A NOVEL SORT OF COMPLEX SYNCHRONIZATIONS

Emad E. Mahmoud

Department of Mathematics, Faculty of Science, Sohag University
Sohag 82524, Egypt
and
Department of Mathematics, Faculty of Science, Taif University
Taif, Kingdom of Saudi Arabia
emad_eluan@yahoo.com

(Received May 29, 2017; accepted September 8, 2017)

Our primary goal of this work is to exhibit and examine a novel kind of complex synchronization. We may call it a complex phase synchronization (CPHS). There are bizarre properties of the CPHS and do not exist in the writing, for example, *(i)* this sort of synchronization can be investigated just for complex nonlinear systems; *(ii)* the CPHS contains or includes two sorts of synchronizations (anti-phase synchronization APS and phase synchronization PHS); *(iii)* the state variable of the main system synchronizes with a different state variable of the slave system. A description of the CPHS is presented for two identical chaotic or hyperchaotic complex nonlinear models. In view of the stability theorem, a scheme is intended to fulfill CPHS of chaotic or hyperchaotic attractors of these systems. The effectiveness of the acquired outcomes is shown by a reproduction illustration on the hyperchaotic complex Chen system. Numerical outcomes are plotted to show state variables, modulus errors, phase errors and the development of the attractors of these hyperchaotic models after synchronization to demonstrate that CPHS is achieved.

DOI:10.5506/APhysPolB.48.1441

1. Introduction

Newly, various types of synchronizations are discussed and studied, such as complete synchronization (CS) [1, 2], phase synchronization (PHS) [3, 4], anti-phase synchronization (APS) [3, 5], projective synchronization (PS) [6, 7], modified projective synchronization (MPS) [6, 8], lag synchronization (LS) [9, 10], anti-lag synchronization (ALS) [11], synchronization for fractional chaotic systems [12], modified projective phase synchronization [13] and so on. These types are applicable for real and complex nonlinear dynamical systems.

However, there are some new sorts of synchronization which cannot be considered for real or complex dynamical systems, for instance, module-stage synchronization [14], complex complete synchronization (CCS) [15], complex lag synchronization (CLS) [16], complex projective synchronization (CPS) [17], complex altered (or modified) projective synchronization (CMPS) [18] and complex anti-lag synchronization [19, 20]. These new sorts of synchronizations are explored for chaotic (or hyperchaotic) complex nonlinear systems. In complex circumstance, there are two imperative amounts — module and phase (or stage). Hence, the practices of the module and phase are discussed in [14–20]. The complex nonlinear dynamical models produce voluminous applications in engineering, detuned laser, and communications [21].

In this paper, we present and study the description of CPHS of two identical hyperchaotic complex nonlinear systems with specific parameters. This sort of synchronization is researched for chaotic and hyperchaotic complex nonlinear systems as. The idea of CPHS can be considered as syncretizing between PHS [4] and APS [5]. PHS happens between the real piece of main system and the imaginary piece of the slave system, whereas APS happens between the real piece of slave system and the imaginary piece of the main system. In CPHS, the state variable of the main system synchronizes with a different state variable of the slave system. Thus, the CPHS gives more unusual security in secure communications. We would like to propose a general scheme to consider and achieve the CPHS of two identical hyperchaotic complex nonlinear systems.

The arrangement of this paper is as per the following: The system description is illustrated in Section 2. In Section 3, we introduce the definition of CPHS. The proposed scheme to realize CPHS of two identical complex nonlinear models with hyperchaotic attractors is completed. Section 4 is devoted to performing CPHS as an example of two hyperchaotic Chen models with complex variables. Numerical recreation is utilized to show the legitimacy of this review. At last, the important conclusions of our investigation are condensed in Section 5.

2. The system description

A complex dynamical system is called hyperchaotic if the following conditions are achieved: *(i)* to a great degree is sensitive to initial conditions, *(ii)* has at minimum two positive Lyapunov exponents, *(iii)* the hyperchaotic system is at least 4 dimensional. Due to hyperchaotic complex systems with the advantage of high capability, high security, and high performance, it has a broadly applied potential in nonlinear circuits, secure communications, lasers, neural networks, biological systems and so on [22, 23]. Thus, presently research on complex nonlinear systems with hyperchaotic behavior is crucial.

Suppose the complex nonlinear model with chaotic or hyperchaotic attractors as regard:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x})\,, \tag{1}$$

where $\boldsymbol{x} = (x_1,\ x_2, \ldots,\ x_n)^{\mathrm{T}}$ is a state complex vector, $\boldsymbol{x} = \boldsymbol{x}^{\mathrm{r}} + j\boldsymbol{x}^{\mathrm{i}}$, $\boldsymbol{x}^{\mathrm{r}} = (u_1,\ u_3, \ldots,\ u_{2n-1})^{\mathrm{T}}$, $\boldsymbol{x}^{\mathrm{i}} = (u_2,\ u_4, \ldots,\ u_{2n})^{\mathrm{T}}$, $j = \sqrt{-1}$, T indicates transpose, $\boldsymbol{A} \in R^{n\times n}$ is real (or complex) matrix of system parameters, $\boldsymbol{f} = (f_1,\ f_2, \ldots,\ f_n)^{\mathrm{T}}$ is a vector of nonlinear complex functions. The superscripts r and i represent the real and unreal members of the state complex vector $\boldsymbol{x}$, respectively.

Remark 1. Most of the hyperchaotic complex systems can be qualified by (1) [1–6], such as the hyperchaotic complex Chen system. In order to authenticate the results of our scheme, we consider the hyperchaotic complex Chen system which has been introduced and studied in [24].

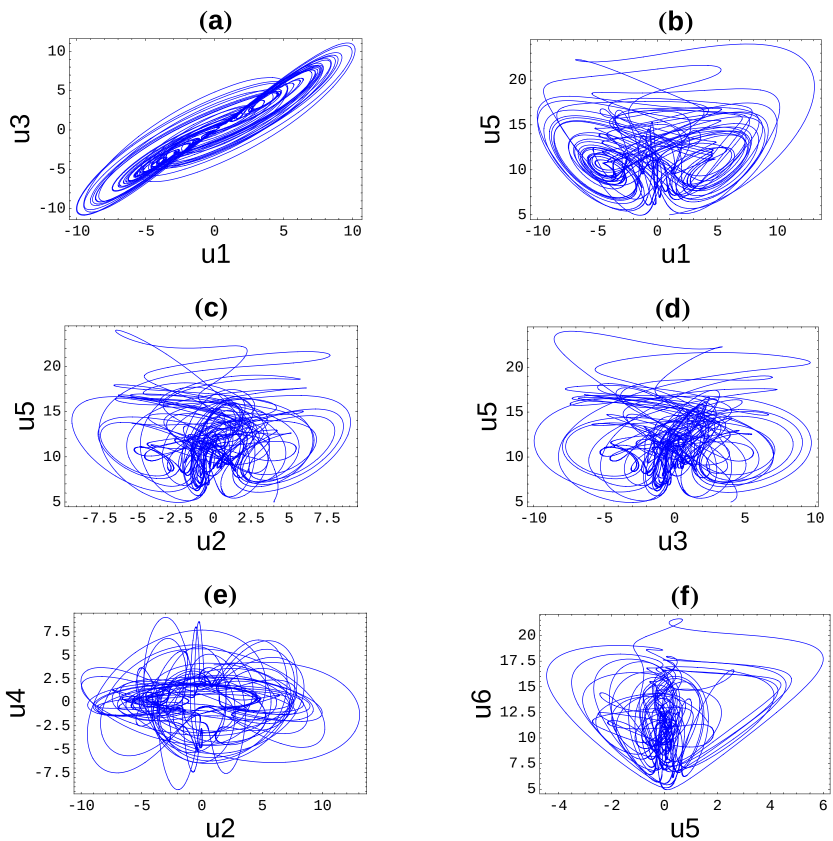

Fig. 1. Hyperchaotic attractors of system (2) in some planes when $\alpha = 22$, $\beta = 2$, $\gamma = 16$.

The hyperchaotic complex Chen system is

$$\begin{aligned}\dot{x} &= \alpha(y-x)\,,\\ \dot{y} &= (\gamma-\alpha)x - xz + \gamma y\,,\\ \dot{z} &= 1/2\,(\bar{x}y + x\bar{y}) + jx^{\mathrm{r}}y^{\mathrm{i}} - \beta z\,,\end{aligned} \tag{2}$$

where $\boldsymbol{x} = (x_1,\ x_2,\ x_3)^{\mathrm{T}} = (x, y, z)^{\mathrm{T}}$, α, β, γ are positive parameters, $x = u_1 + ju_2$, $y = u_3 + ju_4$, $z = u_5 + ju_6$ are complex functions. Dots explain derivatives with regard to time and an overbar means complex conjugate variables.

The hyperchaotic complex Chen is 6-dimensional continuous real autonomous system. For the case of $\alpha = 22$, $\beta = 2$, $\gamma = 16$, system (2) has hyperchaotic attractors [24] (see Fig. 1).

3. A scheme to achieve CPHS

We consider two identical complex nonlinear models with chaotic or hyperchaotic behavior of the form of (1). The first is the main system (we denote the main system with the subscript m) as

$$\dot{\boldsymbol{x}}_{\mathrm{m}} = \dot{\boldsymbol{x}}^{\mathrm{r}}_{\mathrm{m}} + j\dot{\boldsymbol{x}}^{\mathrm{i}}_{\mathrm{m}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{m}} + \boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}})\,, \tag{3}$$

and the second is the controlled slave system (with subscript s) as

$$\dot{\boldsymbol{x}}_{\mathrm{s}} = \dot{\boldsymbol{x}}^{\mathrm{r}}_{\mathrm{s}} + j\dot{\boldsymbol{x}}^{\mathrm{i}}_{\mathrm{s}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}} + \boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}) + \boldsymbol{L}\,, \tag{4}$$

with the complex controller $\boldsymbol{L} = (L_1,\ L_2, \ldots,\ L_n)^{\mathrm{T}} = \boldsymbol{L}^{\mathrm{r}} + j\boldsymbol{L}^{\mathrm{i}}$, $\boldsymbol{L}^{\mathrm{r}} = (v_1,\ v_3, \ldots,\ v_{2n-1})^{\mathrm{T}}$, $\boldsymbol{L}^{\mathrm{i}} = (v_2,\ v_4, \ldots,\ v_{2n})^{\mathrm{T}}$.

Definition 3.1. Two complex dynamical models with chaotic or hyperchaotic behavior in a main–slave configuration can display CPHS if there is a vector of the complex error function δ known as

$$\delta = \delta^{\mathrm{r}} + j\delta^{\mathrm{i}} = \boldsymbol{x}_{\mathrm{s}} - j\boldsymbol{x}_{\mathrm{m}} = j\boldsymbol{p} \quad \text{as} \quad t \to \infty\,, \tag{5}$$

where $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^{\mathrm{T}}$, $\boldsymbol{x}_{\mathrm{m}}$ and $\boldsymbol{x}_{\mathrm{s}}$ are the state complex vectors of the main and slave systems $\lim\limits_{t\to\infty} \delta^{\mathrm{r}} = \lim\limits_{t\to\infty} \left\|\boldsymbol{x}^{\mathrm{r}}_{\mathrm{s}} + \boldsymbol{x}^{\mathrm{i}}_{\mathrm{m}}\right\| = 0$, $\lim\limits_{t\to\infty} \delta^{\mathrm{i}} = \lim\limits_{t\to\infty} \left\|\boldsymbol{x}^{\mathrm{i}}_{\mathrm{s}} - \boldsymbol{x}^{\mathrm{r}}_{\mathrm{m}}\right\| = \boldsymbol{P}$, $\delta^{\mathrm{r}} = (\delta_{u_1}, \delta_{u_3}, \ldots, \delta_{u_{2n-1}})^{\mathrm{T}}$, $\delta^{\mathrm{i}} = (\delta_{u_2}, \delta_{u_4}, \ldots, \delta_{u_{2n}})^{\mathrm{T}}$ and $\boldsymbol{P}$ is vector of real constants.

Remark 2. When $\boldsymbol{P} = \boldsymbol{0}$ in Eq. (5), we define complex complete synchronization (CCS) between systems (3) and (4) [15].

Remark 3. The sum of the imaginary part of main system $\boldsymbol{x}_{\mathrm{m}}^{\mathrm{i}}$ and real part of slave system $\boldsymbol{x}_{\mathrm{s}}^{\mathrm{r}}$ is vanishing when $t \longrightarrow \infty$. This indicates the definition of APS [3, 5].

Remark 4. The error between the imaginary part of slave system $\boldsymbol{x}_{\mathrm{s}}^{\mathrm{i}}$ and the real part of main system $\boldsymbol{x}_{\mathrm{m}}^{\mathrm{r}}$ equals real constants as $t \to \infty$. This appears in the definition of PHS [3, 4].

Remark 5. CPHS incorporate between APS and PHS as shown from **Remarks 3** and **4**.

Remark 6. The distinction between CPHS and PHS can be clarified from:

Complex Phase Synchronization (CPHS)

In CPHS, we define the error in simple case:

$$x_s - jx_m = jp, \text{ as } t \longrightarrow \infty,$$

where $x = u_1 + ju_2$, p is real constant

$$\begin{gathered}
[u_{1s} + ju_{2s}] - j[u_{1m} + ju_{2m}] = jp\,, \\
[u_{1s} + ju_{2s}] - [-u_{2m} + ju_{1m}] = jp\,, \\
[u_{1s} + u_{2m}] + j[u_{2s} - u_{1m}] = jp\,, \\
\left.\begin{array}{l} \Longrightarrow u_{1s} + u_{2m} = 0 \Longrightarrow \textbf{(APS)} \\ \Longrightarrow u_{2s} - u_{1m} = p \Longrightarrow \textbf{(PHS)} \end{array}\right\} \Longrightarrow \textbf{(CPHS)}
\end{gathered}$$

Phase Synchronization (PHS)

In PHS, we define the error in simple case:

$$x_s - x_m = p_1 + jp_2, \text{ as } t \longrightarrow \infty,$$

where $x = u_1 + ju_2$, p_1, p_2 are real constants

$$\begin{gathered}
[u_{1s} + ju_{2s}] - [u_{1m} + ju_{2m}] = p_1 + jp_2\,, \\
[u_{1s} + ju_{2s}] - [u_{1m} + ju_{2m}] = p_1 + jp_2\,, \\
[u_{1s} - u_{1m}] + j[u_{2s} - u_{2m}] = p_1 + jp_2\,, \\
\left.\begin{array}{l} \Longrightarrow u_{1s} - u_{1m} = p_1 \Longrightarrow \textbf{(PHS)} \\ \Longrightarrow u_{2s} - u_{2m} = p_2 \Longrightarrow \textbf{(PHS)} \end{array}\right\} \Longrightarrow \textbf{(PHS)}
\end{gathered}$$

Theorem 3.2. *If the nonlinear controller is designed as:*

$$\begin{aligned}
\boldsymbol{L} &= \boldsymbol{L}^{\mathrm{r}} + j\boldsymbol{L}^{\mathrm{i}} = -\boldsymbol{A}\boldsymbol{x}_{\mathrm{s}} - \boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}) + j[\boldsymbol{A}\boldsymbol{x}_{\mathrm{m}} + \boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}})] + \boldsymbol{\Omega}\delta^{\mathrm{r}} + j\boldsymbol{\Psi}\delta^{\mathrm{i}} \\
&= -\boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}^{\mathrm{r}} - \boldsymbol{f}^{\mathrm{r}}(\boldsymbol{x}_{\mathrm{s}}) - \boldsymbol{A}\boldsymbol{x}_{\mathrm{m}}^{\mathrm{i}} - \boldsymbol{f}^{\mathrm{i}}(\boldsymbol{x}_{\mathrm{m}}) + \boldsymbol{\Omega}\delta^{\mathrm{r}} \\
&\quad + j\left[-\boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}^{\mathrm{i}} - \boldsymbol{f}^{\mathrm{i}}(\boldsymbol{x}_{\mathrm{s}}) + [\boldsymbol{A}\boldsymbol{x}_{\mathrm{m}}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{r}}(\boldsymbol{x}_{\mathrm{m}})] + \boldsymbol{\Psi}\delta^{\mathrm{i}}\right]\,, \qquad (6)
\end{aligned}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{\Psi}$ are diagonal matrix of real diagonal elements ζ_l and ψ_l, $l = 1, 2, \ldots, n$, respectively. Then the slave system (4) and the main system (3) can be achieved CPHS when $\zeta_l < 0$, $\psi_l = 0$.

Proof. From the definition of CPHS

$$\delta = \delta^{\mathrm{r}} + j\delta^{\mathrm{i}} = \boldsymbol{x}_{\mathrm{s}} - j\boldsymbol{x}_{\mathrm{m}} \,. \tag{7}$$

So,

$$\begin{aligned} \dot{\delta} &= \dot{\delta}^{\mathrm{r}} + j\dot{\delta}^{\mathrm{i}} = \dot{\boldsymbol{x}}_{\mathrm{s}} - j\dot{\boldsymbol{x}}_{\mathrm{m}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}} + \boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}) - j\left[\boldsymbol{A}\boldsymbol{x}_{\mathrm{m}} + \boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}})\right] + \boldsymbol{L} \\ &= \dot{\boldsymbol{x}}_{\mathrm{s}}^{\mathrm{r}} + j\dot{\boldsymbol{x}}_{\mathrm{s}}^{\mathrm{i}} - j\left[\dot{\boldsymbol{x}}_{\mathrm{m}}^{\mathrm{r}} + j\dot{\boldsymbol{x}}_{\mathrm{m}}^{\mathrm{i}}\right] = \dot{\boldsymbol{x}}_{\mathrm{s}}^{\mathrm{r}} + \dot{\boldsymbol{x}}_{\mathrm{m}}^{\mathrm{i}} + j\left[\dot{\boldsymbol{x}}_{\mathrm{s}}^{\mathrm{i}} - \dot{\boldsymbol{x}}_{\mathrm{m}}^{\mathrm{r}}\right] \,. \end{aligned} \tag{8}$$

By using the hyperchaotic complex systems (3) and (4), we get the error complex dynamical system as

$$\begin{aligned} \dot{\delta} &= \dot{\delta}^{\mathrm{r}} + j\dot{\delta}^{\mathrm{i}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{r}}(\boldsymbol{x}_{\mathrm{s}}) + \boldsymbol{A}\boldsymbol{x}_{\mathrm{m}}^{\mathrm{i}} + \boldsymbol{f}^{\mathrm{i}}(\boldsymbol{x}_{\mathrm{m}}) + \boldsymbol{L}^{\mathrm{r}} \\ &\quad + j\left[\boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}^{\mathrm{i}} + \boldsymbol{f}^{\mathrm{i}}(\boldsymbol{x}_{\mathrm{s}}) - \left[\boldsymbol{A}\boldsymbol{x}_{\mathrm{m}}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{r}}(\boldsymbol{x}_{\mathrm{m}})\right] + \boldsymbol{L}^{\mathrm{i}}\right] \,. \end{aligned} \tag{9}$$

Via separating the real and the unreal components in Eq. (9), the error complex system is written as

$$\begin{cases} \dot{\delta}^{\mathrm{r}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{r}}(\boldsymbol{x}_{\mathrm{s}}) + \boldsymbol{A}\boldsymbol{x}_{\mathrm{m}}^{\mathrm{i}} + \boldsymbol{f}^{\mathrm{i}}(\boldsymbol{x}_{\mathrm{m}}) + \boldsymbol{L}^{\mathrm{r}} \,, \\ \dot{\delta}^{\mathrm{i}} = \boldsymbol{A}\boldsymbol{x}_{\mathrm{s}}^{\mathrm{i}} + \boldsymbol{f}^{\mathrm{i}}(\boldsymbol{x}_{\mathrm{s}}) - \left[\boldsymbol{A}\boldsymbol{x}_{\mathrm{m}}^{\mathrm{r}} + \boldsymbol{f}^{\mathrm{r}}(\boldsymbol{x}_{\mathrm{m}})\right] + \boldsymbol{L}^{\mathrm{i}} \,. \end{cases} \tag{10}$$

By substituting $\boldsymbol{L}^{\mathrm{r}}$, $\boldsymbol{L}^{\mathrm{i}}$ from Eq. (6) into Eq. (10), we obtain

$$\begin{cases} \dot{\delta}^{\mathrm{r}} = \boldsymbol{\Omega}\delta^{\mathrm{r}} \,, \\ \dot{\delta}^{\mathrm{i}} = \boldsymbol{\Psi}\delta^{\mathrm{i}} \,, \end{cases} \quad \rightarrow \dot{\delta} = \boldsymbol{\Omega}\delta^{\mathrm{r}} + j\boldsymbol{\Psi}\delta^{\mathrm{i}} \,. \tag{11}$$

If we select $\zeta_l < 0$ ($e.g.$ $\zeta_l = -1$), $\psi_l = 0$, $l = 1, 2, \ldots, n$, then $\boldsymbol{\Omega} = \mathrm{diag}(-1, -1, \ldots, -1)$, $\boldsymbol{\Psi} = \mathrm{diag}(0, 0, \ldots, 0)$ and Eq. (11) takes the form of

$$\begin{cases} \dot{\delta}^{\mathrm{r}} = -\delta^{\mathrm{r}} \,, \\ \dot{\delta}^{\mathrm{i}} = \mathbf{0} \,, \end{cases} \quad \rightarrow \dot{\delta} = -\delta^{\mathrm{r}} \,. \tag{12}$$

Using the Lyapunov stability, we get $\delta^{\mathrm{r}} \longrightarrow \mathbf{0}$ and $\delta^{\mathrm{i}} \longrightarrow$ constant vector as $t \longrightarrow \infty$, and the CPHS is achieved for all state variables. This completes the confirmation. □

Remark 7. If we choose $\zeta_l < 0$ and $\psi_l < 0$ ($e.g.$ $\zeta_l = -1$), then Eq. (11) becomes

$$\begin{cases} \dot{\delta}^{\mathrm{r}} = -\delta^{\mathrm{r}} \,, \\ \dot{\delta}^{\mathrm{i}} = -\delta^{\mathrm{i}} \,, \end{cases} \quad \rightarrow \dot{\delta} = -\delta \tag{13}$$

and the CCS is achieved for all state variables [15].

Remark 8. In numerical simulation, one can calculate the phases and modulus errors of the main and slave systems. For any complex number $\xi = \xi^{\mathrm{r}} + j\xi^{\mathrm{i}}$, the phase θ_ξ and modulus P_ξ are calculated as follows [14–20]:

$$\theta_\xi = \begin{cases} \arctan\left(\xi^{\mathrm{i}}/\xi^{\mathrm{r}}\right), & \xi^{\mathrm{r}} > 0, \quad \xi^{\mathrm{i}} \geq 0, \\ 2\pi + \arctan\left(\xi^{\mathrm{i}}/\xi^{\mathrm{r}}\right), & \xi^{\mathrm{r}} > 0, \quad \xi^{\mathrm{i}} < 0, \\ \pi + \arctan\left(\xi^{\mathrm{i}}/\xi^{\mathrm{r}}\right), & \xi^{\mathrm{r}} < 0, \\ \pi - \arctan\left(\xi^{\mathrm{i}}/\xi^{\mathrm{r}}\right), & \xi^{\mathrm{r}} = 0. \end{cases} \tag{14}$$

and

$$P_\xi = \sqrt{(\xi^{\mathrm{r}})^2 + (\xi^{\mathrm{i}})^2}\,. \tag{15}$$

The CPHS occurs if the phases errors are bounded and can move chaotically or hyperchaotically within a small range [3, 13]. Whereas our systems achieve CPHS in cases where these modulus errors are chaotic or hyperchaotic and uncorrelated (linearly independent). In the all types of complex synchronizations, the phases and the modulus errors convert to zero or to particular values. But in the CPHS, these errors have a new property as explained.

Lastly, our scheme is demonstrated by employing it for two hyperchaotic complex Chen models in Section 4.

4. Example

In line to display the performance of the suggested scheme, a representative example is specified. One of the typical continuous hyperchaotic systems of the form of (1), hyperchaotic complex Chen system (2), is taken as an example with numerical simulations carried out.

Suppose the main system as

$$\begin{aligned} \dot{x}_{\mathrm{m}} &= \alpha\left(y_{\mathrm{m}} - x_{\mathrm{m}}\right), \\ \dot{y}_{\mathrm{m}} &= \left(\gamma - \alpha\right) x_{\mathrm{m}} + \gamma y_{\mathrm{m}} - x_{\mathrm{m}} z_{\mathrm{m}}, \\ \dot{z}_{\mathrm{m}} &= 1/2\left(\bar{x}_{\mathrm{m}} y_{\mathrm{m}} + x_{\mathrm{m}} \bar{y}_{\mathrm{m}}\right) + j x^{\mathrm{r}}_{\mathrm{m}} y^{\mathrm{i}}_{\mathrm{m}} - \beta z_{\mathrm{m}}, \end{aligned} \tag{16}$$

which can be rewritten as $\boldsymbol{A}\boldsymbol{x}_{\mathrm{m}} + \boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}})$, where

$$\boldsymbol{A} = \begin{pmatrix} -\alpha & \alpha & 0 \\ (\gamma - \alpha) & \gamma & 0 \\ 0 & 0 & -\beta \end{pmatrix},$$

$$\boldsymbol{f}(\boldsymbol{x}_{\mathrm{m}}) = \begin{pmatrix} 0 \\ -x_{\mathrm{m}} z_{\mathrm{m}} \\ 1/2\left(\bar{x}_{\mathrm{m}} y_{\mathrm{m}} + x_{\mathrm{m}} \bar{y}_{\mathrm{m}}\right) + j x^{\mathrm{r}}_{\mathrm{m}} y^{\mathrm{i}}_{\mathrm{m}} \end{pmatrix}. \tag{17}$$

Let, in the simulation, the parameters of (16) be chosen as $\alpha = 22$, $\beta = 2$, $\gamma = 16$ and the initial values of the main system state vector be taken as $(x_{\mathrm{m}}(0), y_{\mathrm{m}}(0), z_{\mathrm{m}}(0))^{\mathrm{T}} = (1 + 2j, 3 + 4j, 5 + j6)^{\mathrm{T}}$.

Consider the slave system as

$$\begin{aligned}
\dot{x}_{\mathrm{s}} &= \alpha(y_{\mathrm{s}} - x_{\mathrm{s}}) + L_1\,,\\
\dot{y}_{\mathrm{s}} &= (\gamma - \alpha)\, x_{\mathrm{s}} + \gamma y_{\mathrm{s}} - x_{\mathrm{s}} z_{\mathrm{s}} + L_2\,,\\
\dot{z}_{\mathrm{s}} &= 1/2\,(\bar{x}_{\mathrm{s}} y_{\mathrm{s}} + x_{\mathrm{s}} \bar{y}_{\mathrm{s}}) + j x_{\mathrm{s}}^{\mathrm{r}} y_{\mathrm{s}}^{\mathrm{i}} - \beta z_{\mathrm{s}} + L_3\,,
\end{aligned} \tag{18}$$

where

$$\boldsymbol{f}(\boldsymbol{x}_{\mathrm{s}}) = \begin{pmatrix} 0 \\ -x_{\mathrm{s}} z_{\mathrm{s}} \\ 1/2\,(\bar{x}_{\mathrm{s}} y_{\mathrm{s}} + x_{\mathrm{s}} \bar{y}_{\mathrm{s}}) + j x_{\mathrm{s}}^{\mathrm{r}} y_{\mathrm{s}}^{\mathrm{i}} \end{pmatrix}. \tag{19}$$

The initial values of the slave system state vector are selected as $(x_{\mathrm{s}}(0), y_{\mathrm{s}}(0), z_{\mathrm{s}}(0))^{\mathrm{T}} = (-3 + 18j, -4 + 24j, -5 + j19)^{\mathrm{T}}$.

According to Theorem 3.2, the complex control functions L_1, L_2 and L_3 are computed as follows:

$$\begin{aligned}
\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= \begin{pmatrix} -\alpha(y_{\mathrm{s}} - x_{\mathrm{s}}) + j\alpha(y_{\mathrm{m}} - x_{\mathrm{m}}) \\ -(\gamma - \alpha)x_{\mathrm{s}} - \gamma y_{\mathrm{s}} + x_{\mathrm{s}} z_{\mathrm{s}} + j((\gamma - \alpha)x_{\mathrm{m}} + \gamma y_{\mathrm{m}} - x_{\mathrm{m}} z_{\mathrm{m}}) \\ \beta z_{\mathrm{s}} - \varGamma_1 - x_{\mathrm{m}}^{\mathrm{r}} y_{\mathrm{m}}^{\mathrm{i}} + j\left(\varGamma_2 - \beta z_{\mathrm{m}} - x_{\mathrm{s}}^{\mathrm{r}} y_{\mathrm{s}}^{\mathrm{i}}\right) \end{pmatrix} \\
&\quad + \boldsymbol{\varOmega} \delta^{\mathrm{r}} + j \boldsymbol{\varPsi} \delta^{\mathrm{i}}\,,
\end{aligned} \tag{20}$$

where $\varGamma_1 = 1/2(\bar{x}_{\mathrm{s}} y_{\mathrm{s}} + x_{\mathrm{s}} \bar{y}_{\mathrm{s}})$, $\varGamma_2 = 1/2(\bar{x}_{\mathrm{m}} y_{\mathrm{m}} + x_{\mathrm{m}} \bar{y}_{\mathrm{m}})$, $\boldsymbol{\varOmega} = \mathrm{diag}(-1, -1, -1)$, $\boldsymbol{\varPsi} = \mathrm{diag}(0, 0, 0)$, $\delta^{\mathrm{r}} = (\delta_{u_1}, \delta_{u_3}, \delta_{u_5})^{\mathrm{T}}$, $\delta^{\mathrm{i}} = (\delta_{u_2}, \delta_{u_4}, \delta_{u_6})^{\mathrm{T}}$.

Therefore, the complex controllers in (20) can be transcribed as

$$\begin{aligned}
\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} &= \begin{pmatrix} -\alpha(u_{3s} + u_{4m}) + \alpha(u_{1_{\mathrm{s}}} + u_{2_{\mathrm{m}}}) + \zeta_1 \delta_{u_1} \\ (\alpha - \gamma)(u_{1s} + u_{2m}) - \gamma(u_{3_{\mathrm{s}}} + u_{4m}) + \varGamma_3 + \zeta_2 \delta_{u_3} \\ -u_{1s} u_{3s} - u_{2s} u_{4s} - u_{1m} u_{4m} + \beta(u_{5_{\mathrm{s}}} + u_{6m}) + \zeta_3 \delta_{u_5} \end{pmatrix} \\
&\quad + j \begin{pmatrix} -\alpha(u_{4s} - u_{3_{\mathrm{m}}}) - \alpha(u_{1_{\mathrm{m}}} - u_{2_{\mathrm{s}}}) \\ (\alpha - \gamma)(u_{1_{\mathrm{m}}} - u_{2s}) + \gamma(u_{3_{\mathrm{m}}} - u_{4_{\mathrm{s}}}) + \varGamma_4 \\ (u_{1m} u_{3m} + u_{2m} u_{4m}) - u_{1s} u_{4s} - \beta(u_{5m} - u_{6s}) \end{pmatrix},
\end{aligned} \tag{21}$$

where $\varGamma_3 = -u_{2s} u_{6s} + u_{1m} u_{6m} + u_{2m} u_{5m} + u_{1s} u_{5s}$, $\varGamma_4 = u_{2s} u_{5s} + u_{2m} u_{6m} - u_{1m} u_{5m} + u_{1s} u_{6s}$.

In the numerical simulations, systems (16) and (18) with controllers (21) are solved numerically. In Fig. 2, the CPHS is depicted. It is clear the CPHS is incorporated or combined between the PHS and the APS. The PHS appears in Figs. 2 (a), (c), (e), while the APS occurs in Figs. 2 (b), (d), (f).

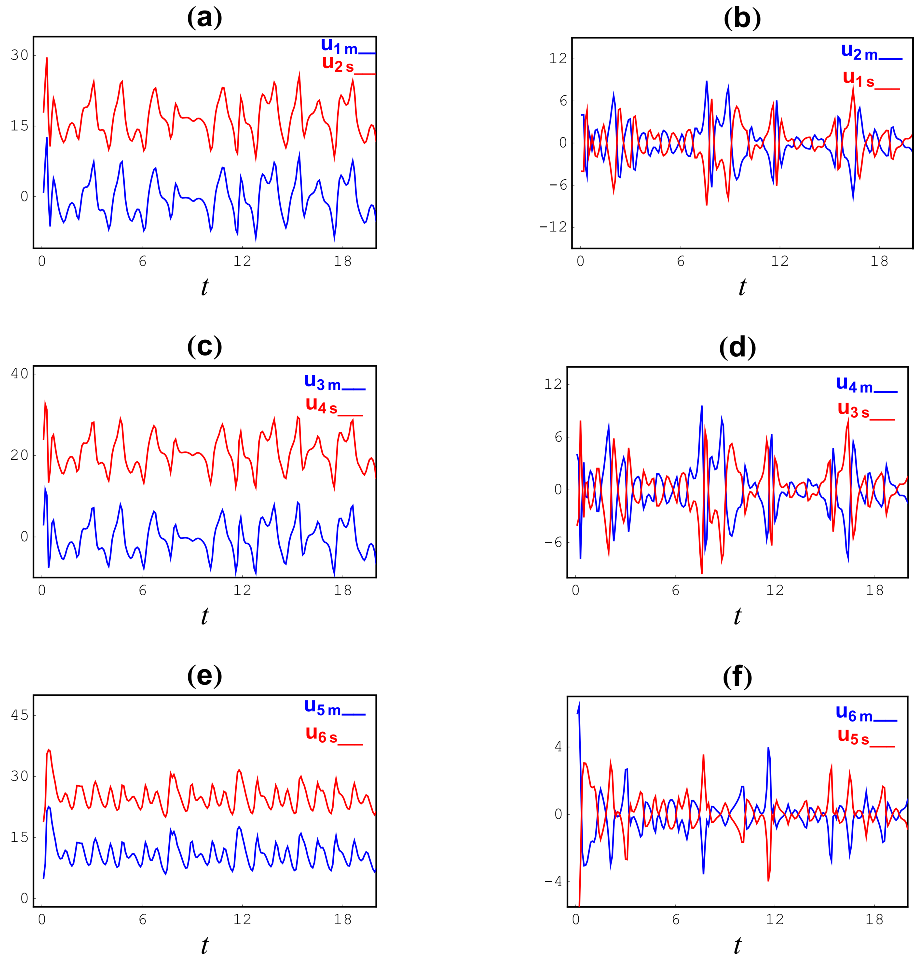

Fig. 2. CPHS between two systems (16) and (18). (a) u_{1m} and u_{2s} *versus* t; (b) u_{2m} and u_{1s} *versus* t; (c) u_{3m} and u_{4s} *versus* t; (d) u_{4m} and u_{3s} *versus* t; (e) u_{5m} and u_{6s} *versus* t; (f) u_{6m} and u_{5s} *versus* t.

It is clear from Fig. 2 that the state variable of the main system synchronizes with a different state variable of the slave system. In this way, the CPHS gives more prominent security in secure communications. The CPHS errors are plotted in Fig. 3. These errors go to zero in Figs. 3 (a), (c), (e) which means the APS is achieved as required of the analytical plan. However, these errors approach constant values in Figs. 3 (b), (d), (f) as expected from the analytical consideration of our scheme and the PHS appears, since $\dot{\delta}^{\mathrm{i}} = \mathbf{0}$ and $\delta^{\mathrm{i}} = (\delta_{u_2}, \delta_{u_4}, \delta_{u_6})^{\mathrm{T}} = \boldsymbol{x}_{\mathrm{s}}^{\mathrm{i}}(\mathbf{0}) - \boldsymbol{x}_{\mathrm{m}}^{\mathrm{r}}(\mathbf{0}) = (17, 21, 14)$.

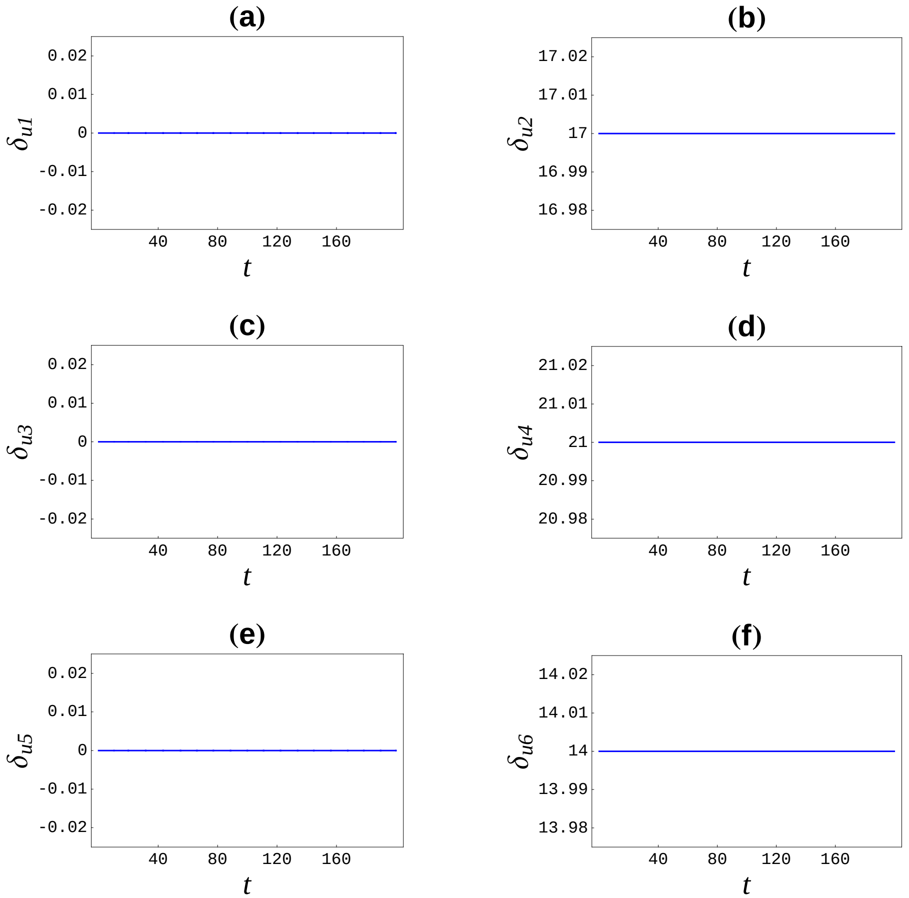

Fig. 3. CPHS errors between two systems (16) and (18). (a) δ_{u_1} *versus* t; (b) δ_{u_2} *versus* t; (c) δ_{u_3} *versus* t; (d) δ_{u_4} *versus* t; (e) δ_{u_5} *versus* t; (f) δ_{u_6} *versus* t.

The phases errors $\theta_{x_{\rm m}} - \theta_{x_{\rm s}}$, $\theta_{y_{\rm m}} - \theta_{y_{\rm s}}$, $\theta_{z_{\rm m}} - \theta_{z_{\rm s}}$ are computed by using **Remark 8** and plotted in Figs. 4 (a), (b), (c). It is clear that the phases errors $\theta_{x_{\rm m}} - \theta_{x_{\rm s}}$, $\theta_{y_{\rm m}} - \theta_{y_{\rm s}}$, $\theta_{z_{\rm m}} - \theta_{z_{\rm s}}$ are bounded and oscillate about the origin in the chaotic or hyperchaotic manner which means that CPHS is achieved. Figures 4 (d), (e), (f) show the relations between $\theta_{x_{\rm m}} - \theta_{x_{\rm s}}$, $\theta_{y_{\rm m}} - \theta_{y_{\rm s}}$, $\theta_{z_{\rm m}} - \theta_{z_{\rm s}}$. We refined these relations are chaotic or hyperchaotic attractors and this detection implies that $\theta_{x_{\rm m}} - \theta_{x_{\rm s}}$, $\theta_{y_{\rm m}} - \theta_{y_{\rm s}}$, $\theta_{z_{\rm m}} - \theta_{z_{\rm s}}$ move chaotically or hyperchaotically and CPHS is efficacious.

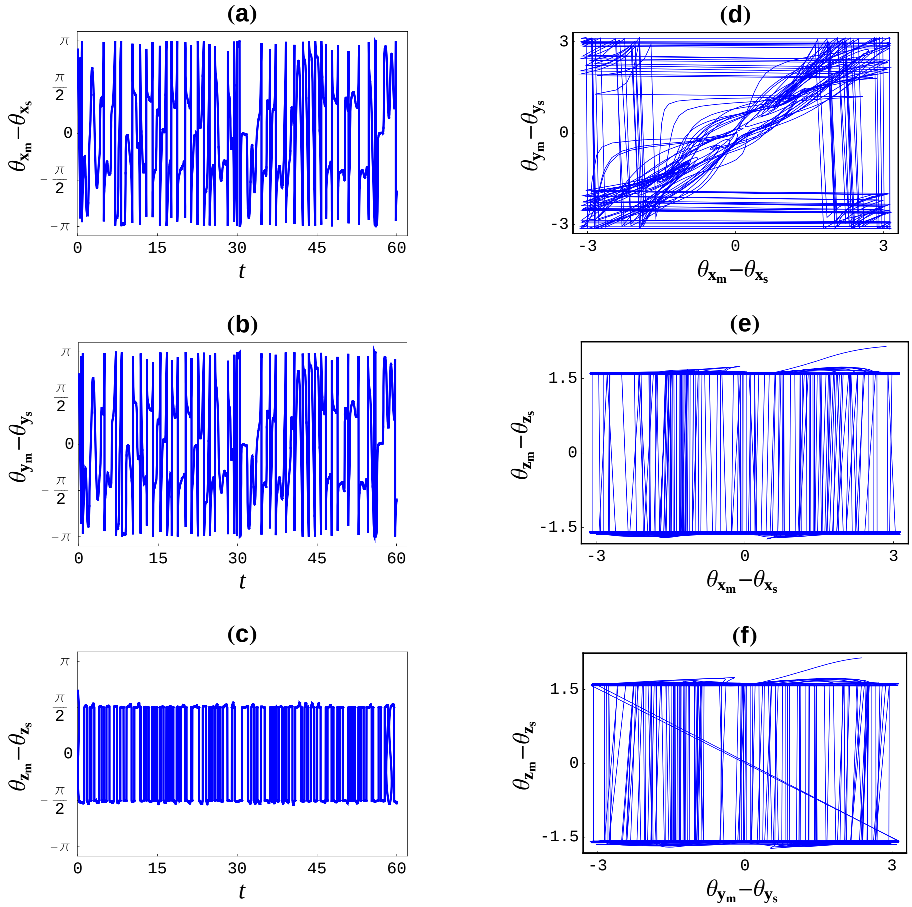

Fig. 4. The phases errors of systems (16) and (18) and the relations between these errors.

The modulus errors $p_{x_{\rm m}} - p_{x_{\rm s}}$, $p_{y_{\rm m}} - p_{y_{\rm s}}$, $p_{z_{\rm m}} - p_{z_{\rm s}}$ are shown *versus* t in Figs. 5 (a), (b), (c), while $p_{x_{\rm m}} - p_{x_{\rm s}}$ *versus* $p_{y_{\rm m}} - p_{y_{\rm s}}$, $p_{x_{\rm m}} - p_{x_{\rm s}}$ *versus* $p_{z_{\rm m}} - p_{z_{\rm s}}$, $p_{y_{\rm m}} - p_{y_{\rm s}}$ *versus* $p_{z_{\rm m}} - p_{z_{\rm s}}$ are illustrated in Figs. 5 (d), (e), (f). Obviously, from Fig. 5, the modulus errors are chaotic or hyperchaotic and uncorrelated (linearly independent) as expected from our scheme.

In [15, 16, 19, 20], the phases errors and the modulus errors convert to $\frac{\pm\pi}{2}$ and zero, respectively. This illustrates the distinction among the CPHS, and the additional sorts of the complex synchronizations.

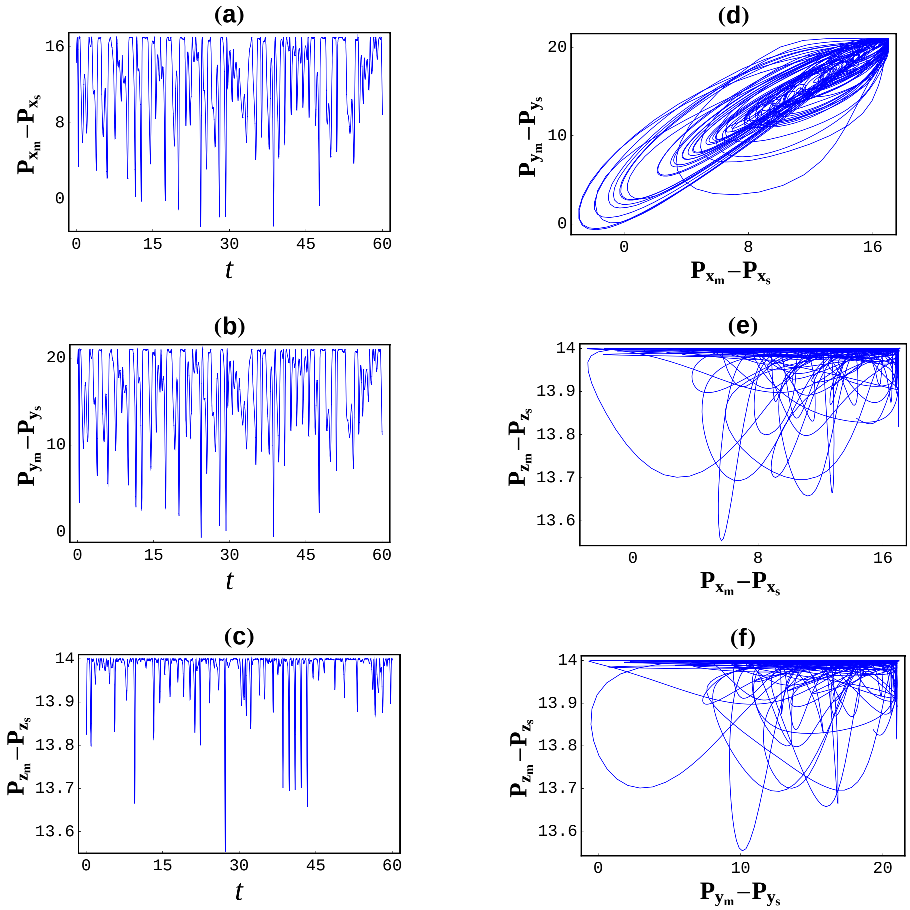

Fig. 5. The modulus errors of systems (16) and (18) and the relations between these errors.

5. Conclusions

In this work, we present another sort of complex synchronization which is called complex phase synchronization CPHS. We dissect and concentrate the CPHS concerning two identical chaotic or hyperchaotic complex nonlinear systems. The CPHS can be concentrated just in complex nonlinear systems. The CPHS can be considered as syncretizing amongst APS and PHS (see Fig. 2). In CPHS, the state variable of the main system synchronizes with a different state variable of the slave system (see Fig. 2). In this way, CPHS gives more prominent security in secure communications.

A novel scheme is proposed to realize CPHS of two identical complex nonlinear systems with chaotic or hyperchaotic behavior based on the stability theorem. In this scheme, we decided analytically the complex control functions which produced CPHS. We implemented this scheme, as an example, to study CPHS of two identical hyperchaotic complexes Chen systems. An excellent agreement is found as shown in Figs. 2, 3, 4, 5. In these figures, we compute the phases errors and modulus errors because, in the complex nonlinear dynamical system, the observable or measurable physical quantities usually are module and phase. The speed and accuracy of the CPHS are illustrated by means of computer simulation.

I can hardly express my gratitude to the referees for very useful suggestions, which helped me to improve considerably the presentation of my results. I am heartily thankful to my supervisor, Prof. G.M. Mahmoud, whose encouragement and guidance enabled me to achieve this manuscript.

REFERENCES

[1] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **62**, 875 (2010).
[2] E.E. Mahmoud, M.A. AL-Adwani, *Results Phys.* **7**, 1346 (2017).
[3] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **61**, 141 (2010).
[4] J. Meng, X.-y. Wang, *Phys. Lett. A* **369**, 294 (2007).
[5] W. Liu, J. Xiao, X. Qian, J. Yang, *Phys. Rev. E* **73**, 057203 (2006).
[6] G.M. Mahmoud, E.E. Mahmoud, *Math. Comput. Simul.* **80**, 2286 (2010).
[7] G.M. Mahmoud, E.E. Mahmoud, A.A. Arafa, *Math. Meth. Appl. Sci.* **40**, 1214 (2017).
[8] G.H. Li, *Chaos Solitons Fract.* **32**, 1786 (2007).
[9] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **67**, 1613 (2012).
[10] D. Tong *et al.*, *Int. J. Control Autom. Syst.* **14**, 706 (2016).
[11] E.E Mahmoud, *J. Franklin Inst.* **349**, 1247 (2012).
[12] Q. Wang, D.L. Qi, *Int. J. Control Autom. Syst.* **14**, 211 (2016).
[13] E.E. Mahmoud, *Math. Comput. Simul.* **89**, 69 (2013).
[14] F. Nian, X. Wang, Y. Niu, D. Lin, *Appl. Math. Comput.* **217**, 2481 (2010).
[15] E.E. Mahmoud, *Math. Methods Appl. Sci.* **37**, 321 (2014).
[16] E.E. Mahmoud, K.M. Abualnaja, *Centr. Eur. J. Phys.* **12**, 63 (2014).
[17] Z. Wu, J. Duan, X. Fu, *Nonlinear Dyn.* **69**, 771 (2012).
[18] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **73**, 2231 (2013).
[19] E.E. Mahmoud, F.S. Abood, *Complexity* **2017**, 3848953 (2017).

[20] E.E. Mahmoud, F.S. Abood, *Results Phys.*, 2017, in press, DOI:10.1016/j.rinp.2017.07.050.

[21] G.M. Mahmoud, T. Bountis, E.E. Mahmoud, *Int. J. Bifurcat. Chaos* **17**, 4295 (2007).

[22] E.E. Mahmoud, *Math. Comput. Model.* **55**, 1951 (2012).

[23] S. Kim, *Int. J. Control Autom. Syst.* **15**, 1 (2017).

[24] E.E. Mahmoud, *Appl. Math. Model.* **38**, 4445 (2014).