CLASSICAL BRST CHARGE AND OBSERVABLES IN REDUCIBLE GAUGE THEORIES*

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(Received March 2, 2017; accepted June 26, 2017)

We study the construction of the classical Becchi–Rouet–Stora–Tyutin (BRST) charge and observables for arbitrary reducible gauge theory. Using a special coordinate system in the extended phase space, we obtain an explicit expression for the Koszul–Tate differential and show that the BRST charge can be found by a simple iterative method. We also give a formula for the classical BRST observables.

DOI:10.5506/APhysPolB.48.1335

1. Introduction

The modern quantization method for gauge theories is based on the BRST symmetry [1, 2]. In the framework of the canonical formalism, this symmetry is generated by the BRST charge. If the quantum BRST charge exists, it is essentially determined by the corresponding classical one. The classical BRST charge is defined as a solution to the Poisson-bracket master equation with certain boundary conditions. A solution to this equation for irreducible gauge theories [3] was described in [4]. The BRST construction in the case of reducible gauge theories was given in [5]. The global existence of the classical BRST charge and observables in the reducible case was proved in [6] (see also [7]). It is based on the nilpotency and aciclicity of the Koszul–Tate differential. In [8], the question of quantization of reducible gauge theories with constraints linear in the momenta is studied.

In this paper, we present a solution to the Poisson-bracket master equation for the arbitrary reducible gauge theory. To this aim, we find a new coordinate system in the extended phase space and transform the master equation by changing variables. This enables us to construct the Koszul–Tate differential. Then the BRST charge can be obtained by using a simple iterative method. We also give a formula for the classical BRST observables. As an example of computing the BRST charge, we consider an SU(2) gauge invariant reducible theory. In this paper, we extend the analysis of [9] to cover the Hamiltonian formalism.

The paper is organized as follows. In Section 2, we review the BRST construction. In Section 3, we introduce a new coordinate system in the extended phase space. With respect to the new coordinates, the Koszul–Tate differential takes a standard form [10]. The construction of the BRST charge is given in Section 4. A formula for the BRST observables is obtained in Section 5. In Section 6, we find the BRST charge for a reducible theory of the order of $L = 1$.

In what follows, the Grassman parity and ghost number of a function X are denoted by $\epsilon(X)$ and $\mathrm{gh}(X)$, respectively. The Poisson superbracket in phase space $\mathit{\Gamma} = (P_A, Q^A), \epsilon(P_A) = \epsilon(Q^A)$ is given by

$$\{X, Y\} = \frac{\partial X}{\partial Q^A}\frac{\partial Y}{\partial P_A} - (-1)^{\epsilon(X)\epsilon(Y)}\frac{\partial Y}{\partial Q^A}\frac{\partial X}{\partial P_A}\,. \tag{1}$$

Derivatives with respect to generalized momenta P_A are always understood as left-hand, and those with respect to generalized coordinates Q^A (unless specified) as right-hand ones. Superbracket (1) possesses the following algebraic properties:

$$\begin{aligned}
&\{X, Y\} = -(-1)^{\epsilon(X)\epsilon(Y)}\{Y, X\}\,,\\
&\{X, YZ\} = \{X, Y\}Z + (-1)^{\epsilon(X)\epsilon(Y)}Y\{X, Z\}\,,\\
&(-1)^{\epsilon(X)\epsilon(Z)}\{\{X, Y\}, Z\} + \text{cycl. perm.}(X, Y, Z) = 0\,.
\end{aligned}$$

The last relation is the Jacobi identity for the superbracket.

2. Generating equations for the gauge algebra

Let P be a phase space with the phase-space coordinates ξ_a, $\epsilon(\xi_a) = \epsilon_a$, $a = 1, \ldots, 2m$, and let G_{a_0}, $a_0 = 1, \ldots, m_0$, be the first class constraints which satisfy the following Poisson brackets:

$$\{G_{a_0}, G_{b_0}\} = U^{c_0}_{a_0 b_0} G_{c_0}\,,$$

where $U^{c_0}_{a_0 b_0}$ are phase-space functions. The constraints are assumed to be of the definite Grassmann parity ϵ_{a_0}, $\epsilon(G_{a_0}) = \epsilon_{a_0}$.

We shall consider a reducible gauge theory of L^{th} order [5]. That is, there exist phase-space functions

$$Z^{a_k}_{a_{k+1}}\,, \qquad k = 0, \ldots, L-1\,, \qquad a_k = 1, \ldots, m_k$$

such that at each stage, the Zs form a complete set

$$\begin{aligned}
&Z^{a_k}_{a_{k+1}}\lambda^{a_{k+1}} \approx 0 \Rightarrow \lambda^{a_{k+1}} \approx Z^{a_{k+1}}_{a_{k+2}}\lambda^{a_{k+2}}\,, \qquad k = 0, \dots, L-2\,,\\
&Z^{a_{L-1}}_{a_L}\lambda^{a_L} \approx 0 \Rightarrow \lambda^{a_L} \approx 0\,,\\
&G_{a_0}Z^{a_0}_{a_1} = 0\,, \qquad Z^{a_{k-2}}_{a_{k-1}}Z^{a_{k-1}}_{a_k} = V^{a_{k-2}a_0}_{a_k}G_{a_0}\,, \qquad k = 2, \dots, L\,m\,.
\end{aligned} \tag{2}$$

The weak equality, $\approx$, means the equality on the constraint surface

$$\Sigma: \qquad G_{a_0} = 0\,.$$

Following the BRST method, the ghost pairs $(\mathcal{P}_{a_k}, c^{a_k}), k = 0, \dots, L$ are introduced

$$\epsilon\,(\mathcal{P}_{a_k}) = \epsilon\,(c^{a_k}) = \epsilon_{a_k} + k + 1\,, \qquad -\mathrm{gh}\,(\mathcal{P}_{a_k}) = \mathrm{gh}(c^{a_k}) = k + 1\,.$$

The BRST charge Ω is defined as a solution to the equations

$$\{\Omega, \Omega\} \;=\; 0\,, \tag{3}$$

$$\epsilon(\Omega) \;=\; 1\,, \qquad \mathrm{gh}(\Omega) = 1\,, \tag{4}$$

and satisfying the boundary conditions [5]

$$\left.\frac{\partial \Omega}{\partial c^{a_0}}\right|_{c=0} = G_{a_0}\,, \qquad \left.\frac{\partial^2 \Omega}{\partial \mathcal{P}_{a_{k-1}}\partial c^{a_k}}\right|_{\mathcal{P}=c=0} = Z^{a_{k-1}}_{a_k}\,.$$

One can write

$$\Omega = \Omega^{(1)} + M\,, \qquad M = \sum_{n\geq 2}\Omega^{(n)}\,, \qquad \Omega^{(n)} \sim c^n\,, \tag{5}$$

where

$$\Omega^{(1)} = G_{a_0}c^{a_0} + \sum_{k=1}^{L}(\mathcal{P}_{a_{k-1}}Z^{a_{k-1}}_{a_k} + N_{a_k})c^{a_k}\,, \tag{6}$$

$N_{a_1} = 0$ and N_{a_k}, $k > 1$, only involves $\mathcal{P}_{a_s}$, $s \leq k-2$. Equation (4) implies $N_{a_k}|_{\mathcal{P}=0} = 0$, $M|_{\mathcal{P}=0} = 0$.

Denote by $\mathcal{B}$ the algebra of polynomials in $(\mathcal{P}_{a_0}, c^{a_0}, \dots, \mathcal{P}_{a_L}, c^{a_L})$ with phase-space functions coefficients, $\mathcal{B} = \mathbb{C}[\mathcal{P}_{a_0}, \dots, \mathcal{P}_{a_L}] \otimes C^\infty(P) \otimes \mathbb{C}[c^{a_0}, \dots, c^{a_L}]$. Define the subspace

$$\mathcal{U} = \{X \in \mathcal{B} : X|_{\mathcal{P}=0,\,\Sigma} = 0\}\,.$$

The space $\mathcal{U}$ can be decomposed as $\mathcal{U} = \bigoplus_{n\geq 0}\mathcal{U}_n$, where $\mathcal{U}_n$ is the space of homogeneous polynomials in $(c^{a_0}, \dots, c^{a_L})$ of degree n.

For any $X, Y \in \mathcal{U}$, we have $XY \in \mathcal{U}$, $\{X, Y\} \in \mathcal{U}$ and, therefore, $\mathcal{U}$ is a Poisson subalgebra of $\mathcal{B}$. It is easily verified that $\Omega \in \mathcal{U}$.

The bracket $\{.,.\}$ splits as

$$\{X,Y\} = \{X,Y\}_\xi + \{X,Y\}_\diamond - (-1)^{\epsilon(X)\epsilon(Y)}\{Y,X\}_\diamond\,,$$

where $\{.,.\}_\xi$ refers to the Poisson bracket in the original phase space and

$$\{X,Y\}_\diamond = \sum_{k=0}^{L} \frac{\partial X}{\partial c^{a_k}} \frac{\partial Y}{\partial \mathcal{P}_{a_k}}\,.$$

Substituting (5) in (3), one obtains the equations

$$\begin{aligned} \delta\Omega^{(1)} &= 0\,, && (7)\\ \delta M + D &= 0\,, && (8)\end{aligned}$$

where

$$\begin{aligned} \delta &= \left\{\Omega^{(1)}, .\right\}_\diamond = G_{a_0}\frac{\partial}{\partial \mathcal{P}_{a_0}} + \sum_{k=1}^{L}\left(\mathcal{P}_{a_{k-1}} Z_{a_k}^{a_{k-1}} + N_{a_k}\right)\frac{\partial}{\partial \mathcal{P}_{a_k}}\,,\\ D &= \tfrac{1}{2}F + AM + \tfrac{1}{2}\{M,M\}\,, \qquad F = \left\{\Omega^{(1)}, \Omega^{(1)}\right\}_\xi\,, \qquad (9)\end{aligned}$$

and the operator A is given by

$$AX = \left\{\Omega^{(1)}, X\right\}_\xi - (-1)^{\epsilon(X)}\left\{X, \Omega^{(1)}\right\}_\diamond\,.$$

The left-hand side of (7) depends linearly on c^{a_k}, while (8) contains terms of the order of at least two in these variables. Equation (7) is equivalent to the nilpotency of δ

$$\delta^2 = 0\,. \qquad (10)$$

δ is called the Koszul–Tate differential.

The Poisson algebra of the first class functions is defined by

$$\mathcal{A} = \{X(\xi):\ \{X, G_\alpha\}|_\Sigma = 0\}\,. \qquad (11)$$

The functions that vanish on Σ form an ideal in $\mathcal{A}$. We denote this ideal by $\mathcal{J}$. Elements of $\mathcal{A}/\mathcal{J}$ are called observables.

A function $\Phi \in \mathcal{B}$ is called a BRST-invariant extension of $\Phi_0 \in \mathcal{A}$ if

$$\begin{gathered} \Phi = \Phi_0 + \Pi\,, \qquad \Pi = \textstyle\sum_{n\geq 1}\Phi_n\,, \qquad \Phi_n \in \mathcal{U}_n, \quad n \geq 1\,,\\ \mathrm{gh}(\Phi) = 0\,, \qquad \{\Omega, \Phi\} = 0\,. \qquad (12)\end{gathered}$$

The Poisson algebra $\mathcal{A}/\mathcal{J}$ is isomorphic to the set of equivalence classes of BRST-closed functions modulo BRST-exact functions with zero ghost number $(\mathrm{Ker}\,\Omega/\mathrm{Im}\Omega)^0$ [6]. Elements of $(\mathrm{Ker}\,\Omega/\mathrm{Im}\Omega)^0$ are called the BRST observables.

3. Reduction of δ

In this section we reduce δ to a standard form. For $k = L$, Eq. (2) reads

$$Z^{a_{L-2}}_{a'_{L-1}} Z^{a'_{L-1}}_{a_L} + Z^{a_{L-2}}_{a''_{L-1}} Z^{a''_{L-1}}_{a_L} \approx 0\,, \tag{13}$$

where $\{a'_{L-1}\}, \{a''_{L-1}\}$ are increasing index sets, such that $\{a'_{L-1}\} \cup \{a''_{L-1}\} = \{a_{L-1}\}$, $|\{a'_{L-1}\}| = |\{a_L\}|$ and $\operatorname{rank} Z^{a'_{L-1}}_{a_L} = |\{a_L\}|$. For an index set $i = \{i_1, i_2, \ldots, i_n\}$, we denote $|i| = n$. From (13), it follows that $\operatorname{rank} Z^{a_{L-2}}_{a_{L-1}} = |\{a_{L-1}\}| - |\{a_L\}| = |\{a''_{L-1}\}|$, and $\operatorname{rank} Z^{a_{L-2}}_{a''_{L-1}} = |\{a''_{L-1}\}|$.

One can split the index set $\{a_{L-2}\}$ as $\{a_{L-2}\} = \{a'_{L-2}\} \cup \{a''_{L-2}\}$, such that $|\{a'_{L-2}\}| = |\{a''_{L-1}\}|$, and $\operatorname{rank} Z^{a'_{L-2}}_{a''_{L-1}} = |\{a''_{L-1}\}|$. For $k = L-1$, Eq. (2) implies

$$Z^{a_{L-3}}_{a'_{L-2}} Z^{a'_{L-2}}_{a''_{L-1}} + Z^{a_{L-3}}_{a''_{L-2}} Z^{a''_{L-2}}_{a''_{L-1}} \approx 0\,.$$

From this it follows that

$$\operatorname{rank} Z^{a_{L-3}}_{a''_{L-2}} = \operatorname{rank} Z^{a_{L-3}}_{a_{L-2}} = |\{a_{L-2}\}| - \left|\{a''_{L-1}\}\right| = \left|\{a''_{L-2}\}\right|\,.$$

Using induction on k, we obtain a set of nonsingular matrices $Z^{a'_{k-1}}_{a''_k}$, $k = 1, \ldots, L$, and a set of matrices $Z^{a_{k-1}}_{a''_k}$, $k = 1, \ldots, L$, such that

$$\operatorname{rank} Z^{a_{k-1}}_{a''_k} = \operatorname{rank} Z^{a_{k-1}}_{a_k} = \left|\{a''_k\}\right|\,.$$

Here, $\{a'_k\} \cup \{a''_k\} = \{a_k\}$, $k = 1, \ldots, L-1$.

Equation (2) implies

$$G_{a'_0} Z^{a'_0}_{a''_1} + G_{a''_0} Z^{a''_0}_{a''_1} = 0\,. \tag{14}$$

From this it follows that $G_{a''_0}$ are independent. We assume that $G_{a''_0}$ satisfy the regularity conditions. It means that there are some functions $F_\alpha(\xi)$, $\{\alpha\} \cup \{a''_0\} = \{a\}$, such that $(F_\alpha, G_{a''_0})$ can be locally taken as new coordinates in the original phase space.

Let $f : \{a''_{k+1}\} \to \{a_k\}$, $k = 0, \ldots, L-1$ be an embedding, $f(j) = j$, and let $\{\alpha_k\}$ be defined by $\{a_k\} = \{f(a''_{k+1})\} \cup \{\alpha_k\}$. Since $|\{a''_k\}| = |\{\alpha_k\}|$, one can write $\alpha_k = g(a''_k)$ for some function g, and hence

$$\{a_k\} = \left\{f\left(a''_{k+1}\right)\right\} \cup \left\{g\left(a''_k\right)\right\}\,, \qquad k = 0, \ldots, L-1\,.$$

Lemma 3.1 *The nilpotent operator δ is reducible to the form of*

$$\delta = \xi'_{a''_0} \frac{\partial}{\partial \mathcal{P}'_{g(a''_0)}} + \sum_{k=1}^{L} \mathcal{P}'_{f(a''_k)} \frac{\partial}{\partial \mathcal{P}'_{g(a''_k)}} \tag{15}$$

by the change of variables: $(\xi_a, \mathcal{P}_{a_0}, \dots, \mathcal{P}_{a_L}) \to (\xi'_a, \mathcal{P}'_{a_0}, \dots, \mathcal{P}'_{a_L})$,

$$\begin{aligned} \xi'_\alpha &= F_\alpha \,, & \xi'_{a''_0} &= G_{a''_0} \,, \\ \mathcal{P}'_{f(a''_{k+1})} &= \delta \mathcal{P}_{a''_{k+1}} \,, & \mathcal{P}'_{g(a''_k)} &= \mathcal{P}_{a''_k} \,, \\ \mathcal{P}'_{a_L} &= \mathcal{P}_{a_L} \,, \end{aligned} \tag{16}$$

where $k = 0, \dots, L-1$, $g(a''_L) = a_L$.

Proof. To prove this statement, we first observe that Eqs. (16) are solvable with respect to $(\xi_a, \mathcal{P}_{a_0}, \dots, \mathcal{P}_{a_L})$. The original variables can be represented as

$$\xi_a = \xi_a \left(\xi' \right) , \qquad \mathcal{P}_{a_k} = \mathcal{P}_{a_k} \left(\xi'_a, \mathcal{P}'_{a_0}, \dots, \mathcal{P}'_{a_k} \right) , \qquad k = 0, \dots, L \,.$$

Here, we have used the fact that the $\mathcal{P}_{a_k}$ depends only on the functions $\mathcal{P}'_{a_s}$ with $s \leq k$. Assume that the functions $\xi_a(\xi')$ have been constructed. Then from (16), it follows that

$$\begin{aligned} \mathcal{P}_{a'_k} &= \left(\mathcal{P}'_{f(a''_{k+1})} - \mathcal{P}'_{g(a''_k)} Z'^{a''_k}_{a''_{k+1}} - N'_{a''_{k+1}} \right) \left(Z'^{(-1)} \right)^{a''_{k+1}}_{a'_k} \,, \\ \mathcal{P}_{a''_k} &= \mathcal{P}'_{g(a''_k)} \,, \qquad k = 0, \dots, L-1 \,, \\ \mathcal{P}_{a_L} &= \mathcal{P}'_{a_L} \,. \end{aligned} \tag{17}$$

Here and in what follows

$$X' \left(\xi', \mathcal{P}'_{a_0}, \dots, \mathcal{P}'_{a_L} \right) = X \left(\xi, \mathcal{P}_{a_0}, \dots, \mathcal{P}_{a_L} \right) \,.$$

Using (9) and (10), one gets

$$\begin{aligned} \delta \xi'_a &= \delta \mathcal{P}'_{f(a''_1)} = \ldots = \delta \mathcal{P}'_{f(a''_L)} = 0 \,, \\ \delta \mathcal{P}'_{g(a''_0)} &= \xi'_{a''_0} \,, \qquad \delta \mathcal{P}'_{g(a''_k)} = \mathcal{P}'_{f(a''_k)} \,, \qquad k = 1, \dots, L \,. \end{aligned} \tag{18}$$

Equations (18) are equivalent to (15). $\square$

With respect to the new coordinate system, the condition $X \in \mathcal{U}$ implies

$$X \big|_{\xi'_{a''_0} = \mathcal{P}' = 0} = 0 \,.$$

4. The BRST charge

Constructing of $\boldsymbol{\delta}$. The Koszul–Tate differential δ is determined by (7). Equation (7) is equivalent to the following system of recurrent equations:

$$\delta N_{a_k} = D_{a_k}\,, \qquad k = 2, \ldots, L \tag{19}$$

with $N_{a_1} = 0$, where

$$D_{a_k} = -\left(\mathcal{P}_{a_{k-2}} Z^{a_{k-2}}_{a_{k-1}} + N_{a_{k-1}}\right) Z^{a_{k-1}}_{a_k}\,.$$

Denote by $\mathcal{V}_k$ the subspace of $\mathcal{U}$ which consists of the functions depending only on $(\xi_a, \mathcal{P}_{a_0}, \ldots, \mathcal{P}_{a_k})$. The restriction of δ on $\mathcal{V}_k$ is given by

$$\delta_k = \xi'_{a''_0} \frac{\delta}{\delta \mathcal{P}'_{g(a''_0)}} + \sum_{s=1}^{k} \mathcal{P}'_{f(a''_s)} \frac{\delta}{\delta \mathcal{P}'_{g(a''_s)}}\,.$$

Define

$$\sigma_k = \mathcal{P}'_{g(a''_0)} \frac{\delta_l}{\delta \xi'_{a''_0}} + \sum_{s=1}^{k} \mathcal{P}'_{g(a''_s)} \frac{\delta}{\delta \mathcal{P}'_{f(a''_s)}}\,.$$

Straightforward calculations show that

$$\delta_k^2 = \sigma_k^2 = 0\,, \qquad \delta_k \sigma_k + \sigma_k \delta_k = n_k\,, \qquad n_k \delta_k = \delta_k n_k\,, \qquad n_k \sigma_k = \sigma_k n_k\,, \tag{20}$$

where n_k is the counting operator

$$n_k = \xi'_{a''_0} \frac{\delta_l}{\delta \xi'_{a''_0}} + \mathcal{P}'_{g(a''_0)} \frac{\delta}{\delta \mathcal{P}'_{g(a''_0)}} + \sum_{s=1}^{k} \left(\mathcal{P}'_{f(a''_s)} \frac{\delta}{\delta \mathcal{P}'_{f(a''_s)}} + \mathcal{P}'_{g(a''_s)} \frac{\delta}{\delta \mathcal{P}'_{g(a''_s)}} \right).$$

The space $\mathcal{V}_k$ splits as

$$\mathcal{V}_k = \mathcal{V}_k^{(0)} \oplus \widetilde{\mathcal{V}}_k\,, \qquad \widetilde{\mathcal{V}}_k = \mathcal{V}_k^{(1)} \oplus \mathcal{V}_k^{(2)} \oplus \ldots\,,$$

with $n_k X = nX$ for $X \in \mathcal{V}_k^{(n)}$. It is clear that

$$\begin{aligned} \mathcal{V}_k^{(0)} &= \left\{ \varPhi \in \mathcal{V}_k \,|\, \varPhi = \varPhi\left(\xi'_{a'_0}, \mathcal{P}_{f\left(a''_{k+1}\right)}\right) \right\}, \qquad k < L\,, \\ \mathcal{V}_L^{(0)} &= 0\,. \end{aligned} \tag{21}$$

The subspace $\widetilde{\mathcal{V}}_k$ is invariant under the action of δ_k, σ_k and n_k. The operator $n_k : \widetilde{\mathcal{V}}_k \to \widetilde{\mathcal{V}}_k$ is invertible. It follows from (20) that $\delta_k^+ : \widetilde{\mathcal{V}}_k \to \widetilde{\mathcal{V}}_k$, defined by $\delta_k^+ = \sigma_k n_k^{-1}$, is a generalized inverse of δ_k

$$\delta_k \delta_k^+ \delta_k = \delta_k\,, \qquad \delta_k^+ \delta_k \delta_k^+ = \delta_k^+\,, \tag{22}$$

and for any $X \in \widetilde{\mathcal{V}}_k$,

$$X = \delta_k^+ \delta_k X + \delta_k \delta_k^+ X \,. \tag{23}$$

Equation (19) can be written as

$$\delta_{k-2} N_{a_k} = D_{a_k} \tag{24}$$

since $N_{a_k} \in \mathcal{V}_{k-2}$. The operator δ_{k-2} and the right-hand side of (24) only involve the functions N_{a_s} with $s < k$.

To find a solution to (24), we assume that the functions $N_{a_s} \in \widetilde{\mathcal{V}}_{s-2}$, $s < k$ have been constructed. Changing variables in (24) $(\xi^a, \mathcal{P}_{a_0}, \ldots, \mathcal{P}_{a_{k-2}}) \to (\xi'_a, \mathcal{P}'_{a_0}, \ldots, \mathcal{P}'_{a_{k-2}})$, we get

$$\delta_{k-2} N'_{a_k} = D'_{a_k} \,, \tag{25}$$

where

$$D'_{a_k} = -\left(\mathcal{P}_{a_{k-2}} Z'^{a_{k-2}}_{a_{k-1}} + N'_{a_{k-1}}\right) Z'^{a_{k-1}}_{a_k} \,, \qquad \mathcal{P}_{a_{k-2}} = \mathcal{P}_{a_{k-2}}\left(\xi', \mathcal{P}'\right) \,.$$

Equation (2) reads

$$Z'^{a_{k-2}}_{a_{k-1}} Z'^{a_{k-1}}_{a_k} = V'^{a_{k-1} a'_0}_{a_k} G'_{a'_0} V'^{a_{k-1} a''_0}_{a_k} \xi'_{a''_0} \,.$$

It follows from (14) that

$$G'_{a'_0} = -\xi'_{a''_0} Z'^{a''_0}_{a''_1} (Z'^{(-1)})^{a''_1}_{a'_0} \,,$$

and hence

$$Z'^{a_{k-2}}_{a_{k-1}} Z'^{a_{k-1}}_{a_k} \in \widetilde{\mathcal{V}}_{k-2} \,.$$

Therefore, $D'_{a_k} \in \widetilde{\mathcal{V}}_{k-2}$. It is straightforward to check that $\delta_{k-2} D'_{a_k} = 0$, or equivalently, using (23), $\delta_{k-2}\delta^+_{k-2} D'_{a_k} = D'_{a_k}$. Then the general solution to (25) is given by

$$N'_{a_k} = Y'_{a_k} + \delta^+_{k-2} D'_{a_k} \,, \tag{26}$$

where Y_k is an arbitrary cocycle, $\delta_{k-2} Y_k = 0$, subject only to the restrictions

$$Y_{a_k} \in \widetilde{\mathcal{V}}_{k-2} \,, \qquad \epsilon(Y_{a_k}) = \epsilon(N_{a_k}) \,, \qquad \mathrm{gh}(Y_{a_k}) = \mathrm{gh}(N_{a_k}) \,.$$

By construction, $N'_{a_k} \in \widetilde{\mathcal{V}}_{k-2}$. In the original variables, (26) takes the form of

$$N_{a_k} = Y_k - \delta^+_{k-2}\left(\left(\mathcal{P}_{a_{k-2}} Z^{a_{k-2}}_{a_{k-1}} + N_{a_{k-1}}\right) Z^{a_{k-1}}_{a_k}\right) \,.$$

Higher orders. Our next task is to find a solution to (8). Since $\mathcal{U} = \mathcal{V}_L \otimes \mathbb{C}[c^{a_0}, \dots, c^{a_L}]$, it follows from (21) that the operator $n_L \colon \mathcal{U} \to \mathcal{U}$ is invertible. Denote $\delta^+ = \delta^+_L$. The space $\mathcal{U}$ can be decomposed as

$$\mathcal{U} = \operatorname{Ker} \delta \oplus \operatorname{Ker} \delta^+ \,, \tag{27}$$

where the corresponding orthogonal projectors are given by

$$P_{\operatorname{Ker}\delta} = \delta\delta^+ , \qquad P_{\operatorname{Ker}\delta^+} = I - \delta\delta^+ = \delta^+\delta$$

and I is the identity map. The last relation follows from (23).

In accordance with decomposition (27), Eq. (8) splits as

$$\delta M + \delta\delta^+ D \;=\; 0 \,, \tag{28}$$

$$\left(I - \delta\delta^+\right) R \;=\; 0 \,, \tag{29}$$

where R denotes the left-hand side of (8)

$$R = \delta M + D \,. \tag{30}$$

From (28), it follows that

$$M + \delta^+ D = W \,, \tag{31}$$

where W is an arbitrary cocycle, $\delta W = 0$, subject only to the restrictions

$$\epsilon(W) = 1 \,, \qquad \operatorname{gh}(W) = 1 \,, \qquad W \in \bigoplus_{n \geq 2} \mathcal{U}_n \,. \tag{32}$$

To prove that the solution to (31) satisfies (29), we use the approach of Ref. [11]. If (7) holds, then $R = \{\Omega, \Omega\}$. It is clear that $R \in \mathcal{U}$. From the Jacobi identity $\{\Omega, \{\Omega, \Omega\}\} = 0$, it follows that $\{\Omega, R\} = 0$, which is equivalent to

$$\delta R + AR + \{M, R\} = 0 \,. \tag{33}$$

Consider (33) and the condition

$$\delta^+ R = 0 \,. \tag{34}$$

Applying δ^+ to (33) and using (27), we get

$$R = -\delta^+ (AR + \{M, R\}) \,. \tag{35}$$

From (35) by iterations, it follows that $R = 0$.

It remains to check (34). The solution to (31) satisfies $\delta^+ M = \delta^+ W$, which implies

$$M = \delta^+ \delta M + W\,. \tag{36}$$

By definition (30), we have $\delta^+ R = \delta^+ \delta M + \delta^+ D$, and therefore by (31) and (36), $\delta^+ R = 0$.

One can rewrite (31) in the form of

$$M = M_0 + \tfrac{1}{2}\langle M, M\rangle\,, \tag{37}$$

where

$$\begin{aligned} M_0 &= \left(I + \delta^+ A\right)^{-1}\left(W - \tfrac{1}{2}\delta^+ F\right)\,,\\ \langle M, M\rangle &= -\left(I + \delta^+ A\right)^{-1}\delta^+\left(\{M, M\}\right)\,,\\ \left(I + \delta^+ A\right)^{(-1)} &= \sum_{m\geq 0}(-1)^m(\delta^+ A)^m\,. \end{aligned}$$

Equation (37) can be iteratively solved as

$$M = M_0 + \tfrac{1}{2}\langle M_0, M_0\rangle + \ldots \tag{38}$$

Series (38) can be obtained by using a diagram technique [12].

5. The BRST observables

To construct the BRST observables, it is necessary to obtain a solution to (12). This equation is rather difficult to analyze. However, in the variables $(\xi', \mathcal{P}', c)$, it takes a special form and can be solved recursively for Φ_n by repeating exactly the same steps as in the irreducible case. In this section, we use an alternative approach. Solving (12), we obtain a general compact expression for Φ which has only three terms (Eq. (49)).

Equation (12) can be written as

$$\delta\Pi + Q = 0\,, \tag{39}$$

where $Q = A\Pi + \{M, \Pi\} + \{\Omega, \Phi_0\}$. Denote by Γ the left-hand side of (39)

$$\Gamma = \{\Omega, \Phi\} = \delta\Pi + Q \in \mathcal{U}\,. \tag{40}$$

Using (27), we split (39) into the following two equations:

$$\begin{aligned} \delta\Pi + \delta\delta^+ Q &= 0\,, &(41)\\ \left(I - \delta\delta^+\right)\Gamma &= 0\,. &(42) \end{aligned}$$

Equation (41) is equivalent to

$$\varPi + \delta^+ Q = \varUpsilon \,, \tag{43}$$

where $\varUpsilon$ is a cocycle, $\delta\varUpsilon = 0$. Setting $\varUpsilon = 0$, we get from (43) the particular solution

$$\varPi_p = -\left(I + \delta^+(A + \mathrm{ad}\, M)\right)^{-1} \delta^+\{\varOmega, \varPhi_0\} \,, \tag{44}$$

where $\mathrm{ad}\, X$ denotes $\{X, \,.\,\}$.

Now, let us show that (44) satisfies (42). From the Jacoby identity $\{\varOmega, \{\varOmega, \varPhi\}\} = 0$, it follows that

$$\delta\varGamma + A\varGamma + \{M, \varGamma\} = 0 \,. \tag{45}$$

It is straightforward to check that

$$\delta^+\varGamma = 0 \,. \tag{46}$$

Indeed, (40) implies that

$$\delta^+\varGamma = \delta^+(\delta\varPi_p + Q) = \delta^+\delta\varPi_p - \varPi_p = 0 \,,$$

which gives (46) since $\varPi_p \in \mathrm{Ker}\, \delta^+ \cap \mathcal{U}$. Applying δ^+ to (45) and using (46), we get

$$\varGamma = -\delta^+(A\varGamma + \{M, \varGamma\}) \,, \tag{47}$$

from which it follows that $\varGamma = 0$. We conclude that (42) is satisfied by $\varPi_p$ (44).

Any solution to the homogeneous equation

$$\{\varOmega, \varPi\} = 0 \,,$$

is given by [13]

$$\varPi = \{\varOmega, \varUpsilon\} \,,$$

where

$$\varUpsilon \in \mathcal{U} \,, \qquad \epsilon(\varUpsilon) = 1 \,, \qquad \mathrm{gh}(\varUpsilon) = -1 \,.$$

Therefore, the BRST invariant extension of $\varPhi_0$ is given by

$$\varPhi = \varPhi_0 - (I + \delta^+(A + \mathrm{ad}\, M))^{-1}\delta^+\{\varOmega, \varPhi_0\} + \{\varOmega, \varUpsilon\} \,. \tag{48}$$

Since $\mathrm{ad}\, \varOmega = \delta + A + \mathrm{ad}\, M$, (48) can be rewritten as

$$\varPhi = \varPhi_0 - \left(I + \delta^+(\mathrm{ad}\, \varOmega - \delta)\right)^{-1} \delta^+\{\varOmega, \varPhi_0\} + \{\varOmega, \varUpsilon\} \,. \tag{49}$$

Using (49), we can effectively construct elements of $(\mathrm{Ker}\, \varOmega / \mathrm{Im}\, \varOmega)^0$ for the arbitrary gauge theory.

6. SU(2) gauge invariant reducible theory of the order of $L = 1$

To illustrate the method of computing the BRST charge, let us consider a simple reducible model. The model is described by three pairs of canonically conjugate variables (φ_a, π_a). It is subject to the first class constraints

$$G_a = \varepsilon_{abc}\varphi_b\pi_c \,. \tag{50}$$

The algebra of these functions is the $su(2)$ Lie algebra

$$\{G_a, G_b\} = \varepsilon_{abc}G_c \,.$$

Constraints (50) appear in the Yang–Mills quantum mechanics [14]. The reducibility condition reads

$$G_a\pi_a = 0 \,.$$

$\Omega^{(1)}$, F and δ are given by

$$\begin{aligned} \Omega^{(1)} &= G_a c^a + \mathcal{P}_a\pi_a c \,, \qquad F = G_a\varepsilon_{abc}c^b c^c - 2\mathcal{P}_a\varepsilon_{abc}\pi_b c^c c \,, \\ \delta &= G_a\frac{\partial}{\partial\mathcal{P}_a} + \mathcal{P}_a\pi_a\frac{\partial}{\partial\mathcal{P}} \,, \end{aligned}$$

where $(c^a, \mathcal{P}_a)$ and $(c, \mathcal{P})$ are auxiliary canonically conjugated variables

$$\begin{aligned} \epsilon(c^a) &= \epsilon(\mathcal{P}_a) = 1 \,, \qquad \epsilon(c) = \epsilon(\mathcal{P}) = 0 \,, \\ \mathrm{gh}(c^a) &= -\mathrm{gh}\,(\mathcal{P}_a) = 1 \,, \qquad \mathrm{gh}(c) = -\mathrm{gh}(\mathcal{P}) = 2 \,. \end{aligned}$$

The change of variables

$$\pi'_i = G_i \,, \qquad \pi'_3 = \pi_3 \,, \qquad \mathcal{P}'_i = \mathcal{P}_i \,, \qquad \mathcal{P}'_3 = \mathcal{P}_a\pi_a \,, \qquad \mathcal{P}' = \mathcal{P} \,, \tag{51}$$

where $i = 1, 2$, yields

$$\begin{aligned} \delta &= \pi'_i\frac{\partial}{\partial\mathcal{P}'_i} + \mathcal{P}'_3\frac{\partial}{\partial\mathcal{P}'} \,, \qquad \sigma = \mathcal{P}'_i\frac{\partial}{\partial\pi'_i} + \mathcal{P}'\frac{\partial}{\partial\mathcal{P}'_3} \,, \\ n &= \pi'_i\frac{\partial}{\partial\pi'_i} + \mathcal{P}'_i\frac{\partial}{\partial\mathcal{P}'_i} + \mathcal{P}'\frac{\partial}{\partial\mathcal{P}'} \,. \end{aligned}$$

In the domain with $\varphi_3 \neq 0$, $\pi_3 \neq 0$, transformation (51) is invertible

$$\pi_i = \frac{1}{\varphi_3}\left(\varepsilon_{ij}\pi'_j + \varphi_i\pi'_3\right) \,, \quad \pi_3 = \pi'_3 \,, \quad \mathcal{P}_i = \mathcal{P}'_i \,, \quad \mathcal{P}_3 = \frac{1}{\pi'_3}\left(\mathcal{P}'_3 - \mathcal{P}'_i\pi_i\right) \,.$$

Here, $\varepsilon_{ij} = \varepsilon_{ij3}$, $\pi_i = \pi_i(\pi')$.

One gets

$$\begin{aligned}\delta^+ F' &= \left(\mathcal{P}'_i\varepsilon_{ibc} - \frac{1}{\varphi_3}\mathcal{P}'_i\varphi_i\varepsilon_{3bc}\right) c^b c^c - \\ &\quad -\frac{2}{\varphi_3}\left(\frac{1}{\pi'_3}\left(\varepsilon_{ij}\mathcal{P}'_i\mathcal{P}'_j + \varphi_3\mathcal{P}'\right)\varepsilon_{ij}\pi_i(\pi') - \mathcal{P}_3(\pi',\mathcal{P}')\mathcal{P}'_j\right) c^j c\,.\end{aligned}$$

To obtain a regular expression for Ω, we take

$$W' = -\frac{1}{\pi'_3}\left(\mathcal{P}'_3 + \frac{1}{\varphi_3}\,\varepsilon_{ij}\pi'_i\mathcal{P}'_j\right)\varepsilon_{kl}c^k c^l\,.$$

Then, one finds

$$W' - \tfrac{1}{2}\delta^+ F' = -\tfrac{1}{2}\left(I + \delta^+ A\right)\mathcal{P}_a\varepsilon_{abc}c^b c^c\,. \tag{52}$$

Substitution (52) in (38) yields

$$M_0 = -\tfrac{1}{2}\mathcal{P}_a\varepsilon_{abc}c^b c^c\,.$$

Since $\{\mathcal{P}_a\varepsilon_{abc}c^b c^c, \mathcal{P}_d\varepsilon_{def}c^e c^f\} = 0$, it follows from (38) that $M = M_0$, and hence

$$\Omega = G_a c^a + \mathcal{P}_a\pi_a c - \tfrac{1}{2}\mathcal{P}_a\varepsilon_{abc}c^b c^c\,.$$

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