x})$ and the lapse function $N(t,\boldsymbol{x})$ as [18]

$$g_{\mu\nu}(t,\boldsymbol{x}) = \begin{pmatrix} -N^2(t,\boldsymbol{x}) + N_a(t,\boldsymbol{x})N^a(t,\boldsymbol{x}) & N_b(t,\boldsymbol{x}) \\ N_a(t,\boldsymbol{x}) & \gamma_{ab}(t,\boldsymbol{x}) \end{pmatrix}. \tag{2}$$

If the lapse function is a function of t only, the theory is projectable, otherwise, in the case where N is a function of $(t,\boldsymbol{x})$, theory is called non-projectable. General cases in which the lapse function is taken as a non-projectable function are studied in Refs. [19, 20]. However, we assume that the lapse function is constrained to be a function only of the time coordinate $N=N(t)$ [15].

The most general action for HL gravity (without the detailed balance condition) is given by $S_{\mathrm{HL}} = S_K + S_V$, where S_K is kinetic part

$$S_K \sim \int \mathrm{d}^4\boldsymbol{x}\,\sqrt{-g}\left(K_{ij}K^{ij} - \lambda K^2\right)\,, \tag{3}$$

in which K_{ij} is the extrinsic curvature tensor (with trace K) defined by

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\,. \tag{4}$$

Also, for the potential part, the following general form is proposed:

$$S_V = \int \mathrm{d}^4x\,\sqrt{-g}\,V[\gamma_{ij}]\,, \tag{5}$$

in which

$$V[\gamma_{ij}] = g_0\zeta^6 + g_1\zeta^4 R + g_2\zeta^2 R^2 + g_3\zeta^2 R_{ij}R^{ij} + g_4 R^3 + g_5 R R_{ij}R^{ij} + g_6 R^i{}_j R^j{}_k R^k_i + g_7 R\nabla^2 R + g_8 \nabla_i R_{jk}\nabla^i R^{jk} \,. \tag{6}$$

The constants λ and g_i $(i = 0, 1, \ldots, 8)$ in the above relations denote the HL corrections to the usual Einstein gravity and ζ is introduced to make the constants g_ks dimensionless. Under these conditions, the full HL action that we shall study is [21–23]

$$S_{\rm HL} = \frac{M_{\rm Pl}^2}{2}\int_{\mathcal{M}} \mathrm{d}^4\boldsymbol{x}\sqrt{-g}\Big[K_{ij}K^{ij} - \lambda K^2 + R - 2\Lambda - \frac{g_2}{M_{\rm Pl}^2}R^2 - \frac{g_3}{M_{\rm Pl}^2}R_{ij}R^{ij} - \frac{g_4}{M_{\rm Pl}^4}R^3 - \frac{g_5}{M_{\rm Pl}^4}RR_{ij}R^{ij} - \frac{g_6}{M_{\rm Pl}^4}R_{ij}R^{jk}R^i_k - \frac{g_7}{M_{\rm Pl}^4}R\nabla^2 R - \frac{g_8}{M_{\rm Pl}^4}\nabla_i R_{jk}\nabla^i R^{jk}\Big] , \tag{7}$$

in which $M_{\rm Pl} = \frac{1}{\sqrt{8\pi G}}$ and we have set $c = 1$, $\zeta = 1$, $\Lambda = g_0 M_{\rm Pl}^2/2$ and $g_1 = -1$.

All cosmological evidences have revealed that the universe is undergoing an accelerated expansion which can be described by exotic cosmic fluid, the so-called dark energy, one of the first models of the cosmological constant. On the other hand, scalar fields play an important role in unified theories of interactions and also in inflationary scenarios in cosmology. Indeed, a rich variety of dark energy and inflationary models can be accommodated phenomenologically by scalar fields in which the inflatons produce the initial acceleration. Another attempt, originally raised in the string theory [24], is to change the equation of state from an ordinary matter to the Chaplygin gas, an exotic fluid with negative pressure. The Chaplygin gas as a candidate behind the current observation of cosmic acceleration has been thoroughly investigated in recent years. The generalized Chaplygin gas with negative pressure is described by an exotic equation of state

$$P = -\frac{A}{\rho^\alpha} \,, \tag{8}$$

where P is the pressure, A is a positive constant, and $0 \le \alpha \le 1$ is the equation-of-state parameter such that $\alpha = 1$ denotes the standard Chaplygin gas [25, 26]. In this sense, since the string theory deals with the high-energy phenomena such as very early universe, considering the Chaplygin gas quantum cosmology may have physical grounds. It is shown [27] that the generalized Chaplygin gas (8) can play the role of a mixture of cosmological constant and radiation by means of which the cosmological dynamics shows a transition from a dust dominated era to a de Sitter phase and thus it interpolates between dust matter and the cosmological constant. Cosmology with the generalized Chaplygin gas (8) results in an expanding universe which begins from a non-relativistic matter dominated phase and ends at a cosmological constant dominated era [26]. Also, the idea of this fluid is used to find a solution to the coincidence problem in cosmology [28–34]. Quantum cosmological models with the Chaplygin gas have been studied in Refs. [35–37], especially in Ref. [38], a scalar field is also added to the Chaplygin gas quantum cosmology and its effects are investigated. In summary, since the Chaplygin gas models are able to describe the smooth transition from a decelerated expansion to an accelerated universe and also since they try to give a unified picture of dark matter and dark energy, one may use them as an alternative to the traditional ΛCDM models.

In this paper, we shall consider a cosmological model in the framework of a projectable HL gravity without detailed balance condition. A Chaplygin gas will play the role of the matter source and a scalar field is coupled to metric with a generic coupling function $F(\phi)$. The classical version of such models is used to answer the missing-matter problem in cosmology [39], and their quantum cosmology is studied in Refs. [38, 40, 41]. Since our aim in the quantum part of the model is to investigate the time evolution of the wave function, we prefer to use the Chaplygin gas in the framework of the Schutz formalism [42, 43]. In such a setup, the Hamiltonian of the gas consists of a linear momentum, the variable canonically conjugate to which may play the role of a time parameter (see Refs. [21, 36, 37, 40, 44, 45] for details of this formalism).

The paper in organized as follows: In Sec. 2, we construct the action of HL gravity with the Chaplygin gas and scalar field in terms of minisuperspace variables. In Secs. 3 and 4, we approximate the super-Hamiltonian in two cases $Sp_\epsilon^{\alpha+1} \gg Aa^{3(\alpha+1)}$ and $Sp_\epsilon^{\alpha+1} \ll Aa^{3(\alpha+1)}$ separately. The Schutz formalism for the Chaplygin gas allows us to get a Schrödinger–Wheeler–DeWitt (SWD) equation in which the only remaining matter degree of freedom plays the role of time. After choosing the coupling function between the scalar field and metric as $F(\phi) = \lambda\phi^m$, we obtain the classical dynamics of the scale factor and scalar field in terms of the Schutz time parameter and see that they exhibit some types of singularities. We then deal with the quantization of the model and, by computing the expectation values of the scale factor and scalar field, we show that the evolution of the universe based on the quantum picture is free of classical singularities. Section 5 is devoted to summary and conclusions.

2. The model

The total action (without the detailed balance condition) of our model consists of three parts, that are gravitational Hořava–Lifshitz gravity action, scalar field and Chaplygin gas actions parts as

$$S = S_{\rm HL} + S_\phi + S_P\,, \tag{9}$$

where $S_{\rm HL}$, S_ϕ and S_P are the Hořava–Lifshitz, scalar field and Chaplygin gas actions, respectively. Now, we expand them separately.

2.1. Hořava–Lifshitz action

The action for the projectable HL gravity without detailed balance is given in (7). In a quasi-spherical polar coordinate system, we assume that the geometry of space-time is described by the FRW metric

$$\begin{aligned} {\rm d}s^2 &= g_{\mu\nu}{\rm d}x^\mu {\rm d}x^\nu \\ &= -N^2(t){\rm d}t^2 + a^2(t)\left[\frac{{\rm d}r^2}{1-kr^2} + r^2\left({\rm d}\vartheta^2 + \sin^2\vartheta {\rm d}\varphi\right)\right]\,, \end{aligned} \tag{10}$$

in which $N(t)$ is the lapse function, $a(t)$ is the scale factor and $k=-1,0,+1$ denotes the open, flat, and closed universes, respectively. Now, in the language of the ADM variables, the above metric can be rewritten as

$${\rm d}s^2 = -N^2(t){\rm d}t^2 + \gamma_{ij}{\rm d}x^i{\rm d}x^j\,,$$

where

$$\gamma_{ij} = a^2(t){\rm diag}\left(\frac{1}{1-kr^2}, r^2, r^2\sin^2\vartheta\right) \tag{11}$$

is the induced intrinsic metric on the 3-dimensional spatial hypersurfaces from which we obtain the Ricci and extrinsic curvature tensors as

$$R_{ij} = \frac{2k}{a^2}\gamma_{ij}\,, \qquad K_{ij} = \frac{\dot a}{Na}\gamma_{ij}\,. \tag{12}$$

The gravitational part for the model may now be written by substituting the above results into action (7) giving

$$\begin{aligned} S_{\rm HL} &= \frac{3(3\lambda-1)M_{\rm Pl}^2V_0}{2}\int {\rm d}t\, Na^3\left[-\frac{\dot a^2}{N^2a^2} + \frac{6k}{3(3\lambda-1)}\frac{1}{a^2}\right. \\ &\quad \left. -\frac{2\Lambda}{3(3\lambda-1)} - \frac{12k^2}{a^4}\frac{3g_2+g_3}{3(3\lambda-1)M_{\rm Pl}^2} - \frac{24k^3}{a^6}\frac{9g_4+3g_5+g_6}{3(3\lambda-1)M_{\rm Pl}^4}\right] \\ &= \int {\rm d}t\, N\left(-\frac{a\dot a^2}{N^2} + g_c a - g_\Lambda a^3 - \frac{g_r}{a} - \frac{g_s}{a^3}\right)\,, \end{aligned} \tag{13}$$

where $V_0 = \int \mathrm{d}^3x \frac{r^2 \sin\vartheta}{\sqrt{1-kr^2}}$ is the integral over spatial dimensions. Also, we have defined the coefficients g_c, g_Λ, g_r and g_s as

$$\left\{ \begin{array}{l} g_c = \frac{6k}{3(3\lambda-1)}\,, \\ g_\Lambda = \frac{2\Lambda}{3(3\lambda-1)}\,, \\ g_r = \frac{12k^2(3g_2+g_3)}{3(3\lambda-1)M_{\mathrm{Pl}}^2}\,, \\ g_s = \frac{24k^3(9g_4+3g_5+g_6)}{3(3\lambda-1)M_{\mathrm{Pl}}^4}\,, \end{array} \right. \tag{14}$$

in which we have set $3V_0M_{\mathrm{Pl}}^2(3\lambda-1)/2 = 1$. Now, the gravitational part of the Hamiltonian for this model can be obtained from its standard procedure. Noting that

$$p_a = -\frac{2a\dot{a}}{N}\,, \tag{15}$$

we get

$$\begin{aligned} H_{\mathrm{HL}} &= p_a\,\dot{a} - \mathcal{L}_{\mathrm{HL}}\,, \\ &= N\left(-\frac{p_a^2}{4a} - g_c a + g_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3}\right). \end{aligned} \tag{16}$$

2.2. *The Chaplygin gas*

In the Schutz formalism, the four velocity of a fluid can be expressed in terms of six scalar potentials as [42, 43]

$$u_\nu = \frac{1}{\mu}(\partial_\nu\epsilon + \varpi\partial_\nu\beta + \theta\partial_\nu S)\,, \tag{17}$$

where μ and S are specific enthalpy and entropy, respectively, while the potentials ϖ and β are related to torsion and are absent in FRW models. The potentials ϵ and θ have no direct physical interpretation in this formalism. The four-velocity obeys the condition $u_\nu u^\nu = 1$. Hence, the four-velocity of the fluid in its rest frame reads

$$u_\nu = N\delta_\nu^0 \quad \Rightarrow \quad \mu = \frac{\dot{\epsilon} + \theta\dot{S}}{N}\,. \tag{18}$$

Following the thermodynamical description of [36, 44], the basic thermodynamic relations of the Chaplygin gas are given by

$$\rho = \rho_0(1+\Pi), \qquad \mu = 1 + \Pi + \frac{P}{\rho_0}\,, \tag{19}$$

where ρ_0 and Π are the rest mass density and the specific internal energy of the gas, respectively. These quantities together with the temperature τ of the system obey the first law of the thermodynamics, which can be rewritten as

$$\begin{aligned}\tau \mathrm{d}S &= \mathrm{d}\Pi + P\mathrm{d}\left(\frac{1}{\rho_0}\right) \\ &= \frac{1}{(1+\alpha)(1+\Pi)^\alpha}\mathrm{d}\left[(1+\Pi)^{1+\alpha} - \frac{A}{\rho_0^{1+\alpha}}\right],\end{aligned} \tag{20}$$

where we have used the equation of state (8). Therefore, the temperature and entropy of the gas are obtained as

$$\tau = \frac{1}{(1+\alpha)(1+\Pi)^\alpha}, \qquad S = (1+\Pi)^{1+\alpha} - \frac{A}{\rho_0^{1+\alpha}}. \tag{21}$$

Now, we can express the energy density and pressure as functions of μ and S

$$\rho = \left[\frac{1}{A}\left(1 - \frac{\mu^{\frac{\alpha+1}{\alpha}}}{S^{\frac{1}{\alpha}}}\right)\right]^{\frac{-1}{\alpha+1}}, \tag{22}$$

$$P = -A\left[\frac{1}{A}\left(1 - \frac{\mu^{\frac{\alpha+1}{\alpha}}}{S^{\frac{1}{\alpha}}}\right)\right]^{\frac{\alpha}{\alpha+1}}. \tag{23}$$

Finally, with the help of these relations, the action of the Chaplygin gas takes the form of

$$\begin{aligned}S_P &= \int \mathrm{d}t\, \mathrm{d}^3x\, N\sqrt{\gamma}\, P \\ &= -A\int \mathrm{d}t\, Na^3\left[\frac{1}{A}\left(1 - \frac{(\dot{\epsilon}+\theta\dot{S})^{\frac{\alpha+1}{\alpha}}}{N^{\frac{\alpha+1}{\alpha}}S^{\frac{1}{\alpha}}}\right)\right]^{\frac{\alpha}{\alpha+1}}.\end{aligned} \tag{24}$$

Now, in terms of the conjugate momenta

$$\begin{aligned}&p_a = p_\theta = 0, \\ &p_\epsilon = a^3\left(\tfrac{\dot{\epsilon}+\theta\dot{S}}{NS}\right)^{\frac{1}{\alpha}}\left[\tfrac{1}{A}\left(1 - \tfrac{(\dot{\epsilon}+\theta\dot{S})^{\frac{\alpha+1}{\alpha}}}{N^{\frac{\alpha+1}{\alpha}}S^{\frac{1}{\alpha}}}\right)\right]^{\frac{-1}{\alpha+1}}, \\ &p_S = \theta p_\epsilon,\end{aligned} \tag{25}$$

the Chaplygin gas Hamiltonian can be written as follows:

$$
\begin{aligned}
H_P &= \left(\dot{\epsilon} + \theta \dot{S}\right) p_\epsilon - \mathcal{L}_P \\
&= N a^3 \left[\frac{1}{A}\left(1 - \frac{\left(\dot{\epsilon} + \theta \dot{S}\right)^{\frac{\alpha+1}{\alpha}}}{N^{\frac{\alpha+1}{\alpha}} S^{\frac{1}{\alpha}}}\right)\right]^{\frac{-1}{\alpha+1}} \\
&= N\left(S p_\epsilon^{\alpha+1} + A a^{3(\alpha+1)}\right)^{\frac{1}{\alpha+1}} .
\end{aligned} \tag{26}
$$

2.3. The scalar field

As mentioned before, we consider a non-linear self-coupling scalar field minimally coupled to geometry by the coupling function $F(\phi)$ [40]. The action of such a scalar field is

$$S_\phi = -\frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^4 x \sqrt{-g}\, F(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \,, \tag{27}$$

where by substituting metric (10), one gets

$$S_\phi = \int \mathrm{d}t\, \frac{1}{N} F(\phi) a^3 \dot{\phi}^2 \,. \tag{28}$$

Noting that the momentum congugate to ϕ is

$$p_\phi = \frac{2}{N} F(\phi) a^3 \dot{\phi} \,, \tag{29}$$

the Hamiltonian of the scale field is obtained as

$$H_\phi = \frac{N p_\phi^2}{4F(\phi) a^3} \,. \tag{30}$$

Now, we are ready to write the total Hamiltonian for our model as

$$
\begin{aligned}
H &= H_{\mathrm{HL}} + H_P + H_\phi \\
&= N\bigg[-\frac{p_a^2}{4a} - g_c a + g_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3} + \frac{p_\phi^2}{4F(\phi) a^3} \\
&\quad + \left(S p_\epsilon^{\alpha+1} + A a^{3(\alpha+1)}\right)^{\frac{1}{\alpha+1}} \bigg] .
\end{aligned} \tag{31}
$$

The setup for constructing the phase space and writing the Lagrangian and Hamiltonian of the model is now complete. However, the resulting classical (and quantum) equations of motion do not seem to have analytical solutions. To extract exact solutions, we first apply some approximation to the above Hamiltonian [36], and then will deal with the behavior of its classical and quantum pictures.

3. The $Sp_\epsilon^{\alpha+1} \gg Aa^{3(\alpha+1)}$ limit

In the early times of cosmic evolution when the scale factor is small, we can use the following expansion [46, 47]:

$$
\begin{aligned}
\left(Sp_\epsilon^{\alpha+1} + Aa^{3(\alpha+1)}\right)^{\frac{1}{\alpha+1}} &= S^{\frac{1}{\alpha+1}} p_\epsilon \left(1 + \frac{Aa^{3(\alpha+1)}}{Sp_\epsilon^{\alpha+1}}\right)^{\frac{1}{\alpha+1}} \\
&= S^{\frac{1}{\alpha+1}} p_\epsilon \left(1 + \frac{1}{\alpha+1}\frac{Aa^{3(\alpha+1)}}{Sp_\epsilon^{\alpha+1}} + \ldots\right) \\
&\simeq S^{\frac{1}{\alpha+1}} p_\epsilon \,. \qquad (32)
\end{aligned}
$$

Therefore, the super-Hamiltonian takes the form of

$$
H = N\left(-\frac{p_a^2}{4a} - g_c a + g_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3} + \frac{p_\phi^2}{4F(\phi)a^3} + S^{\frac{1}{\alpha+1}} p_\epsilon\right) . \qquad (33)
$$

Now, consider the following canonical transformation [36, 48]

$$
\begin{aligned}
T &= -(\alpha+1) S^{\frac{\alpha}{\alpha+1}} p_\epsilon^{-1} p_S \,, \\
p_{\mathrm{T}} &= S^{\frac{1}{\alpha+1}} p_\epsilon \,, \qquad (34)
\end{aligned}
$$

under the act of which Hamiltonian (33) takes the form of

$$
H = N\left(-\frac{p_a^2}{4a} - g_c a + g_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3} + \frac{p_\phi^2}{4F(\phi)a^3} + p_{\mathrm{T}}\right) . \qquad (35)
$$

We see that the momentum p_{T} is the only remaining canonical variable associated with the Chaplygin gas and appears linearly in the Hamiltonian.

3.1. The classical model

The classical dynamics of the system is governed by the Hamiltonian equation of motion $\dot{q} = \{q, H\}$ for each variable. The result is

$$
\begin{cases}
\dot{a} = \frac{Np_a}{2a} \,, \\
\dot{p}_a = N\left(-\frac{p_a^2}{4a^2} + g_c - 3g_\Lambda a^2 + \frac{g_r}{a^2} + \frac{3g_s}{a^3} + \frac{3p_\phi^2}{4Fa^4}\right) , \\
\dot{\phi} = \frac{Np_\phi}{2Fa^3} \,, \\
\dot{p}_\phi = \frac{Np_\phi^2}{4a^3}\frac{F'}{F^2} \,, \\
\dot{T} = N \,, \\
\dot{p}_{\mathrm{T}} = 0 \to p_{\mathrm{T}} = \mathrm{const} \,,
\end{cases} \qquad (36)
$$

where $F' = \frac{\mathrm{d}F(\phi)}{\mathrm{d}\phi}$. Up to this point, the cosmological model, in view of the concerning issue of time, has been, of course, under-determined. Before trying to solve these equations, we must decide on a choice of time in the theory. The under-determinacy problem at the classical level may be resolved by using the gauge freedom via fixing the gauge. A glance at the above equations shows that choosing the gauge $N = 1$, we have $\dot{T} = 1 \Rightarrow T = t$, which means that variable T may play the role of time in the model. With this time gauge, we obtain the following equation of motion for ϕ:

$$2\frac{\ddot{\phi}}{\dot{\phi}} + \frac{F'}{F}\dot{\phi} + 6\frac{\dot{a}}{a} = 0\,. \tag{37}$$

This equation can easily be integrated to yield

$$F(\phi)\dot{\phi}^2 = Ca^{-6}\,, \tag{38}$$

where C is an integration constant. Also, eliminating the momenta from system (36) results in

$$\dot{a}^2 + g_c - g_\Lambda a^2 - \frac{g_r}{a^2} - \frac{g_s + C}{a^4} - \frac{p_{\mathrm{T}}}{a} = 0\,, \tag{39}$$

in which we have used Eq. (38). In general, this equation does not seem to have an exact solution, so we restrict ourselves to the special case in which $g_c = g_\Lambda = g_r = 0$, $g_s \neq 0$, for which the solution to Eq. (39) reads

$$a(t) = \left(\frac{9p_{\mathrm{T}}}{4}t^2 - \frac{g_s + C}{p_{\mathrm{T}}}\right)^{\frac{1}{3}}\,. \tag{40}$$

What remains to be found is an expression for the scalar field $\phi(t)$. In the following, we shall consider the case of a coupling function in the form of $F(\phi) = \lambda\phi^m$. With this choice for the function $F(\phi)$ and with the help of Eqs. (38) and (40), we are able to calculate the time evolution of the scalar field as

$$\phi(t) = \left[\phi_0 - \frac{m+2}{6}\sqrt{\frac{C}{(g_s + C)\lambda}}\,\ln\frac{3p_{\mathrm{T}}t - 2\sqrt{g_s + C}}{3p_{\mathrm{T}}t + 2\sqrt{g_s + C}}\right]^{\frac{2}{m+2}}\,, \tag{41}$$

where ϕ_0 is an integration constant and we assumed $m \neq -2$. Finally, to understand the relation between the Big-Bang singularity $a \to 0$ and the blow up singularity $\phi \to \pm\infty$, we are going to find a classical trajectory in configuration space (a, ϕ), where the time parameter t is eliminated. From (38) and (39), one gets

$$\phi^m\left(\frac{\mathrm{d}\phi}{\mathrm{d}a}\right)^2 = \frac{Ca^{-6}}{\lambda}\left(-g_c + g_\Lambda a^2 + \frac{g_r}{a^2} + \frac{g_s + C}{a^4} + \frac{p_{\mathrm{T}}}{a}\right)^{-1}\,, \tag{42}$$

where for the case of $g_c = g_\Lambda = g_r = 0, g_s \neq 0$, after integration, it reads

$$\phi(a) = \left[\phi_0 - \frac{m+2}{6}\sqrt{\frac{C}{(g_s+C)\lambda}}\ \ln\frac{\sqrt{p_{\mathrm{T}}a^3+g_s+C}-\sqrt{g_s+C}}{\sqrt{p_{\mathrm{T}}a^3+g_s+C}+\sqrt{g_s+C}}\right]^{\frac{2}{m+2}}. \tag{43}$$

We see that the evolution of the universe based on (40) has Big-Bang-like singularities at $t = \pm t_*$, where $t_* = \frac{2}{3P_{\mathrm{T}}}\sqrt{g_s+C}$. Indeed, the condition $a(t) \geq 0$ separates two sets of solutions each of which is valid for $t \leq -t_*$ and $t \geq +t_*$, respectively. For the former, we have a contracting universe which decreases its size according to a power law relation and ends its evolution in a singularity at $t = -t_*$, while for the latter, the evolution of the universe begins with a Big-Bang singularity at $t = +t_*$ and then follows the power law expansion $a(t) \sim t^{2/3}$ at late time of cosmic evolution. On the other hand, the scalar field has a monotonically decreasing behavior coming from $\phi \to +\infty$ at early times and reaches zero as time grows, see Fig. 1. We shall see in the next subsection how this classical picture may be modified if one takes into account the quantum mechanical considerations.

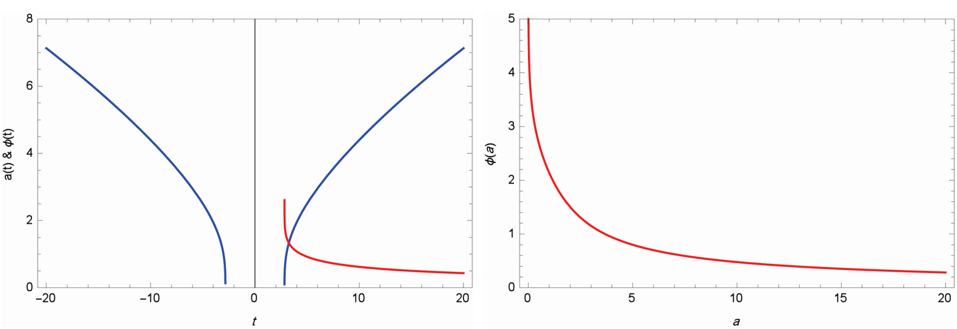

Fig. 1. (Color online) Left: The classical scale factor $a(t)$ (darker/blue line) and $\phi(t)$ (lighter/red line). Right: The classical trajectory in a–ϕ plane. The figures are plotted for the numerical values $g_s = \frac{1}{20}$, $p_{\mathrm{T}} = \frac{4}{9\sigma^2}$, $C = 3$, $\lambda = 1$ and $m = 2$.

3.2. The quantum model

We now focus our attention on the study of the quantum cosmology of the model described above. We start by writing the Wheeler–DeWitt equation from Hamiltonian (35). Since the lapse function appears as a Lagrange multiplier in the Hamiltonian, we have the Hamiltonian constraint $H = 0$. Thus, application of the Dirac quantization procedure demands that the quantum states of the universe should be annihilated by the operator version of H, that is $H\Psi(a,\phi,T)=0$, where $\Psi(a,\phi,T)$ is the wave function of the universe. If we use the usual representation $P_q \to -i\partial_q$, we are led to the following SWD equation:

$$\frac{1}{4a}\left(\frac{\partial^2}{\partial a^2}+\frac{\beta}{a}\frac{\partial}{\partial a}\right)\Psi(a,\phi,T)+\left(-g_c a+g_\Lambda a^3+\frac{g_r}{a}+\frac{g_s}{a^3}\right)\Psi(a,\phi,T)$$
$$-\frac{1}{4Fa^3}\left(\frac{\partial^2}{\partial\phi^2}+\frac{\kappa F'}{F}\frac{\partial}{\partial\phi}\right)\Psi(a,\phi,T)=i\frac{\partial\Psi(a,\phi,T)}{\partial\, T}\,, \tag{44}$$

where the parameters β and κ represent the ambiguity in the ordering of factors (a,P_a) and (ϕ,P_ϕ) respectively. This equation takes the form of a Schrödinger equation $i\partial\Psi/\partial T=H\Psi$, in which the Hamiltonian operator is Hermitian with the standard inner product

$$\langle\Psi_1\mid\Psi_2\rangle=\int\limits_{(a,\phi)}\mathrm{d}a\,\mathrm{d}\phi\, a\,\Psi_1^*\Psi_2\,. \tag{45}$$

We separate the variables in the above equation as $\Psi(a,\phi,T)=e^{iET}\psi(a,\phi)$ leading to

$$\frac{1}{4a}\left(\frac{\partial^2}{\partial a^2}+\frac{\beta}{a}\frac{\partial}{\partial a}\right)\psi(a,\phi)-\frac{1}{4Fa^3}\left(\frac{\partial^2}{\partial\phi^2}+\frac{\kappa F'}{F}\frac{\partial}{\partial\phi}\right)\psi(a,\phi)$$
$$+\left(-g_c a+g_\Lambda a^3+\frac{g_r}{a}+\frac{g_s}{a^3}+E\right)\psi(a,\phi)=0\,, \tag{46}$$

where E is a separation constant. The solutions of the above differential equation are separable and may be written in the form of $\psi(a,\phi)=A(a)\Phi(\phi)$ which yields

$$\frac{\mathrm{d}^2A(a)}{\mathrm{d}a^2}+\frac{\beta}{a}\frac{\mathrm{d}A(a)}{\mathrm{d}a}+4\left(-g_c a^2+g_\Lambda a^4+g_r+\frac{g_s+w}{a^2}+Ea\right)A(a)=0\,, \tag{47}$$

$$\frac{\mathrm{d}^2\Phi(\phi)}{\mathrm{d}\phi^2}+\frac{\kappa F'(\phi)}{F(\phi)}\frac{\mathrm{d}\Phi(\phi)}{\mathrm{d}\phi}+4wF(\phi)\Phi(\phi)=0\,, \tag{48}$$

where w is another constant of separation. The factor-ordering parameters do not affect the semiclassical probabilities [49] so in what follows, we have chosen $\beta=0$ and $\kappa=-1$ to make the differential equations solvable. Upon substituting the relation $F(\phi)=\lambda\phi^m$ into (48), its solutions read in terms of the Bessel functions J and Y as

$$\Phi(\phi)=C_1\,\phi^{\frac{1+m}{2}}\,J_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right)+C_2\,\phi^{\frac{1+m}{2}}\,Y_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right) \tag{49}$$

for $m \neq -2$, and

$$\Phi(\phi) = C_1\,\phi^{\frac{-1+\sqrt{1-16\lambda w}}{2}} + C_2\,\phi^{\frac{-1-\sqrt{1-16\lambda w}}{2}} \tag{50}$$

for $m = -2$. Also, if we set (as in the classical solutions) $g_c = g_\Lambda = g_r = 0$, Eq. (47) admits the solution

$$A(a) = c_1\sqrt{a}\,J_\nu\left(\tfrac{4}{3}\sqrt{E}a^{\frac{3}{2}}\right) + c_2\sqrt{a}\,Y_\nu\left(\tfrac{4}{3}\sqrt{E}a^{\frac{3}{2}}\right)\,, \tag{51}$$

where $\nu = \frac{1}{3}\sqrt{1-16(g_s+w)}$. Thus, the eigenfunctions of the SWD equation can be written as

$$\begin{aligned}\Psi_{E,w}(a,\phi,T) &= e^{iET}A(a)\Phi(\phi) \\ &= e^{iET}\,\sqrt{a}\,J_\nu\left(\frac{4}{3}\sqrt{E}a^{\frac{3}{2}}\right)\,\phi^{\frac{m+1}{2}}\,J_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right)\,,\end{aligned} \tag{52}$$

where we have chosen $C_2 = c_2 = 0$ for having well-defined functions in all ranges of variables a and ϕ. We may now write the general solutions to the SWD equations as a superposition of the eigenfunctions, that is

$$\begin{aligned}\Psi(a,\phi,T) &= \int \mathrm{d}E\,\mathrm{d}w\,f(E)\,g(w)\,\Psi_{E,w}(a,\phi,T) \\ &= \sqrt{a}\phi^{\frac{m+1}{2}}\int_0^{w_0}\mathrm{d}w\,g(w)\,J_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right) \\ &\quad\times\int_0^\infty \mathrm{d}E\,f(E)\,e^{iET}\,J_\nu\left(\frac{4}{3}\sqrt{E}a^{\frac{3}{2}}\right)\,,\end{aligned} \tag{53}$$

where $w_0 = \frac{1}{16} - g_s$ and $f(E)$ and $g(w)$ are weight functions suitable to construct the wave packets. By using the equality [50]

$$\int_0^\infty \mathrm{d}x\,e^{-Zx^2}\,x^{\nu+1}\,J_\nu(bx) = \frac{b^\nu}{(2Z)^{\nu+1}}e^{-\frac{b^2}{4Z}}\,, \tag{54}$$

we can evaluate the integral over E in (53) and a simple analytical expression for this integral is found if we choose the function $A(E)$ to be

$$f(E) = E^{\frac{\nu}{2}}e^{-\sigma E}\,, \tag{55}$$

where σ is an arbitrary positive constant. With this procedure, we get

$$\begin{aligned}\Psi(a,\phi,T) &= \sqrt{a}\,\phi^{\frac{m+1}{2}}\int\limits_0^{w_0}\mathrm{d}w\,g(w)\,J_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right)\\ &\quad\times\frac{\left(\frac{4}{3}a^{\frac{3}{2}}\right)^{\frac{1}{3}\sqrt{1-16(g_s+w)}}}{(2Z)^{1+\frac{1}{3}\sqrt{1-16(g_s+w)}}}e^{\frac{-4a^3}{9Z}}\,, \end{aligned}\tag{56}$$

where $Z=\sigma-iT$. To achieve an analytical closed expression for the wave function, we assume that the above superposition is taken over such values of w for which one can use the approximation $\sqrt{1-16(g_s+w)}\simeq\sqrt{1-16g_s}$, that is

$$\begin{aligned}\Psi(a,\phi,T) &= \sqrt{a}\,\phi^{\frac{m+1}{2}}\frac{\left(\frac{4}{3}a^{\frac{3}{2}}\right)^{\frac{1}{3}\sqrt{1-16g_s}}}{(2Z)^{1+\frac{1}{3}\sqrt{1-16g_s}}}e^{\frac{-4a^3}{9Z}}\\ &\quad\times\int\limits_0^{w_0}\mathrm{d}w\,g(w)\,J_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right)\,. \end{aligned}\tag{57}$$

Now, by using the equality [50]

$$\int\limits_0^1\mathrm{d}\nu\,\nu^{r+1}(1-\nu^2)^{s/2}J_r(z\nu)=\frac{2^s\,\Gamma(s+1)}{z^{s+1}}J_{r+s+1}(z)\tag{58}$$

and choosing the weight function $g(w)=\left(\frac{w}{w_0}\right)^{\frac{m+1}{2(m+2)}}\left(1-\frac{w}{w_0}\right)^{s/2}$, we are led to the following expression for the wave function:

$$\begin{aligned}\Psi(a,\phi,T) &= \mathcal{N}\frac{a^{\frac{1+\sqrt{1-16g_s}}{2}}}{(\sigma-iT)^{1+\frac{1}{3}\sqrt{1-16g_s}}}\exp\left(-\frac{4a^3}{9(\sigma-iT)}\right)\\ &\quad\times\phi^{-\frac{1+(m+2)s}{2}}J_{\frac{2m+3}{m+2}+s}\left(\frac{\sqrt{(1-16g_s)\lambda}}{m+2}\phi^{\frac{m+2}{2}}\right), \end{aligned}\tag{59}$$

where $\mathcal{N}$ is a normalization coefficient. Now, having the above expression for the wave function of the universe, we are going to obtain the predictions for the behavior of the dynamical variables in the corresponding cosmological model. In general, one of the most important features in quantum cosmology is the recovery of classical cosmology from the corresponding quantum model or, in other words, how can the WD wave functions predict a classical universe. In this approach, one usually constructs a coherent wave packet with a good asymptotic behavior in the minisuperspace, peaking in the vicinity of the classical trajectory. On the other hand, in another approach to show the correlations between classical and quantum pattern, following the many-worlds interpretation of quantum mechanics, one may calculate the time dependence of the expectation value of a dynamical variable q as

$$\langle q\rangle(t)=\frac{\langle\Psi|q|\Psi\rangle}{\langle\Psi|\Psi\rangle}\,. \tag{60}$$

Following this approach, we may write the expectation value for the scale factor as

$$\begin{aligned}\langle a\rangle(T) &= \frac{\int_{a=0}^{\infty}\int_{\phi=-\infty}^{+\infty}\mathrm{d}a\,\mathrm{d}\phi\,a^2\,|\Psi|^2}{\int_{a=0}^{\infty}\int_{\phi=-\infty}^{+\infty}\mathrm{d}a\,\mathrm{d}\phi\,a\,|\Psi|^2}\\ &= \frac{3}{2}\frac{\Gamma\left(\frac{4+\sqrt{1-16g_s}}{3}\right)}{\Gamma\left(\frac{3+\sqrt{1-16g_s}}{3}\right)}\left(\frac{\sigma^2+T^2}{3\sigma}\right)^{\frac{1}{3}}\,.\end{aligned} \tag{61}$$

It is important to classify the nature of the quantum model as concerns the presence or absence of singularities. For the wave function (59), the expectation value (61) of a never vanishes, showing that these states are non-singular. Indeed, expression (61) represents a bouncing universe with no singularity where its late time behavior coincides to the late time behavior of the classical solution (40), that is $a(t)\sim t^{\frac{2}{3}}$. We have plotted this behavior in Fig. 2. As this figure shows, instead of two separate contracting and expanding classical solutions, the quantum expectation value consists of two branches. In one branch, the universe contracts and when reaches a minimum size undergoes to an expansion period. Therefore, we have bouncing cosmology in which the bounce occurs at classical singularity. In a similar manner, the expectation value for the scalar field reads as

$$\langle\phi\rangle(T)=\frac{\int\mathrm{d}a\,\mathrm{d}\phi\,a\phi\,|\Psi|^2}{\int\mathrm{d}a\,\mathrm{d}\phi\,a\,|\Psi|^2}=\mathrm{const}\,. \tag{62}$$

We see that the expectation value of ϕ does not depend on time. This result is comparable with those obtained in [51] where a constant expectation value for the dilatonic field in a quantum cosmological model based on the string effective action coupled to matter has been obtained.

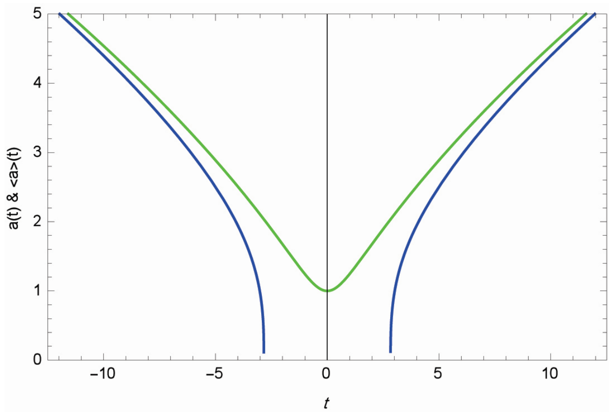

Fig. 2. (Color online) The dynamical behavior of $\langle a\rangle(T)$ (lighter/green line) in comparison with classical scale factor $a(t)$ (darker/blue line). See Eqs. (61) and (40).

4. The $Sp_\epsilon^{\alpha+1} \ll Aa^{3(\alpha+1)}$ limit

Now, let us return to Hamiltonian (31) but this time expand it in the late time limit $Sp_\epsilon^{\alpha+1} \ll Aa^{3(\alpha+1)}$ as

$$\begin{aligned}\left(Sp_\epsilon^{\alpha+1}+Aa^{3(\alpha+1)}\right)^{\frac{1}{\alpha+1}} &= A^{\frac{1}{\alpha+1}}a^3\left(1+\frac{Sp_\epsilon^{\alpha+1}}{Aa^{3(\alpha+1)}}\right)^{\frac{1}{\alpha+1}}\\ &= A^{\frac{1}{\alpha+1}}a^3\left[1+\frac{1}{\alpha+1}\frac{Sp_\epsilon^{\alpha+1}}{Aa^{3(\alpha+1)}}\right.\\ &\quad\left.+\frac{1}{2}\frac{1}{\alpha+1}\left(\frac{1}{\alpha+1}-1\right)\left(\frac{Sp_\epsilon^{\alpha+1}}{Aa^{3(\alpha+1)}}\right)^2+\dots\right]\\ &\simeq A^{\frac{1}{\alpha+1}}a^3+\frac{1}{\alpha+1}\frac{A^{\frac{-\alpha}{\alpha+1}}Sp_\epsilon^{\alpha+1}}{a^{3\alpha}}\,. \end{aligned} \tag{63}$$

Therefore, the super-Hamiltonian takes the form of

$$H=N\left(-\frac{p_a^2}{4a}-g_ca+\bar{g}_\Lambda a^3+\frac{g_r}{a}+\frac{g_s}{a^3}+\frac{p_\phi^2}{4F(\phi)a^3}+\frac{1}{\alpha+1}\frac{A^{\frac{-\alpha}{\alpha+1}}Sp_\epsilon^{\alpha+1}}{a^{3\alpha}}\right), \tag{64}$$

where $\bar{g}_\Lambda=g_\Lambda+A^{\frac{1}{\alpha+1}}$. Now, consider the following canonical transformation [48]:

$$\begin{aligned} T &= -(\alpha+1)A^{\frac{\alpha}{\alpha+1}}p_\epsilon^{-(\alpha+1)}p_S\,,\\ p_{\mathrm{T}} &= \frac{1}{\alpha+1}A^{\frac{-\alpha}{\alpha+1}}Sp_\epsilon^{\alpha+1}\,, \end{aligned} \tag{65}$$

under act of which the above Hamiltonian becomes

$$H = N\left(-\frac{p_a^2}{4a} - g_c a + \bar{g}_\Lambda a^3 + \frac{g_r}{a} + \frac{g_s}{a^3} + \frac{p_\phi^2}{4F(\phi)a^3} + \frac{p_{\rm T}}{a^{3\alpha}}\right). \tag{66}$$

We now may repeat the steps as we have taken in the previous section to obtain the classical and quantum cosmological dynamics based on Hamiltonian (66).

4.1. The classical model

The classical equations of motion from Hamiltonian (66) are

$$\begin{cases} \dot{a} = -\frac{Np_a}{2a}\,, \\ \dot{p}_a = N\left(-\frac{p_a^2}{4a^2} + g_c - 3\bar{g}_\Lambda a^2 + \frac{g_r}{a^2} + \frac{3g_s}{a^4} + \frac{3p_\phi^2}{4Fa^4} + \frac{3\alpha p_{\rm T}}{a^{3\alpha+1}}\right), \\ \dot{\phi} = \frac{Np_\phi}{2Fa^3}\,, \\ \dot{p}_\phi = \frac{Np_\phi^2}{4a^3}\frac{F'}{F^2}\,, \\ \dot{T} = \frac{N}{a^{3\alpha}}\,, \\ \dot{p}_{\rm T} = 0 \rightarrow p_{\rm T} = {\rm const}\,. \end{cases} \tag{67}$$

To have the clock parameter as $T = t$, we should choose the lapse function $N = a^{3\alpha}$. Since the third and fourth equations of this system are the same as their counterparts in system (36), the dynamical equations for the scalar field are the same as Eqs. (37) and (38). Also, with the constraint equation $H = 0$, we obtain

$$\dot{a}^2 + a^{6\alpha}\left(g_c - \bar{g}_\Lambda a^2 - \frac{g_r}{a^2} - \frac{g_s + C}{a^4} - \frac{p_{\rm T}}{a^{3\alpha+1}}\right) = 0\,. \tag{68}$$

To solve this equation, we suppose $g_c = \bar{g}_\Lambda = 0$ and $g_r, g_s \neq 0$ which simplifies the above equation to

$$\dot{a}^2 = a^{6\alpha}\left(\frac{g_r}{a^2} + \frac{g_s + C}{a^4} + \frac{p_{\rm T}}{a^{3\alpha+1}}\right). \tag{69}$$

This equation does not yet have an exact solution for the general case with arbitrary α. So, from now on, we restrict ourselves to the case of $\alpha = \frac{1}{3}$ for which the solution to Eq. (69) is

$$a(t) = \sqrt{(g_r + p_{\rm T})t^2 - \frac{g_s + C}{g_r + p_{\rm T}}}\,. \tag{70}$$

By means of this relation, with the help of (37) and (38), and with the same detail as in previous section, we get the following expressions for $\phi(t)$ and $\phi(a)$:

$$\phi(t) = \left[\phi_0 - \frac{m+2}{4}\sqrt{\frac{C}{(g_s+C)\lambda}}\,\ln\frac{(g_r+p_{\mathrm{T}})t-\sqrt{g_s+C}}{(g_r+p_{\mathrm{T}})t+\sqrt{g_s+C}}\right]^{\frac{2}{m+2}}, \tag{71}$$

and

$$\phi(a) = \left[\phi_0 + \frac{m+2}{2}\sqrt{\frac{C}{(g_s+C)\lambda}}\,\ln\frac{\sqrt{g_s+C}+\sqrt{(g_r+p_{\mathrm{T}})a^2+g_s+C}}{a}\right]^{\frac{2}{m+2}}. \tag{72}$$

4.2. The quantum model

The standard quantization process based on Hamiltonian (66) get us the following SWD equation:

$$\begin{aligned}&\frac{1}{4a}\left(\frac{\partial^2}{\partial a^2}+\frac{\beta}{a}\frac{\partial}{\partial a}\right)\Psi(a,\phi,T)+\left(-g_c a+\bar{g}_\Lambda a^3+\frac{g_r}{a}+\frac{g_s}{a^3}\right)\Psi(a,\phi,T)\\&-\frac{1}{4Fa^3}\left(\frac{\partial^2}{\partial\phi^2}+\frac{\kappa F'}{F}\frac{\partial}{\partial\phi}\right)\Psi(a,\phi,T)=\frac{i}{a^{3\alpha}}\frac{\partial\Psi(a,\phi,T)}{\partial\,T},\end{aligned} \tag{73}$$

where β and κ are again factor ordering parameters which are, as before, set as $\beta=0$ and $\kappa=-1$. This time the Hamiltonian operator is Hermitian with the inner product

$$\langle\Psi_1,\Psi_2\rangle = \int\limits_{(a,\phi)} \mathrm{d}a\,\mathrm{d}\phi\,a^{1-3\alpha}\,\Psi_1^*\Psi_2\,. \tag{74}$$

Separation of variables as $\Psi(a,\phi,T)=e^{iET}A(a)\Phi(\phi)$ leads to Eq. (48) with a solution (49) for the ϕ-sector of the eigenfunctions, while for $A(a)$, we arrive at the following equation (with $g_c=\bar{g}_\Lambda=0$):

$$\frac{\mathrm{d}^2A}{\mathrm{d}a^2}+4\left(g_r+\frac{g_s+w}{a^2}+\frac{E}{a^{3\alpha-1}}\right)A=0\,. \tag{75}$$

For $\alpha=\frac{1}{3}$, this equation has the solutions

$$A(a)=c_1\sqrt{a}\,J_\nu\left(2\sqrt{g_r+E}a\right)+c_2\sqrt{a}\,Y_\nu\left(2\sqrt{g_r+E}a\right), \tag{76}$$

with $\nu = \frac{1}{2}\sqrt{1-16(g_s+w)}$. Therefore, the eigenfunctions of the corresponding SWD equation read

$$\Psi_{E,w}(a,\phi,T) = e^{iET}\,\sqrt{a}\,J_\nu\left(2\sqrt{g_r+Ea}\right)\,\phi^{\frac{m+1}{2}}\,J_{\frac{m+1}{m+2}}\left(\frac{4\sqrt{\lambda w}}{m+2}\phi^{\frac{m+2}{2}}\right), \tag{77}$$

in which we have removed the Bessel functions Y from the solutions. Following the same steps which led us to the wave function (59), we obtain the wave function as

$$\begin{aligned}\Psi(a,\phi,T) &= \mathcal{N}\,e^{-ig_rT}\,\frac{a^{\frac{1+\sqrt{1-16g_s}}{2}}}{(\sigma-iT)^{1+\frac{1}{2}\sqrt{1-16g_s}}}\exp\left(\frac{-a^2}{\sigma-iT}\right)\\ &\quad\times\phi^{-\frac{1+(m+2)s}{2}}\,J_{\frac{2m+3}{m+2}+s}\left(\frac{\sqrt{(1-16g_s)\lambda}}{m+2}\phi^{\frac{m+2}{2}}\right),\end{aligned} \tag{78}$$

from which the expectation values are obtained as

$$\begin{aligned}\langle a\rangle(T) &= \frac{\int \mathrm{d}a\,\mathrm{d}\phi\,a\,|\Psi|^2}{\int \mathrm{d}a\,\mathrm{d}\phi\,|\Psi|^2}\\ &= \frac{\Gamma\left(\frac{3+\sqrt{1-16g_s}}{2}\right)}{\Gamma\left(\frac{2+\sqrt{1-16g_s}}{2}\right)}\left(\frac{\sigma^2+T^2}{2\sigma}\right)^{\frac{1}{2}},\end{aligned} \tag{79}$$

$$\langle\phi\rangle(T) = \frac{\int \mathrm{d}a\,\mathrm{d}\phi\,\phi\,|\Psi|^2}{\int \mathrm{d}a\,\mathrm{d}\phi\,|\Psi|^2} = \mathrm{const}\,. \tag{80}$$

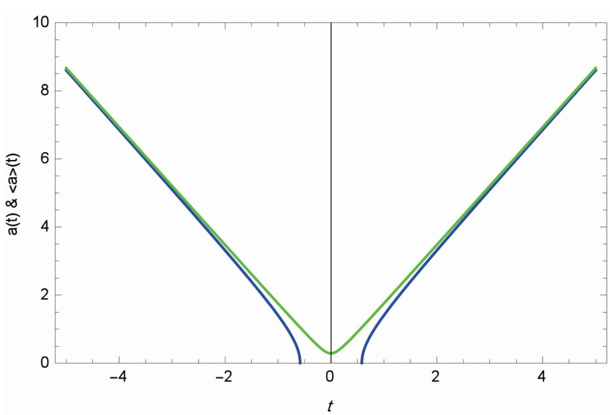

Fig. 3. (Color online) Qualitative behavior of $a(t)$ (darker/blue line) and $\langle a\rangle(t)$ (lighter/green line), see Eqs. (70) and (79).

In Fig. 3, we have plotted the classical scale factor (70) and its quantum expectation value (79). The discussions on the comparison between quantum cosmological solutions and their corresponding form from the classical formalism are the same as in the previous section. Similar discussion as above would be applicable to this case as well.

5. Conclusion

In this paper, we have applied the Hořava theory of gravity to a FRW cosmological model coupled minimally to a scalar field in which a generalized Chaplygin gas, in the context of the Schutz representation, plays the role of the matter field. The use of the Schutz formalism for the Chaplygin gas allowed us to introduce the only remaining matter degree of freedom as a time parameter in the model. After a very brief review of HL theory of gravity, we have considered a FRW cosmological setting in the framework of the projectable HL gravity without detailed balance condition and presented its Hamiltonian in terms of the minisuperspace variables. Though the corresponding classical equations did not have exact solutions, we analyzed their behavior in the limiting cases of the early and late times of cosmic evolution and obtained analytical expressions for the scale factor and the scalar field in these regions. We have seen that these solutions are consisted of two separate branches each of which exhibits some kinds of classical singularities. Indeed, the classical solutions have either contracting or expanding branches which are disconnected from each other by some classically forbidden regions. Another part of the paper is devoted to the quantization of the model described above in which we saw that the classical singular behavior will be modified. In the quantum models, we showed that the SWD equation can be separated and its eigenfunctions can be obtained in terms of analytical functions. By an appropriate superposition of the eigenfunctions, we constructed the corresponding wave packets. Using Schutz's representation for the Chaplygin gas, under a particular gauge choice, we led to the identification of a time parameter which allowed us to study the time evolution of the resulting wave function. Investigation of the expectation value of the scale factor shows a bouncing behavior near the classical singularity. In addition to singularity avoidance, the appearance of bounce in the quantum model is also interesting in its nature due to prediction of a minimal size for the corresponding universe. It is well-known that the idea of existence of a minimal length in nature is supported by almost all candidates of quantum gravity.

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