EFFECT OF ZEALOTS ON THE OPINION DYNAMICS OF RATIONAL AGENTS WITH BOUNDED CONFIDENCE

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This paper incorporates a micro-level decision-making paradigm along with a social interaction model (bounded confidence) in the presence of influences (zealots). Every agent in the society represents a node in a Barabási–Albert network and is given a decision-making ability (to choose from a fixed set of states). The decision making is based on maximization of estimated accumulated rewards gained as a result of an individual's own sequence of choices in the presence of different probabilities of external events. The effects of interactions, and events on the final distribution of decision states are studied with and without the presence of influences. Bounded confidence model parameters (the distance parameter and the convergence parameter) are used to study the final distribution of states, and the time the society needs to reach its equilibrium (convergence time). Finally, effects of network topology on the final distribution of states and convergence time are presented.

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1. Introduction

Inspired by the rapid increase in interactions through social media, opinion evolution in networked societies has become an active field of study. Through interactions, a specific idea, opinion, behavior or decision can initiate, develop, and in some cases, spread to the whole society [1].

Decisions can be described in terms of three essential components: alternatives, anticipated consequences, and uncertainty. Despite vast diversity in the field of judgment and decision making, its boundaries and major theoretical concerns are mostly related to the historically dominant expected utility family of theories made popular by Von Neumann and Morgenstern [2] and Savage [3, 4]. The heart of the theory, sometimes called the rational expectations principle or expectancy-value model [5], proposes that each alternative course of action or choice option should be evaluated by weighting its global expected satisfaction–dissatisfaction with the probabilities that the component consequences will occur and be experienced. For example, in the Logit model, the relative frequency of usage of a strategy is proportional to the number of times it was successful (which implies a trial and error behavior, at least in the beginning). In mathematical terms, this law of relative effect [6] reads $p(i) = N(i)/(\Sigma_{i'} N(i')) = \frac{e^{U(i)/T}}{\Sigma_{i'} e^{U(i')/T}}$, where for any parameter T (sometimes called the 'social temperature'), $U(i) = T \ln N(i)$ is the utility function, somewhat arbitrarily defining a preference scale.

However, although expectancy-value theory has been very successful in explaining central concepts uses and gratifications research, other factors that influence the process have been recognized. For example, the social and psychological origins of needs, which give rise to motives for behavior, may be guided by beliefs, values, and social circumstances into seeking various gratifications through media consumption and other nonmedia behaviors. In a network setting, such as Twitter, one's estimate of rewards are not absolute quantities, but are influenced by opinions of friends and neighbors. Social scientists for many years have developed theories of group position [7], social identity [8], and system justification [9]. Now, such theories can be validated quantitatively by analyses of 'retweets', 'via', 'hat tip' and 'mention' conventions which have been shown to be analogous to broadcasting one's position, and help explain how virality, meme propagation, and opinion formation occur on social networks [10].

To study the effect of social influence and interaction on emergent behavior, a statistical physics approach deals with a single basic question of social dynamics: how do local interactions between social agents create order out of an initial disordered situation? Much theoretical efforts have been devoted to clarify the implications on the macroscopic outcomes, among other aspects, of different interaction mechanisms (modeled by the voter model [11–13], Sznajd model [14, 15], bounded confidence model [16–18], *etc.*), as well as different topologies of the interaction networks (such as the complete network, Erdős–Rényi random graphs [19], the Watts–Strogatz small world network [20, 21] and the Barabási–Albert scale-free topology [22]). All these models implement the phenomenon of 'social validation' and are thus extensions or modifications, in some way, of the Ising spin model.

In all the above-mentioned models, opinions are modelized as numbers, integer or real. Each agent is initialized with a random number as their representative opinions. As interactions proceed, the agents rearrange their opinion variables, through mutual discussions. At some stage, the system reaches a configuration which is stable under the chosen dynamics; this final configuration may represent consensus with all agents sharing the same opinion, polarization with two main clusters of opinions ("parties"), or fragmentation where several opinion clusters survive. In all such evolutionary models of societies, the detailed behavior of each human, inherently the complex outcome of many internal processes, is largely overlooked. This gross simplification is justified by assuming that only higher level features, such as symmetries, dimensionality, and topology of the interaction network are relevant for the global behavior of the society, rather than microscopic details of individual motives, perceptions and judgments.

There has also been a surge of research in the past few years on the impact of "true believers", "zealots", "committed agents", or "inflexibles" (essentially used synonymously) on agent-based models on all kinds of networks. The terminology above varies from paper to paper, but the common theme is the introduction of a special class of agents who never change their states [23]. Non-exhaustibly, some of the notable works are by Galam [24] and Mobilia [25], where the role of inflexible minorities and zealotry in the breaking of democratic opinion dynamics has been studied. Xie [26, 27] has looked at consensus in the presence of competing committed group, while Yildiz [28] describes binary opinion dynamics with stubborn agents.

In this paper, we try to bridge this gap between micro- and macroscopic studies of emergent behavior using an agent-based (AB) model in the presence of influences. We incorporate individual judgment and decision mechanism, parameterized as a probabilistic finite state automata (PFSA) while observing the effect of interaction between these PFSA models and influences linked through a Barabási–Albert scale free network. The basic idea behind

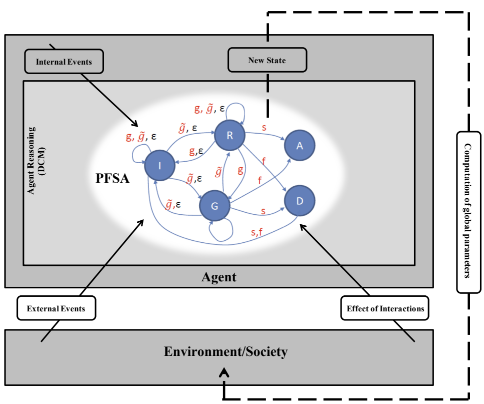


Fig. 1. Schematic diagram of the interaction between PFSA-based individual logic mechanisms and the society. At each decision step, an individual chooses the most attractive state based on an utility maximization principle. This choice influences the reward estimates of each of his neighbors within his confidence bounds, who in turn choose the most attractive state. This cycle of interaction and reward update continues till equilibrium is reached.

this PFSA-based discrete choice model (DCM) is that each decision maker chooses from a finite set of alternatives to maximize the potential cumulative reward she can collect over several steps in the future (Fig. 1) [29]. The perceived reward for the different choices are negotiated by each individual through interaction with a subset of her neighbors, whose reward values do not differ from hers by more than a chosen threshold (confidence bound).

To do so, we consider a society whose population are indecisive about supporting their government or opposing it by joining the rebellion group. The population includes a few influencers who try to convince the population to join the rebellion group. We have created a setup which enables individuals to interact with other agents in their network, take into account different actions of the government, and then decide which of the two groups they should join.

Section 2 discusses the PFSA-based discrete choice model used in this paper. Section 3 explains elements of the social computation such as the interaction model and the network topology. Section 4 presents the results obtained from simulations along with necessary discussions. Finally, in Section 5, findings of this study are summarized and the paper is concluded.

2. Individual Decision Making Algorithms (IDMAs)

Assumption 2.1. Finite set of discrete choices
At each instant, every individual in the network is faced with the same set of finite discrete choices — for example, to vote for political candidate ‘A’ or ‘B’ or not to vote at all [30]. In Markov Decision Modeling [31] as well as in the current framework, the problem is posed as finding the optimal choice policy for maximizing the rewards gained as a result of one’s own choices. It may be noted that this marks a departure from the usual setting in which the Krause and Hegselmann (KH) bounded confidence model is studied. To a degree, our model resembles the study of vector opinion dynamics, such as those done by Axelrod [32] and several studies by Jacobmeier [33] and Fortunato *et al.* [34]. In vector opinion dynamics, the opinion has an integer number of components and the agents occupying the sites of a network, communicate within the KH framework.

Assumption 2.2. Rational perspective
Individuals are assumed to be rational. This means they order the states into which they can reach, and they maximize something, reward function in the case of this paper,[30]. Subscription to the rational perspective does not suggest similar reactions from individuals under the same influence. However, it creates a rational structure for individuals’ behavior. Since this behavioral logic will be encoded as a Probabilistic Finite State Automaton (PFSA) in the next section, rational perspective permits pairs of states without authorized transitions to exist.

Assumption 2.3. Probabilistic individual decisions
In the IDMA, decision making is not assumed to be a deterministic phenomenon, *i.e.* we assume that even when provided with the same conditions and reward vectors, different individuals may choose different decisions, and even the same individual can choose differently on different occasions [30]. The only constraint is that the choices should conform with the *rational perspective* (Assumption 2.2).

Assumption 2.4. Two kinds (external and internal (ε)) of events
External/global events simultaneously affect all individuals often resulting in uncontrollable large scale transitions in the society as a whole. In contrast, *internal/local* events represent the individuals' personal choices.

2.1. Normative perspective modeled as a Probabilistic Finite State Automata (PFSA)

The assumption of *rational perspective* allows individual behavior, to be encoded as a PFSA. Even though the IDMA described in the last section is a generic mathematical structure which can be used to model various scenarios of emergent opinions, in this paper, we study the specific case of a society in the cusp of a rebellion against the existing ruling power. In our simplified depiction of the situation, each individual faces the *internal* decision of supporting the existing government, supporting the rebelling group, or remaining in a state of indecision. Additionally, the individual can reach a state of political advantage or disadvantage, but the uncontrollable transition to these two states can only occur through an *external* event, namely, the success or failure of the revolution. The five PFSA states and events are described in Table I.

TABLE I

List of PFSA states and events.

States	Description	Events	Description
I	State of being undecided/ neutral	g	A popular act by the government
R	State of supporting the revolutionary group	$\tilde{g}$	An unpopular government act
G	State of supporting the government	ε	An internal decision
A	State of political advantage	s	Success of the revolution
D	State of political disadvantage	f	Failure of the revolution

Figure 2 gives a schematic diagram of the assumed rational perspective encoded as a PFSA. It may be noticed that transitions such as $g : G \to R$ or $g : I \to R$ are unauthorized, since it is assumed that a favorable act by the government should not make anyone decide to join the opposing group. Also, the same event can cause alternate transitions from the same state; the actual transition will depend probabilistically on the measure of attractiveness of the possible target states. In the simplified described model, all events of the same type are clubbed together as g (popular) or $\tilde{g}$ (unpopular acts by the government). However, varying degrees of 'popularity' and 'unpopularity' of government acts can be encoded by creating separate groups $g_1, g_2, \ldots, g_n$ and $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m$ with different costs associated with these transitions.

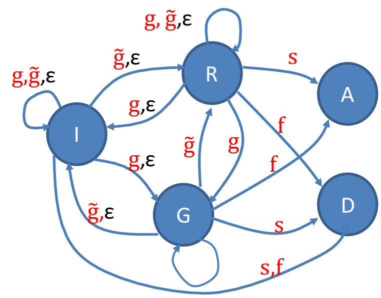


Fig. 2. Schematic diagram of normative perspective encoded as a PFSA.

2.2. Rewards, transition costs and probabilities

The probability of transitioning from one state to another depends on three things — whether the target state is reachable from the current state, whether the current event acts as an impetus for state change, and whether the target state is relatively more attractive compared to the current state. The concept of *positive real measure of a sequence of events* is used to calculate the relative degree of attractiveness of states [35]. This is briefly explained below.

At finite state, machine consists of states, inputs and outputs. The number of states is fixed; when an input is executed, the state is changed and an output is possibly produced. The probabilistic automaton (PFSA) is a generalization of the finite automaton structure and includes two probabilities: the probability P of a particular state transition taking place, and with the initial state q_0 replaced by a stochastic vector giving the probability of the automaton being in a given initial state.

With suitable definitions of the states, inputs and transition matrices, the PFSA structure is well-suited for quantifying the IDMA framework. Let the discrete choice behavior be modeled as a PFSA as

$$G_i \equiv (Q, \Sigma, \delta, q_i, Q_m)\,, \tag{1}$$

where $Q = \{I, R, G, A, D\}$ is the finite set of choices with $|Q| = 5$ and the initial state $q_i \in Q = I$; *i.e.*, the whole society is neutral in the initial stage. The distribution of states may be represented as a coordinate vector of the form of $\bar{v}_i$, defined as the $1 \times N$ vector $\left[v_1^i, v_2^i, \ldots, v_N^i\right]$, given by

$$v_j^i = \begin{cases} 1 & \text{if} \quad i = j\,, \\ 0 & \text{if} \quad i \neq j\,. \end{cases} \tag{2}$$

$\Sigma = \{\varepsilon, g, \tilde{g}, s, f\}$ is the (finite) alphabet of events (inputs to the PFSA) with $|\Sigma| = 5$; the Kleene closure of Σ is denoted as Σ^*; the (possibly partial) function $\delta : Q \times \Sigma \times Q \to [0, 1]$ represents probabilities of state transitions and $\delta^* : Q \times \Sigma^* \times Q \to [0, 1]$ is an extension of δ; and $Q_m \subseteq Q$ is the set of marked (*i.e.* accepted) states. It may be noted that the parameters of the model introduced so far are a physical manifestation of the choices that each rational actor faces in an election scenario (as explored in [35]) — the choice set Q is well-defined by the number of available options (parties to vote for, G and R in this case), and the initial-state choice ($q_i = I$) allows the simulation to start from the same point every time. The effect of initial clustering of similar choices in regional and local neighborhoods can be an important factor and will be explored in subsequent publications. The transition probability δ signifies a parameter that the actors learn or estimate through their experience and understanding.

Definition 2.1. The reward from each state $\chi : Q \to [0, \infty)$ is defined as a characteristic function that assigns a positive real weight to each state q_i, such that

$$\chi(q_j) \in \begin{cases} [0, \infty) & \text{if} \quad q_j \in Q_m\,, \\ \{0\} & \text{if} \quad q_j \notin Q_m\,. \end{cases} \tag{3}$$

Definition 2.2. The event cost, conditioned on a PFSA state at which the event is generated, is defined as $\tilde{\pi} : \Sigma^* \times Q \to [0, 1]$ such that $\forall q_j \in Q$, $\forall \sigma_k \in \Sigma, \forall s \in \Sigma^*$

1. $\tilde{\pi}[\sigma_k, q_j] \equiv \tilde{\pi}_{jk} \in [0, 1)\,; \quad \sum_k \tilde{\pi}_{jk} < 1\,;$
2. $\tilde{\pi}[\sigma, q_j] = 0 \quad \text{if} \quad \delta(q_j, \sigma, q_k) = 0\,\forall k\,; \quad \tilde{\pi}[\epsilon, q_j] = 1\,.$

The event cost matrix, ($\tilde{\Pi}$-matrix), is defined as

$$\tilde{\boldsymbol{\Pi}} = \begin{bmatrix} \tilde{\pi}_{11} & \tilde{\pi}_{12} & \ldots & \tilde{\pi}_{1m} \\ \tilde{\pi}_{21} & \tilde{\pi}_{22} & \ldots & \tilde{\pi}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\pi}_{n1} & \tilde{\pi}_{n2} & \ldots & \tilde{\pi}_{nm} \end{bmatrix}.$$

The characteristic vector $\bar{\chi}$ is a numerical depiction of an individual's perception of expected benefits or 'rewards' to be obtained by being in a particular state. For example, if the states represent various job choices, the remuneration from these jobs can serve as the characteristic vector. On the other hand, the event cost is an intrinsic property of the nominal perspective. The event cost is conceptually similar to the state-based conditional probability of Markov Chains, except $\sum_k \tilde{\pi}_{jk} = 1$ is not allowed to be satisfied. The condition $\sum_k \tilde{\pi}_{jk} < 1$ provides a sufficient condition for the existence of the real signed measure, as discussed in [36].

Definition 2.3. The state transition function of the PFSA is defined as a function $\pi : Q \times Q \to [0,\, 1)$ such that $\forall q_j$, and $q_k \in Q$

1. $\pi(q_j,\, q_k) = \sum_{\sigma \in \Sigma :\, \delta(q_j,\sigma,q_k) \neq 0} \tilde{\pi}(\sigma,\, q_j) \equiv \pi_{jk}\,;$
2. and $\pi_{jk} = 0$ if $\{\sigma \in \Sigma :\, \delta(q_j,\, \sigma) = q_k\} = \emptyset\,.$

The state transition matrix, (Π-matrix), is defined as

$$\boldsymbol{\Pi} = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{n1} & \pi_{n2} & \cdots & \pi_{nn} \end{bmatrix}.$$

2.3. Measure of attractiveness of the states

A real measure ν_θ^i for state i is defined as

$$\nu_\theta^i = \sum_{\tau=0}^{\infty} \theta\,(1-\theta)^\tau\, \bar{v}^i \Pi^\tau \bar{\chi}\,, \tag{4}$$

where $\theta \in (0,1]$ is a user-specified parameter and $\bar{v}^i$ is defined in Eq. (2).

Remark 2.1. Physical significance of real measure
Assuming that the state probability vector is $\bar{v}_i$ corresponding to the current state of the Markov process (i), at an instant τ time steps in the future, the state probability vector is given by $\bar{v}^i \Pi^\tau$. Consequently, the expected value of the characteristic function is given by $\bar{v}^i \Pi^\tau \bar{\chi}$. The measure of state i, described by Eq. (4), is the weighted expected value of χ over all time steps in the future for the Markov process that begins in state i. The weights for each time step $\theta\,(1-\theta)^\tau$ form a decreasing geometric series (sum equals 1). The measure in vector form yields

$$\bar{\nu}_\theta = \theta\,(\mathbb{I} - (1-\theta)\,\Pi)^{-1}\,\bar{\chi} \qquad \text{and} \qquad \bar{\nu}_{\text{norm}} = \frac{1}{\sum_k \nu_k}\bar{\nu}\,. \tag{5}$$

Remark 2.2. The effect of θ
The rate at which the weights decrease with increasing values of τ is controlled by θ. More importance is put on the states reachable in the near future if parameter θ assumes large values (close to 1), because of the fast decay in the weights. The states through time are more uniformly weighted for small values of θ allowing the system to interact with a large neighborhood of the connected states.

The probabilistic decisions are made based on $\bar{\nu}_{\text{norm}}$. The discounted expected reward of a state is proportional to the measure of that state. Higher measure of a state corresponds to a higher discounted expected reward, hence the potential to make a transition to that state is higher.

3. Elements of social computations

The dynamics of the KH bounded confidence model is very simple: One of the agents is chosen at random; then the agent adopts the average opinion of its compatible neighbors. Compatibility between two nodes is determined by the distance between the current opinions held by the two nodes. The procedure is repeated by selecting another agent randomly and so on. The type of final configuration reached by the system depends on the value of the confidence bound d. For a complete graph, consensus is reached for $d > d_{\text{c}}$, where $d_{\text{c}} \simeq 0.2$ or 0.5, depending on whether the average degree of the graph diverges or stays finite when the number of vertices goes to infinity.

In this paper, it is assumed that the interaction is entirely through the characteristic function χ of the states.This assumption is based on the insight that the anticipated reward from a state is the most well-discussed and well-broadcast quantity in a social network [37, 38]. The update rule for the reward vector of agent i due to interactions with its neighbors is as follows:

$$\bar{\chi}^i_{t+1} = \bar{\chi}^i_t + \mu \left(\bar{\chi}^i_{t_{\text{neighbors}}} - \bar{\chi}^i_t \right) , \tag{6}$$

where $\bar{\chi}^i_{t_{\text{neighbors}}}$ is the *mean reward vector* of the first-order neighbors in the network of agent i at time step t. Following the notion of bounded confidence, only those neighbors whose opinions are within $\chi(q_j) \pm d, \forall j$ contribute to the opinion update of agent i. Here, μ (or the convergence parameter) is the weight which determines how much an agent is influenced by the other one.

Since many networks in the real world are conjectured to be scale free, including the World Wide Web, biological networks, and social networks, in this study, a BA extended model network created by the Pajek software program is used. Table II presents the parameters of the network. In addition, one of the experiments is repeated on a complete graph topology for comparing the results related to different networks.

TABLE II

List of parameters used for BA scale-free network.

Number of vertices	100
Number of initial disconnected nodes	3
Number of added/rewired edges at a time	2
Probability to add new lines	0.3333
Probability to rewire edges	0.33335

3.1. Numerical simulations

In this study, a population of 100 people are initialized and assigned a random number drawn from a uniform distribution $U(0,1)$, representing the time remaining before that person makes a decision. This imposes an ordering on the list of people in the network. As soon as someone makes a decision, the time to her next decision, drawn from $U(0,1)$, is assigned and the list is updated. Additionally, external events g and $\tilde{g}$ are also associated with a random time drawn from $U(a,b)$. Choosing a and b, the external events can be interspersed more, or less densely.

At t_0, all 100 individuals are initialized at state I. Initial values of a mean reward vector $\bar{\chi}_m$ and the true event probabilities are fixed. Individuals receive a noisy estimate of the true probabilities and the rewards. At the time epoch t_k, when it is the i^{th} person's turn to make a decision, she updates her personal estimate of the reward vector according to the influence equation (Eq. (6)). She then calculates the degree of attractiveness of the states based on the normalized measure, using Eq. (5). The transition probabilities are calculated as $P(q_{t_{k+1}} = q' | q_{t_k} = q, \sigma = \sigma') = \nu_{\text{norm}}(q') R(q, \sigma', q')$, where $R(q, \sigma', q') = 1$ if $\sigma' : q \to q'$ exists, otherwise 0. The only difference in the case of an external event such as $g, \tilde{g}, s$ or f is that everyone simultaneously updates their states rather than asynchronously, as in the case of internal events. Algorithm 1 describes how numerical simulations of this study have been conducted.

3.1.1. Effect of influencing agents in decision making

In this study, the influencers are treated as indistinguishable except for the fact that they never update or change their $\bar{\chi}$ values; moreover, they do not make decisions and stay in the same state of mind during the entire simulation. It is also typical that the influencers are serving a certain agenda, in this case, trying to mobilize forces to join the rebellion. However, this influence is exerted very passively, by advertising a higher value for $\chi(R)$ and lower value for all other states

Algorithm 1 : Interactions and decision making

Inputs: $N, \mu, d, \theta,$ influence size, network

Initialization;
Randomly order nodes and events;
while Not converged **do**
  for (Node i) **do**
    Update $\bar{\chi}^i_{t+1}$ from Eq. (6);
    Update $\bar{\nu}^i_{\text{norm}}$ from Eq. (5);
    Update state for Node i;
    Insert Node i randomly in the ordered list;
  end for
  if (an external event occurs) **then**
    Update $\bar{\chi}^i_{t+1}$ $\forall i \in N$;
    Update $\bar{\nu}^i_{\text{norm}}$ $\forall i \in N$;
    Update state for all nodes;
  end if
end while

$$\chi^I(q_j) = \begin{cases} \chi_m(q_j) - \Delta & \text{if} \quad j = 1, 3, 4, 5\,, \\ \chi_m(q_j) + \Delta & \text{if} \quad j = 2\,, \end{cases} \tag{7}$$

in which $\chi^I(q_j)$ represents the reward associated with state q_j for influence nodes, and $\bar{\chi}_m$ is an estimate of the reward values expressed by the whole society on an average. Δ is a parameter adjusting the strength of influences (control input). In a situation where influences have to favor other states, the corresponding element in the mean reward vector needs to be strengthened.

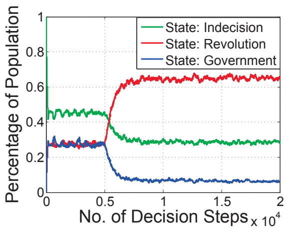
4. Results and discussions

This study investigates the simultaneous interplay between two separate subsystems, namely a logical decision-making subsystem modeled by a PFSA, and the interaction and influence subsystem modeled by bounded confidence. Parameters of each subsystem is investigated separately. Moreover, each simulation is conducted in two phases. During the first phase (*decision epoch* ≤ 5000), the dynamics of the opinion evolution is studied without introducing the effect of influences. In the second phase (*decision epoch* > 5000), a group of 10 influences is activated.

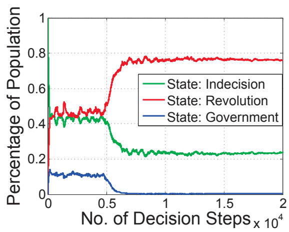
4.1. PFSA parameters

4.1.1. External events

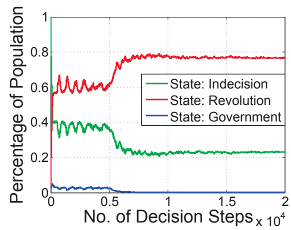
In the first set of experiments, the ratio of occurrences of ‘good’ and ‘bad’ external events, $r = P(g) : P(\tilde{g})$ is varied to observe the effect of long-term government policies on a population. Each simulation was run 30 times and the average of all the runs are showed in Fig. 3.



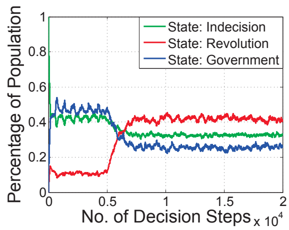
(a) $r = 1 : 1$



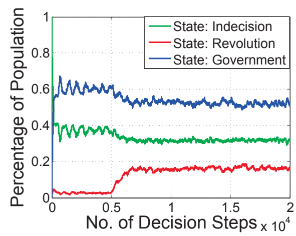
(b) $r = 3 : 7$



(c) $r = 1 : 9$



(d) $r = 7 : 3$



(e) $r = 9 : 1$

Fig. 3. Effect of external events on the final-state distribution of the society with $|I| = 10$ and $\Delta = 0.3$.

Effects of external events can be observed in the first phase (*decision epoch* $\leq$ 5000) of each subfigure. When $r = P(g) : P(\tilde{g}) = 1 : 1$, the percentage of the population in states R and G are equal, and a large part of the population remains undecided (Fig. 3 (a)). In the absence of any deadline for making a decision, this result is only to be expected. As the government starts to push more and more unpopular policies, opinions bifurcate and the rebellion group starts to become more popular. Distribution of the society between the 3 states varies monotonically with r.

The presence of influences in the second phase causes a dramatic change in system behavior. In each of the influenced cases (Fig. 3), the percentage of the population in state R starts to increase rapidly, as soon as the influencing nodes are activated. Interestingly, the length of the transition phase between the initial (uninfluenced) distribution and the final steady state is independent of r (Table III). The final steady state distribution, on the other hand, is affected both by r, I. In Fig. 3 (e), the number of popular policies by the government is high enough to overcome the effect of influences, and therefore, a smaller portion of people are converted to join the rebellion state. It may be noted that the quick transition period is a clear indication of the narrow window of opportunity available to intervene and prevent the society from transitioning into instability.

TABLE III

Convergence time related to different ratios in Fig. 3.

r	1 : 9	3 : 7	1 : 1	7 : 3	9 : 1
Convergence time	6958	6940	6986	6856	6739

4.1.2. Comparison of IDMA with standard interaction models

A focus on network structure and large scale dynamics rather than individual behavior have successfully put forward several elegant theories of emergent social behaviors, such as evolution of opinions, consensus formation, properties of elections, and formation of a common language.

In order to gauge the relative expressive power of such a simple interaction dynamics with the IDMA framework developed in this paper, we directly compare the results obtained by implementing the more complex IDMA framework with that of a well-established but simple model of social interaction, namely the voter model where at each step, an arbitrarily chosen node imitates the state of an arbitrarily chosen neighbor. We use the same parameters to make the comparison reliable, such as the same BA scale-free network, and consider a population of indecisive people alongside a few influencers who promote joining the rebellion group. In this setup, people can stay in the indecisiveness state, or they can join either the rebellion group, or the group which supports the government. Again, influencers enter the simulations after the 5000$^{\text{th}}$ decision step (Fig. 4).

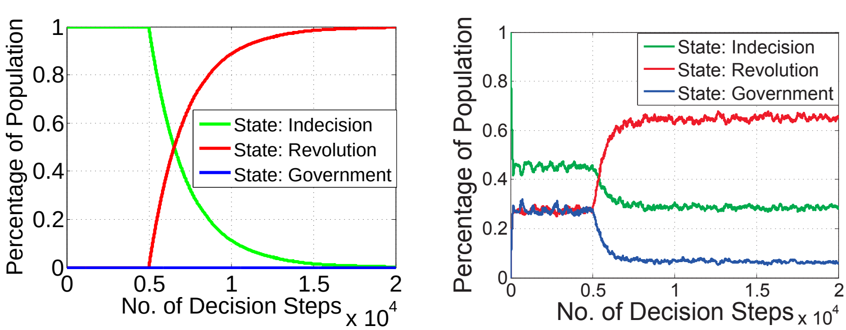


Fig. 4. Effects of incorporating an IDMA; (left) voter model, (right) model incorporating the IDMA.

One of the shortcomings of the voter model becomes apparent from the first phase of the simulations; since the voter model runs purely on imitation, starting from an initial condition where all agents are in the state of indecisiveness, none of them is able to change their state to another one. On the other hand, when IDMA is included in the model, with the help of the reward vector, effects of external events can be taken into account resulting in opinion change of agents in this region.

Another drawback of the voter model can be seen in the second phase of the simulations. As soon as influences are added to the population, the number of people supporting the revolution increases; and, finally, converges to one of the stable equilibria — where all nodes coincide in the R state. This phenomenon is rather unrealistic. However, in the model with IDMA, although this increase is seen, this is limited by the heterogeneity in the probabilistic decision-making logic of individuals. Most importantly, this model can thus predict and converge to multiple equilibria rather than just the three unrealistic pure states where the voter model converges. Considering these factors, the benefits of incorporating rational decision making in opinion change models become apparent, though it comes with added layers of complexity.

4.2. BC model parameters

4.2.1. Distance parameter (d) and the control input Δ

The influencers deliberately advertise biased reward values in an attempt to pull the population slowly towards the state of their choice (R, in this study). The amount of bias (Δ) is vitally important, since too high of an offset would cause the influencers to drift outside the confidence bound of the general populace and, consequently, they will not be able to enter into dialogues with the undecided nodes. On the other hand, a very low Δ will not have a pronounced effect on the dynamics of opinion evolution and will not be able to produce a substantial change on a global scale.

Since Δ may be thought of as an opinion bias adopted by the influencing group in order to 'control' the final distribution of people over the different states, we call Δ the control input. d is the distance parameter that determines the bound in the bounded confidence model. For simplicity, the effects of d and Δ are presented here only with respect to the state R (Fig. 5). The results for other states are qualitatively equivalent. Higher values of Δ mean that stronger influences exist in the society whose reward vectors have been altered significantly (Eq. (7)). Hence, for low values of d, agents do not interact with influences causing no change in the system behavior. On the other hand, for low values of Δ, although agents interact with influences, the influences are not strong enough to seriously affect the system behavior.

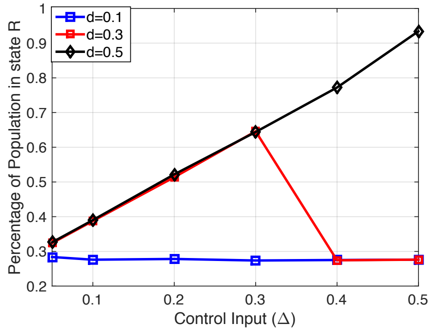


Fig. 5. Dependence of the dynamics of the R state on d and Δ with $|I| = 10$.

As an example, for an intermediate confidence bound ($d = 0.3$), agents can interact with strong influences resulting in noticeable changes in the behavior. However, if the control Δ is too high, influences lie out of the confidence zone of agents. So, although present, influences cannot affect the society. For $d = 0.5$, agents are able to interact with even stronger influences. Consequently, a higher percentage of the population is in state R. From a social psychology perspective, it is very important to estimate the bound of confidence for individual groups in order to be effective in bringing about a positive change. The optimal approach when trying to impact an older population may not be similar to the control parameter suitable for younger demographics.

4.2.2. Convergence parameter

Based on Eq. (6), the convergence parameter determines the influence ability of a node, *i.e.* to what extent a node adheres to his personal estimates of anticipated rewards from a choice as opposed to converging to the mean estimate gathered from his first-order neighbors. $\mu = 0$ implies complete self-reliance while making decisions, while $\mu = 1$ implies complete malleability.

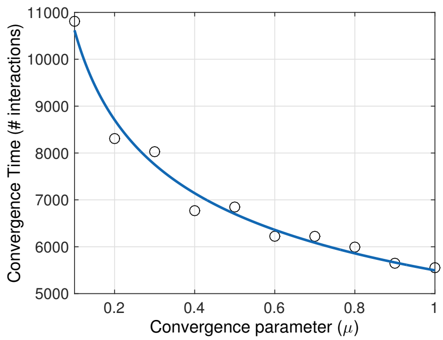
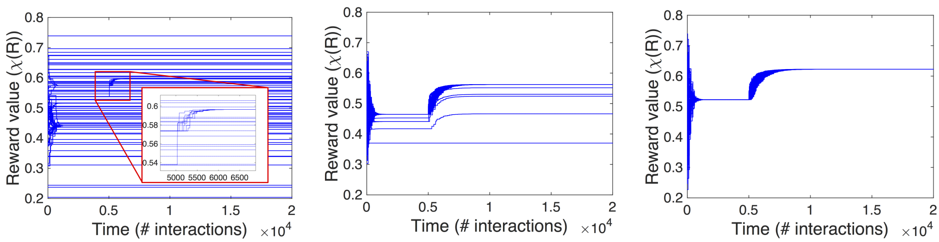


Fig. 6. Effect of μ on convergence time.

It may be conjectured that larger μs might lead to fewer steps required for convergence for the population as a whole. This conjecture is validated in Fig. 6.

4.2.3. Clustering behavior

Formation of clusters is a typical behavior observed in the BC model. The distance parameter d determines the number of clusters in the equilibrium state of a system. Large values of d results in interactions among a large number of nodes, consequently very few clusters emerge [16]. This phenomena is visualized in Fig. 7. As the distance parameter increases, the number of clusters decreases.



(a) $d = 0.1$ (b) $d = 0.3$ (c) $d = 0.5$

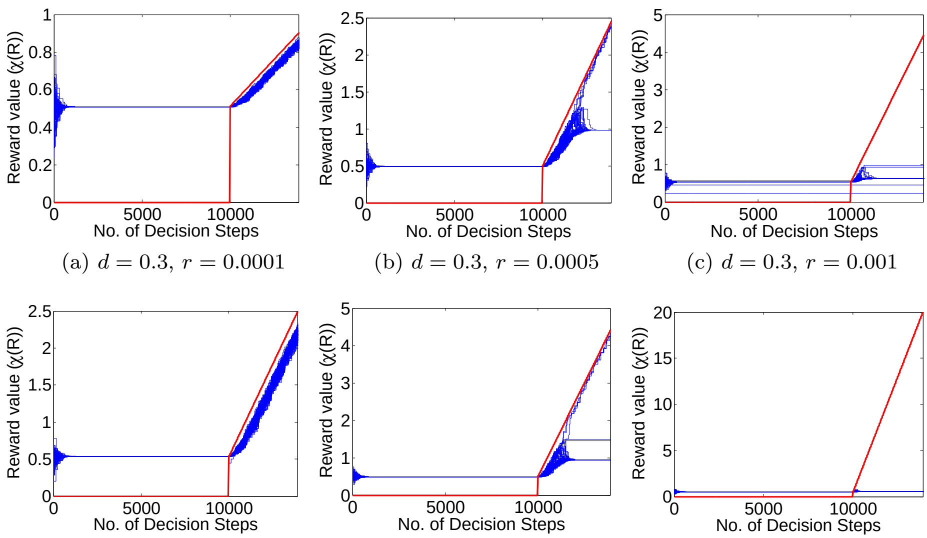
Fig. 7. Clustering behavior of the BC model for different ds with $|I| = 10$.

Addition of influences after 5000 decision epochs adds a layer of complexity to the clustering process. The influencers effectively attract and cluster nodes, but in societies with low confidence bounds ($d = 0.1$), the effect is very local. Most clusters retain their identities and never enter into any interactions outside their own clique. This is demonstrated in Fig. 7 (a), where the small effect of the influencers is illustrated in the inset. Broadly speaking, this implies that societies which are highly clustered due to low confidence bounds are more difficult to influence, unless all the nodes are very homogeneous to begin with. Increase in d allows more agents to interact (with each other as well as with influencers), therefore, a more populated cluster forms around influences ($d = 0.3$). Finally, when $d = 0.5$, the influences are reachable to all agents resulting in only one cluster around influences.

4.2.4. Varying control input $\varDelta(t)$

So far, the influencers have always been treated as static agents, *i.e.* the pro-revolution value of $\chi^I(q_2)$ was fixed and unchanged during the course of the simulation. The motivation behind studying varying control input ($\varDelta$) is to investigate whether a small group of influencers can start from a popular stand-point, cluster the population around themselves, affect them continuously without getting out of populations's confidence bound, and guide them to a different set of rewards ($\bar{\chi}$), which ultimately gets filtered through the IDMA to support of a different state. To do so, a new set of experiments has been conducted using the exact same model; however, the χ value of influencers increases linearly as a function of time steps with a rate of r.

The results of these experiments are shown in Fig. 8. Results are presented based on the distance parameter d and the rate r. The thick black/red graph represents the χ value of influencers at each time step. Figure 8 depicts that for a specific distance parameter, there is threshold for r after which the influencers get strong so fast that the society looses its reach to influencers. For example, in Fig. 8 (a), the increase rate is low, resulting in a homogenous population clustered around influencers. However, as r increases, clusters appear both in and out of reach of influencers (Fig. 8 (b)). Finally, after the threshold, the whole society looses its reach to the influencers (Fig. 8 (c)) rendering their presence in the society obsolete. The same reasoning can be applied for Figs. 8 (d), (e), (f).



(a) $d = 0.3$, $r = 0.0001$ (b) $d = 0.3$, $r = 0.0005$ (c) $d = 0.3$, $r = 0.001$

(d) $d = 0.5$, $r = 0.0005$ (e) $d = 0.5$, $r = 0.001$ (f) $d = 0.5$, $r = 0.005$

Fig. 8. Clustering behavior of a society in the presence of influences with variable control input.

It can also be seen that the threshold for r depends on the distance parameter d. When $r = 0.0005$, $d = 0.3$ produces a cluster which do not get affected by the influencers, but $d = 0.5$ produces a homogenous population grouped around influencers or, in the case of $r = 0.001$, $d = 0.3$ results in a society which does not interact with influencers, however, $d = 0.5$ results in a society with clusters both in and out of reach of influencers. This phenomenon is in line with the findings of the previous section.

The interesting finding of these experiments is the fact that, although for some rates the influencers get out of reach of individuals very fast, in the time interval when they do interact with individuals, the whole society is affected by their presence. For example, in Fig. 8 (b), in the first few time steps, most of the society start lagging the influencers, but they cluster again at a higher level of reward values. This means that the society is more inclined towards the rebellion state (although influencers are out of reach) in comparison to the period when influencers were absent.

4.2.5. Effect of network topology

Figure 9 provides a comparison of the time evolution of the BC interactions deployed on a BA network and on a complete graph. Intuitively, the complete graph promises to have a faster dynamics since the average degree of each node is higher and influence should propagate faster. Results from numerical simulations, on the other hand, indicate the opposite — influence propagation through the BA network is faster (measured by a smaller transition period between initial and final states). This can be qualitatively explained by the fact that the relative influence on each node from the influencers is much lower when the nodes are connected through a complete network. The presence of many links from other regular nodes dilute the effect of the influencing nodes, leading to a slower change and a lower percentage of the population in state R at equilibrium. The convergence times for the two networks under discussion are reported in Table IV.

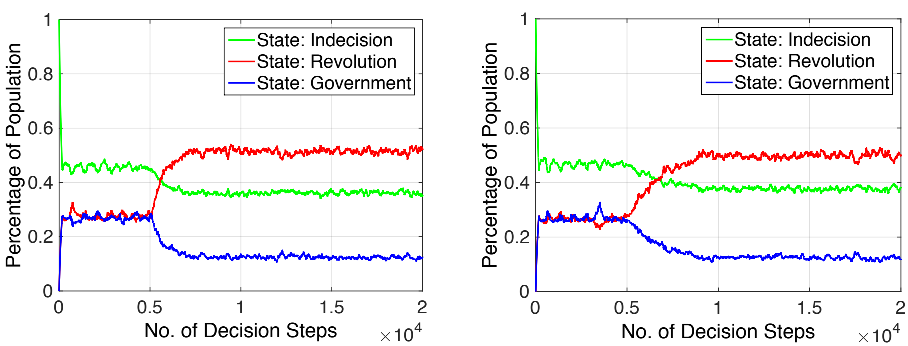


Fig. 9. Effect of different network types on the final-state distribution with $|I| = 10$; (left) BA network, (right) complete graph.

TABLE IV

Comparison of characteristics related to two different networks.

Network	Convergence time	Percentage of population in R
BA	6624	51.51
Complete graph	8156	49.98

5. Conclusion

This paper attempts to incorporate a micro-level decision making paradigm along with a social interaction model (bounded confidence) in the presence of influences (zealots). Every agent in the society is given a decision-making ability (to choose from a fixed set of states). The decision making is based on maximization of accumulated rewards gained as a result of an individual's own choices in the presence of different events.

The effects of interactions and events on the final distribution of decision states are studied with and without the presence of influences. Bounded confidence model parameters (the distance parameter and the convergence parameter) are used to study the final distribution of states, and the time the society needs to reach its equilibrium (convergence time). Moreover, effects of network topology on the final distribution of states, convergence time, and dynamic of the society is presented.

It is observed that without influences, the final distribution of states is purely a function of the external events; more unpopular policies by the government result in higher percentages of people joining the rebellion group. However, presence of influences causes a rapid change in the behavior of the system in favor of the group they support (state R). A short transition period is required for the system to reach equilibrium after the influences are activated.

It is shown that no change in the final distribution of states takes place unless influences are in the confidence bound of the population and at the same time, have large enough offsets. The clustering behavior of the BC model is visualized for different values of the distance parameter. It is concluded that the time interval needed for a system to reach its equilibrium decreases as the convergence parameter increases. In addition, the effect of varying influence ability is studied. Results show that there is a limit for the influence rate so that the society can keep up with the influencers, and that limit depends on the distance parameter.

Finally, simulations reveal that not only is influence propagation slower in the society when agents have higher number of links (complete graph network), but also their effect in the final distribution of state R is weaker.

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