PRODUCTION OF DIRAC PARTICLES IN EXTERNAL ELECTROMAGNETIC FIELDS*

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Pair creation of spin-$\frac{1}{2}$ particles in Minkowski spacetime is investigated by obtaining exact solutions of the Dirac equation in the presence of electromagnetic fields and using them for determining the Bogoliubov coefficients. The resulting particle creation number density depends on the strength of the electric and magnetic fields.

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1. Introduction

After the pioneering works of Sauter [1], Heisenberg and Euler [2] on the particle creation by the strong electromagnetic fields, Schwinger formulated the following pair creation probability per unit volume and time by obtaining the one-loop effective action in a constant and homogeneous classical electric field (in natural units, $\hbar = c = 1$) [3]:

$$\omega = \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{+\infty} \frac{1}{n^2} \exp\left(-\frac{n\pi m^2}{eE}\right) , \tag{1}$$

where m and e are the mass and charge of the electron, E is the electric field, respectively. Since then, this process is called the Schwinger mechanism and has become an important problem in the quantum field theory (QFT). This kind of a classical electric field is assumed to be of the order of $E \sim 10^{16}$ V/cm [4] which is very difficult to generate using the current technology. Strong fields arising from the collisions between the relativistic high-energy particles and heavy ions are called color electric fields and have ability to create particles from the vacuum. These type of collisions are generated at

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the modern colliders, *i.e.* at CERN. The Schwinger mechanism is attributed to the hadronic particle creation and, on the base of Color Glass Condensate (CGS), this phase is called Glasma.

The Schwinger mechanism have been studied in the presence of various stationary and non-stationary external fields [5–9]. The studies about the Schwinger mechanism in gauge fields having both electric and magnetic field components have revealed that electric field has a dominant influence in creating the particles. Therefore, the pair creation mechanism is totally attributed to the pure electric field [10]. This quantum effect of the classical electromagnetic fields is carried out to the curved spacetime as well [11–13].

A considerable number of authors raised in their studies a point that the magnetic field reduces particle creation process. One of the aims of this study is to investigate this phenomenon for a particular choice of the electromagnetic gauge field that has both electric and magnetic field components.

The calculation of fermionic particle creation rate requires to define the positive and negative frequency energy states, namely the "in" and "out" mode vacuum solutions. For the motion of the relativistic charged particles moving in an external field, analysis of mode functions as positive and negative frequency solutions is not easy, since the Lagrangian of the corresponding system depends completely on space-time coordinates. Namely, particle concept becomes indefinite owing to interaction with the external fields. For this reason, we require a condition to define the "particle" concept. In the present study, we will apply a quasiclassical method. We obtain exact solutions of the Hamilton–Jacobi (HJ) equation and discuss their asymptotic behavior in the infinite past and future. Then, asymptotic behavior of the solutions of the Dirac equation in the neighborhood of the time singularities will be identified. With the help of this analysis and comparison of asymptotic solutions of both HJ and Dirac equations in the infinite past and future, the particle picture will be identified.

We define positive and negative frequency mode functions in such a way that the positive frequency mode function approaches $e^{iS(t)}$ and the negative frequency one $e^{-iS(t)}$ in asymptotic regions [4], where $S(t)$ is the solution of the HJ equation for the presence of a 4-vector electromagnetic potential given as

$$A_\nu = B_0\tau[1+\tanh(x/\tau)]\delta_\nu^2 - E_0(\mathit{\Gamma}+\mathit{\Lambda} t)\delta_\nu^3\,, \tag{2}$$

and τ, $\mathit{\Gamma}$ and $\mathit{\Lambda}$ are constants. This new suggested form of the vector potential generates parallel stationary electric [5] and Sauter-type magnetic fields [14] that are persuaded in the Glasma flux tube model of high-energy heavy-ion collisions.

The magnetic current emerging is found to be

$$j_\nu = \frac{1}{4\pi}\left[\frac{2\left|\vec{B}\right|}{\tau}\tanh(x/\tau)\right]\delta_\nu^2 \,. \tag{3}$$

The outline of the paper is as follows: In Section 2, we solve the relativistic HJ equation and obtain the asymptotic behavior of the solutions. In Section 3, we solve the Dirac equation for the considered electromagnetic fields and obtain the asymptotic limits of the solutions to define the vacuum "in" and "out" modes by referring the asymptotic solutions of the HJ equation. We use the Bogoliubov transformation technique to relate the solutions at the boundaries and calculate the particle creation number density for fermions in Section 4. Finally, in Section 5, we discuss the results we obtained. Throughout the paper, the natural units, $\hbar = c = 1$, are used.

2. Solutions of the Hamilton–Jacobi equation

The relativistic HJ equation for the action S is given by [11]

$$\zeta^{\epsilon\theta}\left[\frac{\partial S}{\partial x^\epsilon} - eA_\epsilon\right]\left[\frac{\partial S}{\partial x^\theta} - eA_\theta\right] + m^2 = 0\,, \tag{4}$$

where $\zeta^{\epsilon\theta} = (1,-1,-1,-1)$ is the Minkowski metric, m is the mass of the particle and A_ν is the 4-vector electromagnetic potential.

The electromagnetic potential satisfies the Lorentz gauge, and the Lorentz invariants are determined from the electromagnetic field tensor as follows:

$$F^{\pi\varrho}F_{\pi\varrho} = 2\left(B^2 - E^2\right) = 2B_0^2\,\mathrm{sech}^4(x/\tau) - E_0^2\Lambda^2 \tag{5}$$

and

$$F^{\pi\varrho}F^*_{\pi\varrho} = 4\vec{E}\cdot\vec{B} = 4\Lambda E_0 B_0\,\mathrm{sech}^2(x/\tau)\,. \tag{6}$$

Because of the space-time dependence of the considered electromagnetic field, the solution of the HJ equation can be separated as follows:

$$S(t,\vec{x}\,) = P(x) + Q(t) + (yk_y + zk_z)\;, \tag{7}$$

where k_y and k_z can be viewed as the conserved momenta that exist given the symmetries chosen for the electromagnetic gauge (2). By using (7) in Eq. (4), we obtain

$$\dot{Q}^2 - \acute{P}^2 - [k_z + eE_0(\Gamma + \Lambda t)]^2 - [k_y - eB_0\tau(1+\tanh(x/\tau))]^2 + m^2 = 0\,, \tag{8}$$

where dot and acute denote derivatives with respect to t and x, respectively.

We obtain two first order differential equations as follows:

$$\dot{Q}^2 - [eE_0(\Gamma + \Lambda t)]^2 - 2k_z eE_0(\Gamma + \Lambda t) + m^2 - k_z^2 = \upsilon^2 \tag{9}$$

and

$$\acute{P}^2 + \{eB_0\tau[1 + \tanh(x/\tau)]\}^2 - 2k_y eB_0\tau[1 + \tanh(x/\tau)] = \upsilon^2\,, \tag{10}$$

where υ^2 is the constant of separation.

Time-dependent external fields cause unstable vacuum and this results in the pair creation by vacuum. For this reason, the dynamics involving spatial coordinates affects the solutions only by a constant and we obtain the solution of the HJ equation for electromagnetic gauge (2) as follows:

$$\begin{aligned} S(\varrho, \vec{x}\,) &= S_0(0, \vec{x}\,) + \frac{eE_0}{\Lambda}\int_0^{\varrho}\sqrt{\varrho^2 + \frac{2k_z}{eE_0}\varrho + \left(\frac{k_z^2 + \upsilon^2 - m^2}{e^2E_0^2}\right)}\,\mathrm{d}\varrho \\ &= \left(\frac{\varrho eE_0 + k_z}{2\Lambda}\right)\sqrt{\varrho^2 + \frac{2k_z}{eE_0}\varrho + \left(\frac{k_z^2 + \upsilon^2 - m^2}{e^2E_0^2}\right)} + \left(\frac{\upsilon^2 - m^2}{2eE_0\Lambda}\right) \\ &\quad \times \ln\left\{2\varrho + \frac{2k_z}{eE_0} + 2\sqrt{\varrho^2 + \frac{2k_z}{eE_0}\varrho + \left(\frac{k_z^2 + \upsilon^2 - m^2}{e^2E_0^2}\right)}\right\} + S_0(0, \vec{x}\,)\,, \end{aligned} \tag{11}$$

where $\varrho = (\Gamma + \Lambda t)$.

The dependence of the solution on time is derived by $\psi \to e^{iS(t)}$ and we achieve the following expressions for the asymptotic behavior of the relativistic wave function:

$$\psi_{(t\to\mp\infty)} = e^{iS(t)} \to e^{\pm i\left(\frac{eE_0\Lambda}{2}\right)t^2 \pm i\left(\frac{\upsilon^2 - m^2}{2eE_0\Lambda}\right)\ln(2\Lambda|t|)}\,, \tag{12}$$

where the upper and lower signs represent the negative and positive-frequency states, respectively.

3. Solutions of the Dirac equation

The Dirac equation in external electromagnetic fields is given by [15]

$$[i\gamma^\nu\partial_\nu + eA_\nu\gamma^\nu - m]\,\psi = 0\,, \tag{13}$$

where γ^ν are Dirac matrices, A_ν is the 4-vector electromagnetic potential, m is the mass of electron, e is the charge of the electron and ψ is the four-component spinor.

The Dirac equation yields four coupled differential equations for the spinor and usually it is difficult to obtain the exact analytical solutions, in particular, for mathematically complicated external fields. This difficulty of the problem has been accomplished by Feynmann and Gell-Mann by considering a two-component form of the Dirac equation in the presence of electromagnetic fields as follows [16]:

$$\left[\left(\vec{P}-e\vec{A}\right)^2+m^2-e\vec{\sigma}\cdot\left(\vec{B}+i\vec{E}\right)\right]\phi=(p_0-eA_0)^2\phi\,, \tag{14}$$

where $\vec{\sigma}$ are usual Pauli matrices and $\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ are the solutions of the two-component equation. The four-component spinor can be derived from ϕ as follows:

$$\psi=\begin{pmatrix}\left[\vec{\sigma}\cdot\left(\vec{P}-e\vec{A}\right)+(p_0-eA_0)+m\right]\phi\\ \left[\vec{\sigma}\cdot\left(\vec{P}-e\vec{A}\right)+(p_0-eA_0)-m\right]\phi\end{pmatrix}. \tag{15}$$

Thence, for the purpose of obtaining the analytic solutions, we follow up the two-component formalism and consider the electromagnetic gauge (2). Because the given gauge field depends on x coordinate and t, both k_y and k_z are constants of the motion and solutions can be written in the form of

$$\phi=e^{i(yk_y+zk_z)}\begin{pmatrix}\chi_1(x)T_1(t)\\ \chi_2(x)T_2(t)\end{pmatrix}. \tag{16}$$

Therefore, with the usage of Eqs. (2) and (16), Eq. (14) becomes

$$\begin{aligned}&\Big\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-eB_0(eB_0\tau^2+s)\,\mathrm{sech}^2(x/\tau)+2eB_0(eB_0-\tau k_y)\tanh(x/\tau)\\&+2eB_0\tau(eB_0\tau-k_y)+m^2+k_y^2+k_z^2+\frac{\mathrm{d}^2}{\mathrm{d}t^2}+(eE_0\Lambda t)^2\\&+2eE_0\Lambda(eE_0\Gamma+k_z)t+eE_0\Gamma(eE_0\Gamma+2k_z)-iseE_0\Lambda\Big\}\chi_s(x)T_s(t)=0\,,\end{aligned} \tag{17}$$

where the spin index s has the ±1 eigenvalues corresponding to the spinors ϕ_1 and ϕ_2, respectively. This equation can be written in a simpler form as

$$\left[\widehat{F}(x)+\widehat{Q}(t)\right]\chi_s(x)T_s(t)=0 \tag{18}$$

with the following definitions:

$$\begin{aligned}\widehat{F}(x) &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - eB_0(eB_0\tau^2+s)\,\mathrm{sech}^2(x/\tau) + 2eB_0(eB_0-\tau k_y)\tanh(x/\tau)\\ &\quad +2eB_0\tau(eB_0\tau-k_y)+m^2+k_y^2+k_z^2\,, \qquad (19)\\ \widehat{Q}(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}+(eE_0\Lambda t)^2+2eE_0\Lambda(eE_0\Gamma+k_z)t\\ &\quad \times eE_0\Gamma(eE_0\Gamma+2k_z)-iseE_0\Lambda\,. \qquad (20)\end{aligned}$$

Equation (18) has a separable form, so we get the following two equations:

$$\left[\widehat{F}(x)-\varpi^2\right]\chi_s(x) = 0\,, \qquad (21)$$

$$\left[\widehat{Q}(t)+\varpi^2\right]T_s(t) = 0\,, \qquad (22)$$

where ϖ^2 is the constant of separation.

By defining $x=\tau r$, Eq. (21) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\Sigma\,\mathrm{sech}^2 r-\Upsilon\tanh r-\varepsilon\right]\chi_s(x)=0\,, \qquad (23)$$

where the definitions

$$\begin{aligned}\Sigma &= eB_0\left(eB_0\tau^2+s\right)\,, \qquad \Upsilon=2eB_0\tau^2\left(eB_0-\tau k_y\right)\,,\\ \varepsilon &= 2eB_0\tau^3(eB_0\tau-k_y)+\tau^2\left(m^2+k_y^2+k_z^2-\varpi^2\right) \qquad (24)\end{aligned}$$

were made.

Following Rosen and Morse [17], we set $\chi_s(r)=e^{ra}\cosh^{-b} r f_s(r)$ and obtain the following equation:

$$\begin{aligned}&\Big\{f_s''+2(a-b\tanh r)f_s'+\left[(\Sigma-b(b+1))\,\mathrm{sech}^2 r\right.\\ &\left.-(2ab+\Upsilon)\tanh r+\left(a^2+b^2-\varepsilon\right)\right]f_s\Big\}=0\,. \qquad (25)\end{aligned}$$

In order χ/f to be finite in the range of $-\infty\le r\le+\infty$, $(a^2+b^2-\varepsilon)=0$ and $(2ab+\Upsilon)=0$ conditions are necessary [17]. From these conditions, we derive the following expressions for a and b:

$$a=-\tfrac{1}{2}\left[(\varepsilon+\Upsilon)^{\frac{1}{2}}-(\varepsilon-\Upsilon)^{\frac{1}{2}}\right] \qquad (26)$$

and

$$b=\tfrac{1}{2}\left[(\varepsilon+\Upsilon)^{\frac{1}{2}}+(\varepsilon-\Upsilon)^{\frac{1}{2}}\right]\,. \qquad (27)$$

Therefore, keeping these expressions and by introducing $\eta = \frac{1}{2}(1 + \tanh r)$, we arrive at

$$\left\{\eta(1-\eta)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + [a+b+1-2(b+1)\eta]\frac{\mathrm{d}}{\mathrm{d}\eta} + [\Sigma - b(b+1)]\right\} f = 0 \quad (28)$$

which is the differential equation satisfied by the hypergeometric functions. The hypergeometric function remaining finite at $\eta = 0$ will provide this equation, and a solution will be given as [18]

$$f_s(\eta) = {}_2F_1\left[(b+1/2) - (\Sigma+1/4)^{\frac{1}{2}}; (b+1/2) + (\Sigma+1/4)^{\frac{1}{2}}; a+b+1; \eta\right]. \quad (29)$$

So, we obtain

$$\begin{aligned}\chi_s \;=\; & e^{ra}\cosh^{-b} r\,{}_2F_1\left[(b+1/2) - (\Sigma+1/4)^{\frac{1}{2}};\right.\\ & \left.(b+1/2) + (\Sigma+1/4)^{\frac{1}{2}}; a+b+1; \eta\right]. \end{aligned} \quad (30)$$

For this solution to be convergent at infinity, the following condition must be satisfied [17]:

$$\left[(b+1/2) - (\Sigma+1/4)^{\frac{1}{2}}\right] = -n\,. \quad (31)$$

Then

$$a = -\Upsilon\left[(4\Sigma+1)^{\frac{1}{2}} - (2n+1)\right]^{-1} \quad (32)$$

and

$$b = \frac{1}{2}\left[(4\Sigma+1)^{\frac{1}{2}} - (2n+1)\right]. \quad (33)$$

The constant of separation ϖ can be easily derived from $(a^2 + b^2 - \varepsilon) = 0$.

By introducing a variable $\xi = \sqrt{\frac{2}{eE_0\Lambda}}(eE_0\Lambda t + eE_0\Gamma + k_z)$, we obtain the following equation from Eq. (22):

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{1}{4}\xi^2 - \frac{iesE_0\Lambda + k_z^2 - \varpi^2}{2eE_0\Lambda}\right\} T_s(\xi) = 0\,. \quad (34)$$

Solutions of this differential equation are parabolic cylinder functions [18]

$$T_s(\xi) = \frac{e^{-\frac{\pi\tilde{a}}{4}}}{(2eE_0\Lambda)^{\frac{1}{4}}}\left[D_{-i\tilde{a}-1/2}\left(e^{i\pi/4}\xi\right) + D^*_{-i\tilde{a}-1/2}\left(e^{i\pi/4}\xi\right)\right], \quad (35)$$

where $\tilde{a} = (\frac{iesE_0\Lambda + k_z^2 - \varpi^2}{2eE_0\Lambda})$.

Therefore, exact solutions are obtained and all components of the Dirac spinor can be found with the insertion of Eqs. (30) and (35) into Eq. (16).

4. Particle creation via Bogoliubov transformation method

Due to difficulty of the direct observation of the pair creation in a constant field [10], because the typical $|eE|$ is smaller than m^2, the particle creation will be induced by the time-dependent components of the wavefunction (28), namely the parabolic cylinder functions.

Two solutions of Eq. (34) are given as

$$T_{s_1}(\xi) = \frac{e^{-\frac{\pi\tilde{a}}{4}}}{(2eE_0\Lambda)^{\frac{1}{4}}} D_{-i\tilde{a}-1/2}\left(e^{i\pi/4}\xi\right) \tag{36}$$

and

$$T_{s_2}(\xi) = \frac{e^{-\frac{\pi\tilde{a}}{4}}}{(2eE_0\Lambda)^{\frac{1}{4}}} D^*_{-i\tilde{a}-1/2}\left(e^{i\pi/4}\xi\right) . \tag{37}$$

These are not the only solutions and any of the remaining two-sets can be constructed via Bogoliubov coefficients as follows:

$$\tilde{T}_{s_1}(\xi) = \alpha T_{s_1}(\xi) - \beta^* T_{s_2}(\xi) \tag{38}$$

and

$$\tilde{T}_{s_2}(\xi) = \alpha^* T_{s_2}(\xi) + \beta T_{s_1}(\xi) . \tag{39}$$

The Bogoliubov transformation method is a technique that associates a canonical commutation relation algebra or a canonical anti-commutation relation algebra into another representation, caused by an isomorphism [19].

In the Minkowskian QFT, eigenfunctions of the field equation, ψ, can be written with the help of the mode solutions as [19, 20]

$$\psi = \sum_n \left(a_n\varphi_n + a_n^\dagger\varphi_n^*\right) = \sum_k \left(b_k\Theta_k + b_k^\dagger\Theta_k^*\right) , \tag{40}$$

where we have the relations $(\varphi_i, \varphi_j) = \delta_{ij}$, $(\varphi_i^*, \varphi_j^*) = \delta_{ij}$, $(\varphi_i, \varphi_j^*) = 0$ and $(\Theta_i, \Theta_j) = \delta_{ij}$, $(\Theta_i^*, \Theta_j^*) = \delta_{ij}$, $(\Theta_i, \Theta_j^*) = 0$ for φ and Θ that are mode solutions. The φ and Θ can be expanded in terms of each other.

The creation and annihilation operators $a_n^\dagger$, $b_k^\dagger$ and a_n, b_k are given by the following expressions:

$$a_n = \sum_k \left(\alpha_{kn}b_k + \beta_{kn}^* b_k^\dagger\right) , \tag{41}$$

$$b_k = \sum_n \left(\alpha_{kn}^* a_n - \beta_{kn}^* a_n^\dagger\right) . \tag{42}$$

α_{kn} and β_{kn} are Bogoliubov coefficients determined by $\alpha_{ij} = (\Theta_i, \varphi_j)$, $\beta_{ij} = -(\Theta_i, \varphi_j^*)$. They are related as

$$\sum_i (\alpha_{ni}\alpha_{ki}^* - \beta_{ni}\beta_{ki}^*) = \delta_{nk}\,, \tag{43}$$

$$\sum_i (\alpha_{ni}\beta_{ki} - \beta_{ni}\alpha_{ki}) = 0\,. \tag{44}$$

Let $|0_a\rangle$ and $|0_b\rangle$ be two states of vacuum in the Fock space related to each particle notion in (30). They are represented for all n and k as

$$|0_a\rangle : a_n|0_a\rangle = 0\,, \tag{45}$$

$$|0_b\rangle : b_k|0_b\rangle = 0\,. \tag{46}$$

If $|0_b\rangle$ is introduced as the usual vacuum, then $|0_a\rangle$ is regarded as a many-particle state. Therefore, the number of Θ_n-mode particles in the state of $|0_a\rangle$ is

$$\langle 0_a|b_k^\dagger b_k|0_a\rangle = \sum_n |\beta_{kn}|^2\,. \tag{47}$$

If the $\varphi_n(x)$ are defined as positive frequency modes and the $\Theta_n(x)$ modes are linear unification of them, then $\beta_{jk} = 0$. Then, $b_k|0_b\rangle = 0$ and $a_k|0_a\rangle = 0$. Hence, φ_j and Θ_k modes have a common vacuum state. If $\beta_{jk} \neq 0$, then Θ_k contain a combination of positive-φ_k and negative-φ_k^* frequency modes.

Therefore, we can define the positive- and negative-frequency solutions in order to find the Bogoliubov coefficients. Asymptotic expansion of the parabolic cylinder functions is given by [21]

$$D_\nu(z)_{|z|\to+\infty} \approx z^\nu e^{-z^2/4}\,, \qquad |\arg z| < \frac{3\pi}{4}\,. \tag{48}$$

Taking into account this relation for Eqs. (36) and (37) in the limit of $t \to +\infty$ (namely, $\xi \to +\infty$) and comparing their asymptotic expansion with Eq. (12), we see that the positive- and negative-frequency mode solutions will be, respectively, as follows:

$$T_{s_1}(\xi) \approx \left(\sqrt{2eE_0\Lambda}|t|\right)^{-1/2} e^{\left(-ieE_0\Lambda t^2/2 - i\tilde{a}\ln\left(\sqrt{2eE_0\Lambda}|t|\right)\right)} \tag{49}$$

and

$$T_{s_2}(\xi) \approx \left(\sqrt{2eE_0\Lambda}|t|\right)^{-1/2} e^{\left(ieE_0\Lambda t^2/2 + i\tilde{a}\ln\left(\sqrt{2eE_0\Lambda}|t|\right)\right)}\,. \tag{50}$$

We conclude that the solutions behave as $T_\pm \approx e^{\pm iS(t)}$.

For $t \to -\infty (\xi \to -\infty)$, the solutions are in the form of

$$T_{s_1}(\xi) = \frac{e^{-\frac{\pi\tilde{a}}{4}}}{(2eE_0\Lambda)^{\frac{1}{4}}} D^*_{-i\tilde{a}-1/2}\left(-e^{i\pi/4}\xi\right) \tag{51}$$

and

$$T_{s_2}(\xi) = \frac{e^{-\frac{\pi\tilde{a}}{4}}}{(2eE_0\Lambda)^{\frac{1}{4}}} D_{-i\tilde{a}-1/2}\left(-e^{i\pi/4}\xi\right) \tag{52}$$

so that their asymptotic behavior should be $T_\pm \approx e^{\pm iS(t)}$. It is clear that the solutions are different in the asymptotic regions and this is the nature of the particle creation. Therefore, the solutions for $t \to +\infty$ belong to vacuum "out" mode, whereas are vacuum "in" mode for $t \to -\infty$.

The positive- and negative-frequency vacuum "out" and "in" modes can be related to each other with the Bogoliubov coefficients. By using Eq. (39), we can write

$$D_{-i\tilde{a}-1/2}\left(-e^{i\pi/4}\xi\right) = \alpha^* D^*_{-i\tilde{a}-1/2}\left(e^{i\pi/4}\xi\right) + \beta D_{-i\tilde{a}-1/2}\left(e^{i\pi/4}\xi\right) . \tag{53}$$

Expanding the left-hand side of this expression according to the below formula [21]

$$D_\nu(z) = \left[e^{-i\pi\nu} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-i\pi(\nu+1)/2} D_{-\nu-1}(iz)\right] , \tag{54}$$

and using the result $[D_\nu(z)]^* = D_{-\nu-1}(-iz)$ that can be easily derived by taking the advantage of the relation between the parabolic cylinder function and the Whittaker function given as [21]

$$D_\nu(z) = 2^{\left(\nu+\frac{1}{2}\right)/2} z^{-1/2} W_{\frac{1}{2}\left(\nu+\frac{1}{2}\right),-\frac{1}{4}}\left(z^2/2\right) , \tag{55}$$

we obtain the Bogoliubov coefficients α and β as follows:

$$\alpha = \frac{\sqrt{\frac{2\pi}{\breve{a}}}\, i e^{-\pi\breve{a}/2}}{\Gamma(-i\breve{a})} \tag{56}$$

and

$$\beta = e^{-\pi\breve{a}} , \tag{57}$$

where $\breve{a} = (\frac{k_z^2 - \varpi^2}{2eE_0\Lambda})$ and $|\alpha|^2 + |\beta|^2 = 1$ condition is satisfied.

Then, we find the below expression for the Bogoliubov coefficients

$$\frac{|\alpha|^2}{|\beta|^2} = \frac{2\pi}{\breve{a}} e^{\pi\breve{a}} \frac{1}{|\Gamma(-i\breve{a})|^2} . \tag{58}$$

By considering the following formula for Gamma functions [17]

$$|\Gamma(iq)|^2 = \frac{\pi}{q\sinh(\pi q)}\,, \tag{59}$$

the number density of the created particles can be computed as follows:

$$N \simeq |\beta|^2 = \left[\frac{|\alpha|^2}{|\beta|^2} + 1\right]^{-1} = e^{-2\pi\breve{a}}\,, \tag{60}$$

where the parameter $\breve{a}$ in terms of the physical constants of four-vector potential (2) has been given as below

$$\begin{aligned}\breve{a} \;=\; & \frac{1}{2eE_0\Lambda}\left[\frac{4e\tau^2B_0^2(eB_0-\tau k_y)^2}{\left(-1-2n+\sqrt{1+4eB_0(s+eB_0\tau^2)}\right)^2}\right.\\ & -\frac{1}{4\tau^2}\left(-1-2n+\sqrt{1+4eB_0(s+eB_0\tau^2)}\right)^2\\ & \left. -\left(m^2+k_y^2\right)-2eB_0\tau(eB_0\tau-k_y)\right]. \end{aligned} \tag{61}$$

5. Conclusion

In this study, we used the two-component formalism for the Dirac equation that is proposed by Feynmann and Gell-Mann. This approach to the problem removes the complexity of obtaining the exact solutions. One of the advantages of working with this form of the Dirac equation is that these solutions are valid for the Klein–Gordon particles in the case of $s = 0$. Thus, the results can be used both for scalar and fermionic particles.

The mechanism of particle production by strong electric fields must be significant in order to explain the early stages of the heavy-ion collisions, for example, their effect on the thermalization of quarks and gluons. For the analysis of our problem, we take into account a strong constant electric field and a space-dependent hyperbolic magnetic field. Exact solutions of the Dirac equation were identified in terms of the parabolic cylinder and hypergeometric functions.

Existence of the strong electric fields causes that unstable vacuum is asymptotically static at future. The "in" and "out" vacuum states were determined with the help of the asymptotic solutions of relativistic HJ equation. They were related by the Bogoliubov coefficients that are used to calculate the particle creation number density in Eq. (60). This expression depends on the parameters of electric and magnetic fields and is not in the Fermi–Dirac thermal form. As it is seen by analyzing the formula and also from figure 1, selected form of the magnetic field has a reduction effect on the creation of fermionic particles. This situation is compatible with previously obtained results. Moreover, it can be seen from figure 1 that the particle creation rate increases due to the electric field strength, $(E_0 \Lambda)$.

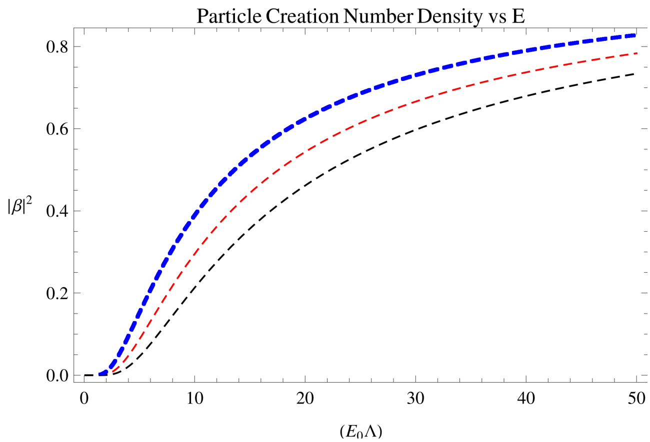

Fig. 1. (Color online) Particle creation number density *versus* electric field strength is depicted. $m = 1; n = 1;\ \tau = 1;\ k_y = 1;\ k_z = 1;\ \Lambda = 1$ and $B_0 : 0$ (thick black/blue), $B_0 : 0.2$ (gray/red), $B_0 : 0.4$ (black).

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