VACUUM POLARIZATION OF MASSIVE FIELDS IN THE SPACETIME OF THE HIGHER-DIMENSIONAL BLACK HOLES

Jerzy Matyjasek, Dariusz Tryniecki

Institute of Physics, Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland

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Using the Schwinger–DeWitt expansion, we construct and study the approximate vacuum polarization, $\langle\phi^2\rangle_D$, of the quantized massive scalar field with a general curvature coupling parameter in higher-dimensional static and spherically-symmetric black hole spacetimes, with a special emphasis put on the electrically charged Tangherlini solutions and the extremal and ultraextremal configurations. For $4 \leq D \leq 7$, the explicit analytic expressions for the vacuum polarization are obtained when the Compton length associated with the quantized field is much less than the characteristic radius of the curvature of the background geometry and the nonlocal contribution to the result ignored. For the conformally coupled fields, the relation between the trace of the stress-energy tensor and the vacuum polarization is examined, which requires knowledge of the higher-order terms in the Schwinger–DeWitt expansion.

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1. Introduction

The vacuum polarization, $\langle\phi^2\rangle$, has often been regarded as a lesser cousin to the stress-energy tensor. However, despite the natural interest, there are a few additional reasons to calculate and study $\langle\phi^2\rangle$. Indeed, it plays an important role in symmetry breaking problem and in the calculation of the trace of the stress-energy tensor of the conformally coupled quantum fields, being much easier to find. For example, to construct the stress-energy tensor within the framework of the Schwinger–DeWitt approach, one has to functionally differentiate the effective action with respect to the metric tensor, whereas to calculate the vacuum polarization, it is sufficient to use the Green function. Moreover, the calculations of the vacuum polarization may reveal conceptual subtleties of the problem and help to choose the best calculational strategy. Generally speaking, if there are some technical problems in the calculations of $\langle\phi^2\rangle$, the same is expected in construction of the stress-energy tensor. On the other hand, if the calculation of the vacuum polarization goes smoothly, the same is expected for the stress-energy tensor. In the latter case, the only difference is the scale of the calculations.

Starting with the seminal work of Candelas [1], the vacuum polarization has been, and currently is, studied in a number of physically interesting cases, as for example, Schwarzschild black holes or FRWL cosmologies [1–18]. The vacuum polarization effects have been analysed in the spacetimes of distorted black holes [19, 20], in the spacetimes of dimension higher (or smaller) than 4 [21–31]. Approximate expressions describing $\langle\phi^2\rangle$ have been constructed in Refs. [32–36].

The aim of this paper is to construct the vacuum polarization of a quantized massive scalar field (with arbitrary curvature coupling) satisfying the covariant equation

$$\left(-\Box + m^2 + \xi R\right)\phi = 0\,, \tag{1}$$

where m is the mass of the field, ξ is the coupling constant and R is the Ricci scalar, in a general D-dimensional static and spherically-symmetric spacetime described by the line element

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + h(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}\,, \tag{2}$$

where $\mathrm{d}\Omega^2_{D-2}$ is the metric on a unit $(D-2)$-dimensional sphere, and to apply the general formulas in the spacetime of the charged black hole. The line element describing such configurations has been constructed by Tangherlini in the early sixties [37]. It has a particularly simple and transparent form when parametrized by the radial coordinates of the event and inner horizons, denoted respectively by r_+ and r_-. The charged Tangherlini solution has the form (2) with

$$f(r) = \frac{1}{h(r)} = \left[1-\left(\frac{r_+}{r}\right)^{D-3}\right]\left[1-\left(\frac{r_-}{r}\right)^{D-3}\right]. \tag{3}$$

Making use of the relations

$$M = \frac{D-2}{16\pi}\Omega_{D-2}\left(r_+^{D-2} + r_-^{D-2}\right) \tag{4}$$

and

$$Q = \pm\left(r_+ r_-\right)^{\frac{D-3}{2}}\sqrt{\frac{(D-3)(D-2)}{8\pi}}\,, \tag{5}$$

where Ω_{D-2} is the area of a unit $(D-2)$-sphere, the line element can be expressed in a standard mass, charge parametrization. The area of the $(D-2)$-dimensional sphere is given by

$$\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}\,. \tag{6}$$

When the horizons merge, *i.e.*, ($r_+ = r_- = r_\pm$), the topology of the closest vicinity of the horizon of the extremal black hole is $\mathrm{AdS}_2 \times S^{D-2}$, and the local geometry is a special case of the general solution

$$\mathrm{d}s^2 = \frac{1}{A}\left(-\sinh^2\chi \mathrm{d}\tau^2 + \mathrm{d}\chi^2\right) + B\mathrm{d}\Omega^2_{D-2}\,, \tag{7}$$

where $A, B \in \mathbb{R}$. It is a product spacetime with maximally symmetric subspaces. Sometimes it is advantageous to work with the

$$\mathrm{d}s^2 = \frac{1}{Ay^2}\left(-\mathrm{d}T^2 + \mathrm{d}y^2\right) + B\mathrm{d}\Omega^2_{D-2}\,. \tag{8}$$

Equation (7) describes the vicinity of the extremal black hole (2) provided $A = f''(r_\pm)/2$ and $B = r_\pm^2$. Consequently,

$$A = \frac{(D-3)^2}{r_\pm^2} \tag{9}$$

and in $D = 4$, one has the Bertotti–Robinson solution.

The calculations of the stress-energy tensor and the vacuum polarization of the quantized fields in curved spacetimes are extremely hard as they exhibit a nonlocal dependence on the spacetime metric. Here, we consider the case when the Compton length associated with the field, λ_c, satisfies the condition

$$\frac{\lambda_\mathrm{c}}{L} \ll 1\,, \tag{10}$$

where L is a characteristic radius of the curvature of the background geometry and, consequently, the nonlocal contribution to the vacuum polarization can be neglected [25, 34, 38, 39].

In the proper-time formalism, one assumes that the Green function, $G(x, x')$, that satisfies the equation

$$\left(\Box - m^2 - \xi R\right) G\left(x, x'\right) = -\delta\left(x, x'\right) \tag{11}$$

is given by

$$G^F\left(x, x'\right) = \frac{i\Delta^{1/2}}{(4\pi)^{D/2}} \int_0^\infty i\mathrm{d}s \frac{1}{(is)^{D/2}} \exp\left[-im^2 s + \frac{i\sigma\left(x, x'\right)}{2s}\right] A\left(x, x'; is\right)\,, \tag{12}$$

where

$$A\left(x,x';is\right)=\sum_{k=0}^{\infty}(is)^{k}a_{k}\left(x,x'\right)\,, \tag{13}$$

s is the proper time and the biscalars $a_k(x,x')$ are the celebrated Hadamard–DeWitt coefficients, $\Delta(x,x')$ is the vanVleck–Morette determinant and the biscalar $\sigma(x,x')$ is defined as the one-half of the geodetic distance between x and x'. Now, let us define

$$A^{(n)}_{\rm reg}\left(x,x';is\right)=A\left(x,x';is\right)-\sum_{k=0}^{\lfloor\frac{D}{2}\rfloor-1}a_{k}\left(x,x'\right)(is)^{k}\,, \tag{14}$$

where $\lfloor x\rfloor$ (a floor function) gives the largest integer less than or equal x, substitute in Eq. (12) $A^{(D)}_{\rm reg}(x,x';is)$ for $A(x,x';is)$ and, finally, denote the thus obtained biscalar by $G^{(D)}_{\rm reg}$. The field fluctuation that characterizes the vacuum polarization in the D-dimensional spacetime is defined as

$$\left\langle\phi^{2}\right\rangle=-i\lim_{x'\to x}G^{(D)}_{\rm reg}\,. \tag{15}$$

Making the substitution $m^2\to m^2-i\varepsilon$ ($\varepsilon>0$), integral (12) can be easily calculated [25, 26]

$$\left\langle\phi^{2}\right\rangle=\frac{1}{(4\pi)^{D/2}}\sum_{k=\lfloor D/2\rfloor}^{N}\frac{a_{k}}{(m^{2})^{k+1-D/2}}\Gamma\left(k+1-\frac{D}{2}\right)\,, \tag{16}$$

where the coincidence limit of the Hadamard–DeWitt biscalars is defined as $a_k=\lim_{x'\to x}a_k(x,x')$ and the upper sum limit, N, depends on how many terms of the expansion we want to use. One expects that if condition (10) holds, Eq. (16) gives a reasonable approximation to the exact $\langle\phi^2\rangle$. Equation (16) is a generalization to arbitrary dimension of the formula derived by Frolov [35] and coincides with the result obtained in Ref. [24].

The plan of the paper is as follows. In the next section (Subsections 2.1 and 2.2), we shall construct the vacuum polarization of the quantized massive field in the general static and spherically-symmetric D-dimensional spacetime ($4\leq D\leq 7$) and use the obtained formulas in the spacetime of the charged Tangherlini black holes[1]. The special emphasis is put on the extremal and ultraextremal black holes, *i.e.*, the configurations in which two or three horizons merge.

[1] Actually, we have calculated the vacuum polarization in $4\leq D\leq 9$. However, the complexity of the formulas describing $\langle\phi^2\rangle_D$ rapidly grows with D and the results in the higher-dimensional case are rather complicated. A brief information about $\langle\phi^2\rangle$ in 8- and 9-dimensional Tangherlini spacetime is given at the end of Sec. 2.2. All results can be obtained on request from the first author.

In Section 2.3, we shall analyse the trace of the stress-energy tensor of the conformally coupled massive fields and analyse its relation to the vacuum polarization. The last section concludes the paper with some final remarks, putting our results in a somewhat broader perspective. Our general results for $\langle\phi^2\rangle$ in D-dimensional black hole spacetime are relegated to Appendix.

Throughout the paper the natural system of unit is used. The signature of the metric is "mainly positive" $(-,+,\ldots,+)$ and our conventions for curvature are $R^a{}_{bcd} = \partial_c \Gamma^a_{bd}\ldots$ and $R^a{}_{bac} = R_{bc}$.

2. $\langle\phi^2\rangle$ in the spacetime of D-dimensional static and spherically-symmetric black hole

Formula (16) shows that the Hadamard–DeWitt coefficients can be used in a twofold way: Firstly, for a given dimension, the lowest coefficient of the expansion gives the leading approximation to the vacuum polarization, whereas the higher order coefficients give the higher-order terms in (16). On the other hand, we can confine ourselves only to the main approximation and use the coefficients in various dimensions. Moreover, for the conformally coupled fields with $\xi = (D-2)/(4D-4)$, one has a very interesting formula that relates the trace of the quantized stress-energy tensor and the vacuum polarization.

In this paper, we shall restrict our analyses to $4 \leq D \leq 7$ and use the first three nontrivial coefficients (a_2, a_3, and a_4) to calculate all the terms from Table I. Since the results for the higher-order terms as well as these for $D > 7$ are rather complicated, to prevent unnecessary proliferation of lengthy and not very illuminating formulas, they will be not presented here. (The only exception is Sec. 2.3).

TABLE I

The rows (from left to right) represent the dimension of the spacetime, the leading terms, the next-to-leading and the next-to-next-to-leading terms of the expansion (16).

D	1^{st}	2^{nd}	3^{rd}
4	a_2	a_3	a_4
5	a_2	a_3	a_4
6	a_3	a_4	
7	a_3	a_4	
8	a_4		
9	a_4		

As can be seen in Table I, the possible applications of the Hadamard–DeWitt coefficients are of course wider. Indeed, the coefficients a_i for $(i \geq 3)$ play a crucial role in calculations of the stress-energy tensor of the quantized massive fields in a large mass limit, giving the unique possibility to study the dependence of the quantum effects on the dimension of the background spacetime. In this case, the entries in Table I should be moved one column to the left, as the main approximation of the stress-energy tensor in $D = 4$ and 5 requires a_3, whereas a_4 is needed in the calculation of the main approximation in $D = 6$ and 7 (see Sec. 2.3).

If there are N scalar fields ϕ_i, each with a different mass m_i, then all formulas remain intact provided the following change is made:

$$\frac{1}{m^2} = \sum_{i=1}^{N} \frac{1}{m_i^2}\,. \tag{17}$$

This also shows that the quantum effects can be made great by taking large number of the quantized fields.

2.1. $D = 4$ and $D = 5$

Inspection of the general formula (16) shows that to calculate the vacuum polarization of the massive scalar field, one needs the coincidence limit of the Hadamard–DeWitt coefficient a_2, which is constructed from the curvature invariants $R_{abcd}R^{abcd}$, $R_{ab}R^{ab}$, R^2 and $\Box R$. Although it looks quite simple, the resulting expression for $\langle \phi^2 \rangle$ constructed for a general metric (2) is complicated. Indeed, taking a_2 in the form of [40]

$$a_2 = \frac{1}{180} R_{abcd}R^{abcd} - \frac{1}{180} R_{ab}R^{ab} + \frac{1}{6}\left(\frac{1}{5} - \xi\right) R_{;a}^{\ a} + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2\,, \tag{18}$$

after some algebra, one has

$$\left\langle \phi^2 \right\rangle_D = \frac{1}{K} \sum_{i=0}^{2} \sum_{k=1}^{13} \alpha_k^i \xi^i F_k(r)\,, \tag{19}$$

where

$$K = \begin{cases} 16\pi^2 m^2 & \text{if} \quad D = 4\,, \\ 32\pi^2 m & \text{if} \quad D = 5 \end{cases} \tag{20}$$

and the functions F_k (the same for both dimensions) as well as the dimension-dependent coefficients α_k^i are shown in Table II, see Appendix.

Since the general form of the vacuum polarization can easily be inferred form Table II, it will not be presented explicitly here. Instead, we shall discuss its behaviour in a few important regimes. First, let us consider the simplest case of the four-dimensional black holes. On general grounds, one expects that for the line element (3), the result falls as r^{-6} and the most interesting region is the vicinity of the event horizon. For the physical values of the coupling constant, *i.e.*, for $\xi = 0$ and $\xi = 1/6$, the general expression calculated at the event horizon reduces to

$$K\left\langle \phi^2 \right\rangle_4 = -\frac{f''}{18r_+^2} + \frac{1}{60}f''^2 - \frac{f'}{3r_+^3} + \frac{13f'^2}{45r_+^2} - \frac{1}{30}f^{(3)}f' - \frac{f'f''}{30r_+} + \frac{1}{15r_+^4} \tag{21}$$

and

$$K\left\langle \phi^2 \right\rangle_4 = \frac{1}{360}f''^2 + \frac{f'^2}{90r_+^2} - \frac{1}{180}f^{(3)}f' - \frac{f'f''}{30r_+} + \frac{1}{90r_+^4}\,, \tag{22}$$

respectively.

Now, let us assume that the black hole is extremal, *i.e.*, the event and the inner horizons coincide and analyze the vacuum polarization on the degenerate horizon. It means that $f(r_\pm) = f'(r_\pm) = 0$ and from the previous analysis, we know that the same result can be obtained calculating the vacuum polarization in the product spacetime with the maximally symmetric subspaces. Inspection of Table II gives the following expression for the vacuum polarization in the spacetime of the extreme black hole:

$$\begin{aligned} K\left\langle \phi^2 \right\rangle_4 &= \frac{1}{15r_\pm^4} + \xi^2\left(-\frac{2f''}{r_\pm^2} + \frac{1}{2}f''^2 + \frac{2}{r_\pm^4}\right) \\ &\quad + \xi\left(\frac{2f''}{3r_\pm^2} - \frac{1}{6}f''^2 - \frac{2}{3r_\pm^4}\right) + \frac{1}{60}f''^2 - \frac{f''}{18r_\pm^2}\,. \end{aligned} \tag{23}$$

If, additionally, the second derivative of the function f at the event horizon vanishes, one has the Plebański–Hacyan geometry with

$$K\left\langle \phi^2 \right\rangle_4 = \frac{2\xi^2}{r_+^4} - \frac{2\xi}{3r_+^4} + \frac{1}{15r_+^4}\,. \tag{24}$$

The Plebański–Hacyan solution is a product of the maximally-symmetric two-dimensional subspaces, such that one of them has zero curvature [41]. Topologically, it is either $M_2 \times S^{D-2}$ or $\mathrm{AdS}_2 \times E^{D-2}$, where M_2 is a two-dimensional Minkowski space and E^{D-2} is $(D-2)$-dimensional Euclidean space. Of course, here, we are interested in the former solution as the latter appears for the charged topological black holes.

Similarly, at the event horizon of the five-dimensional black hole, one has for the minimal coupling,

$$K\left\langle\phi^2\right\rangle_5 = -\frac{f''}{6r_+^2} + \frac{1}{60}f''^2 - \frac{4f'}{3r_+^3} + \frac{59f'^2}{120r_+^2} - \frac{1}{30}f^{(3)}f' - \frac{f'f''}{20r_+} + \frac{1}{2r_+^4} \tag{25}$$

and for the conformal coupling,

$$\begin{aligned} K\left\langle\phi^2\right\rangle_5 &= \frac{1}{64}\left(\frac{1}{2r_+^4} - \frac{f''(r)}{6r_+^2} + \frac{23}{120}f''^2 + \frac{5f'}{3r_+^3} - \frac{f'^2}{30r_+^2}\right. \\ &\left. -\frac{2}{15}f^{(3)}f' - \frac{17f'f''}{10r_+}\right). \end{aligned} \tag{26}$$

The vacuum polarization of the extremal black hole at the event horizon is given by

$$\begin{aligned} K\left\langle\phi^2\right\rangle_5 &= -\frac{f''}{6r_\pm^2} + \xi^2\left(-\frac{6f''}{r_\pm^2} + \frac{1}{2}f''^2 + \frac{18}{r_\pm^4}\right) \\ &+\xi\left(\frac{2f''}{r_\pm^2} - \frac{1}{6}f''^2 - \frac{6}{r_\pm^4}\right) + \frac{1}{60}f''^2 + \frac{1}{2r_\pm^4}, \end{aligned} \tag{27}$$

whereas for the ultraextremal configuration, one has

$$K\left\langle\phi^2\right\rangle_5 = \frac{18\xi^2}{r_\pm^4} - \frac{6\xi}{r_\pm^4} + \frac{1}{2r_\pm^4}. \tag{28}$$

Finally, let us return to the charged Tangherlini black holes and introduce new variables, x and β, defined as $x = r/r_+$ and $\beta = r_-/r_+$. Simple calculation gives

$$K\left\langle\phi^2\right\rangle_4 = \frac{1}{15r_+^4}\left[\frac{1}{x^6}\left(1+2\beta+\beta^2\right) - \frac{4}{x^7}\left(\beta+\beta^2\right) + \frac{13}{3x^8}\beta^2\right] \tag{29}$$

and

$$\begin{aligned} K\left\langle\phi^2\right\rangle_5 &= \frac{1}{5r_+^4}\left[\frac{1}{x^8}\left(2\beta^4 - 4\beta^2 + 40\beta^2\xi + 2\right)\right. \\ &+\frac{1}{x^{10}}\left(2\beta^4 + 2\beta^2 - 60\beta^4\xi - 60\beta^2\xi\right) \\ &\left.+\frac{1}{x^{12}}\left(-\frac{17\beta^4}{6} + 10\beta^4\xi^2 + \frac{230\beta^4\xi}{3}\right)\right]. \end{aligned} \tag{30}$$

The higher order terms of $\langle\phi^2\rangle_4$ constructed from a_3 and a_4 can be found in Ref. [16] and the results presented in this paragraph generalize those of Lemos and Thompson [24].

Inspection of Eq. (29) shows that the result (that is independent of ξ) is always nonnegative on the event horizon and it tends to 0^+ as $x \to \infty$, whereas in the ($D = 5$)-case, the vacuum polarization exhibits the more complicated dependence on the coupling parameter. The asymptotic behaviour of the latter is shown in Fig. 1. The vacuum polarization at the event horizon is always positive for the minimal and the conformal coupling.

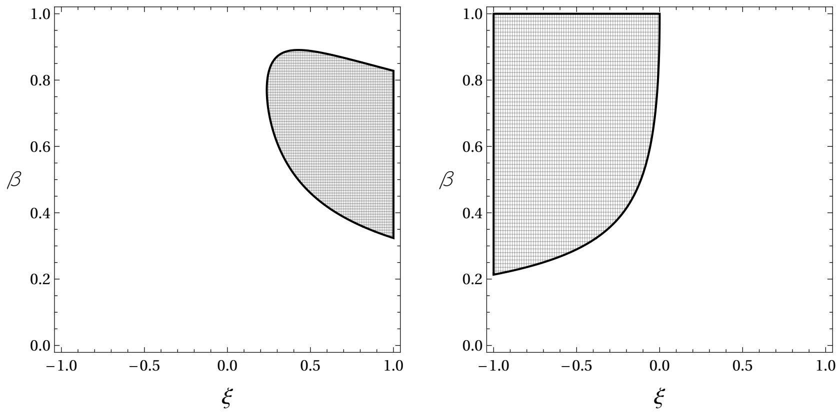

Fig. 1. The shaded regions in the (ξ, β)-space, in which the field fluctuation is negative at the event horizon (left panel). The shaded region in the right panel represents points with the property $\langle\phi^2\rangle_5 \to 0^-$ as $r \to \infty$.

Since the second derivative of f calculated at the event horizon of the extreme four- and five-dimensional Tangherlini black hole is, respectively, $f'' = 2/r_\pm^2$ and $f'' = 8/r_\pm^2$, from (23) and (27), one has

$$K \left\langle\phi^2\right\rangle_4 = \frac{1}{45 r_\pm^4} \tag{31}$$

and

$$K \left\langle\phi^2\right\rangle_5 = \frac{1}{r_\pm^4}\left(\frac{7}{30} - \frac{2}{3}\xi + 2\xi^2\right). \tag{32}$$

Equally well one can put $\beta = 1$ and $x = 1$ in (29) and (30). The same results can be obtained using the line element (7).

2.2. $D = 6$ and $D = 7$

In this section, we shall analyse the approximation to the field fluctuation of the quantized massive fields in $D = 6$ and $D = 7$. The coincidence limit of the Hadamard–DeWitt biscalar $a_3(x, x')$ is much more complicated than the coefficient a_2 and can be written in the form that is valid in any dimension [42]

$$a_3 = \frac{1}{7!} b_3^{(0)} + \frac{1}{360} b_3^{(\xi)}\,, \tag{33}$$

where

$$\begin{aligned} b_3^{(0)} \;=\;& \frac{35}{9} R^3 + 17 R_{;a} R^{;a} - 2 R_{ab;c} R^{ab;c} - 4 R_{ab;c} R^{ac;b} + 9 R_{abcd;e} R^{abcd;e} \\ & -8 R_{ab;c}{}^{c} R^{ab} - \frac{14}{3} R R_{ab} R^{ab} + 24 R_{ab;c}{}^{b} R^{ac} - \frac{208}{9} R_{ab} R_{c}{}^{a} R^{bc} \\ & + \frac{64}{3} R_{ab} R_{cd} R^{acbd} + \frac{16}{3} R_{ab} R_{cde}{}^{a} R^{becd} + \frac{80}{9} R_{abcd} R_{e}{}^{a}{}_{f}{}^{c} R^{bedf} \\ & + \frac{14}{3} R R_{abcd} R^{abcd} + 28 R R_{;a}{}^{a} + 18 R_{;a}{}^{a}{}_{b}{}^{b} + 12 R^{abcd}{}_{;e}{}^{e} R_{abcd} \\ & + \frac{44}{9} R_{abcd} R_{ef}{}^{ab} R^{cdef} \end{aligned} \tag{34}$$

and

$$\begin{aligned} b_3^{(\xi)} \;=\;& -5 R^3 \xi + 30 R^3 \xi^2 - 60 R^3 \xi^3 - 12 \xi R_{;a} R^{;a} + 30 \xi^2 R_{;a} R^{;a} - 22 R \xi R_{;a}{}^{a} \\ & -6 \xi R_{;a}{}^{a}{}_{b}{}^{b} - 4 \xi R_{;ab} R^{ab} + 2 R \xi R_{ab} R^{ab} - 2 R \xi R_{abcd} R^{abcd} \\ & + 60 R \xi^2 R_{;a}{}^{a}\,. \end{aligned} \tag{35}$$

A closer inspection of the coefficient a_3 shows that it is a sum of the curvature invariants constructed from the Riemann tensor, its covariant derivatives and contractions. In general, the coefficient $[a_n]$ (for a given spin) is a linear combination of the Riemann invariants and belongs to $\bigoplus_{q=1}^{n} \mathcal{R}^{0}_{2n,q}$, where $\mathcal{R}^{r}_{s,q}$ is a vector space of Riemannian polynomials of rank r (the number of free tensor indices), order s (number of derivatives) and degree q (number of factors). The type of the field is encoded in the coefficients of the linear combination.

Now, the vacuum fluctuation has a general form of

$$\langle \phi^2 \rangle_D = \frac{1}{K} \sum_{i=0}^{3} \sum_{k=1}^{36} \alpha_k^i \xi^i F_k(r)\,, \tag{36}$$

where

$$K = \begin{cases} 64\pi^3 m^2 & \text{if} \quad D = 6\,, \\ 128\pi^3 m & \text{if} \quad D = 7 \end{cases} \tag{37}$$

and the functions F_k and the (dimension-dependent) coefficients α_k are listed in Tables III–V, see Appendix. Once again, we shall not present the general result for $\langle \phi^2 \rangle_D$ as it can easily be obtained form the tables. Instead, we shall confine ourselves to the physically important limits.

Following the steps form the previous section, for the vacuum polarization at the event horizon of the minimally coupled field, one has

$$
\begin{aligned}
K\left\langle\phi^2\right\rangle_6 \;=\; & -\frac{29f''}{90r_+^4}+\frac{f''^2}{30r_+^2}-\frac{1}{630}f''^3-\frac{58f'}{15r_+^5}+\frac{116f'^2}{45r_+^4}-\frac{2f'^3}{315r_+^3} \\
& -\frac{f^{(3)}f'}{15r_+^2}-\frac{8f^{(3)}f'^2}{315r_+}-\frac{f'f''}{15r_+^3}+\frac{131f'^2f''}{630r_+^2}+\frac{2f'f''^2}{315r_+} \\
& -\frac{1}{140}f^{(4)}f'^2+\frac{74}{63r_+^6}+\frac{1}{210}f^{(3)}f'f'', \qquad (38)
\end{aligned}
$$

whereas the analogous result for the conformally coupled fields is given by

$$
\begin{aligned}
K\left\langle\phi^2\right\rangle_6 \;=\; & \frac{f''}{2250r_+^4}-\frac{f''^2}{750r_+^2}-\frac{2f''^3}{7875}-\frac{2f'}{375r_+^5}-\frac{16f'^2}{1125r_+^4}+\frac{2f'^3}{2625r_+^3} \\
& -\frac{4f^{(3)}f'^2}{525r_+}+\frac{7f'f''}{375r_+^3}-\frac{113f'^2f''}{15750r_+^2}-\frac{13f'f''^2}{7875r_+}+\frac{1}{700}f^{(3)}f'f'' \\
& -\frac{f^{(4)}f'^2}{2100}-\frac{74}{7875r_+^6}. \qquad (39)
\end{aligned}
$$

Usually, the quantum effects of the massive fields are most pronounced at the event horizon and its closest vicinity. For the extremal and ultraextremal configurations, one has

$$
\begin{aligned}
K\left\langle\phi^2\right\rangle_6 \;=\; & -\frac{29f''}{90r_\pm^4}+\frac{f''^2}{30r_\pm^2}-\frac{1}{630}f''^3+\frac{74}{63r_\pm^6} \\
& +\xi^3\left(\frac{72f''}{r_\pm^4}-\frac{6f''^2}{r_\pm^2}+\frac{1}{6}f''^3-\frac{288}{r_\pm^6}\right) \\
& +\xi^2\left(-\frac{36f''}{r_\pm^4}+\frac{3f''^2}{r_\pm^2}-\frac{1}{12}f''^3+\frac{144}{r_\pm^6}\right) \\
& +\xi\left(\frac{89f''}{15r_\pm^4}-\frac{8f''^2}{15r_\pm^2}+\frac{1}{60}f''^3-\frac{116}{5r_\pm^6}\right) \qquad (40)
\end{aligned}
$$

and

$$
K\left\langle\phi^2\right\rangle_6=-\frac{288\xi^3}{r_\pm^6}+\frac{144\xi^2}{r_\pm^6}-\frac{116\xi}{5r_\pm^6}+\frac{74}{63r_\pm^6}, \qquad (41)
$$

respectively.

Thus far our results have been valid for any static and spherically-symmetric metric. Now, let us consider the charged Tangherlini solution. Simple manipulations give

$$\begin{aligned}\left\langle\phi^2\right\rangle_6 \;=\; & \frac{1}{r_+^6}\left[\frac{1}{x^{12}}\left(15\rho-30\beta^3+280\beta^3\xi\right)\right.\\ & +\frac{\beta^3}{x^{18}}\left(\frac{907\rho}{105}-\frac{1250\beta^3}{21}+432\beta^3\xi^2+\frac{5448\rho\xi}{5}+\frac{21576\beta^3\xi}{5}\right)\\ & -\frac{\beta^6\eta}{x^{21}}\left(576\xi^2+\frac{16736\xi}{5}+\frac{88}{35}\right)\\ & +\frac{\beta^9}{x^{24}}\left(36\xi^3+702\xi^2+2392\xi+\frac{6761}{315}\right)\\ & \left.+\frac{\eta}{x^{15}}\left(-\frac{1333\beta^6\rho}{63}+\frac{3382\beta^3}{63}-1232\beta^3\xi\right)\right], \qquad (42)\end{aligned}$$

where $\eta = 1+\beta^3$ and $\rho = 1+\beta^6$. The sign of the vacuum polarization at the event horizon as well as its asymptotic behaviour as $r \to \infty$ is shown in Fig. 2. Specifically, the vacuum polarization at the event horizon is always negative for the minimal coupling. On the other hand, for the conformal coupling, it is positive for $0.683 < \xi < 0.762$.

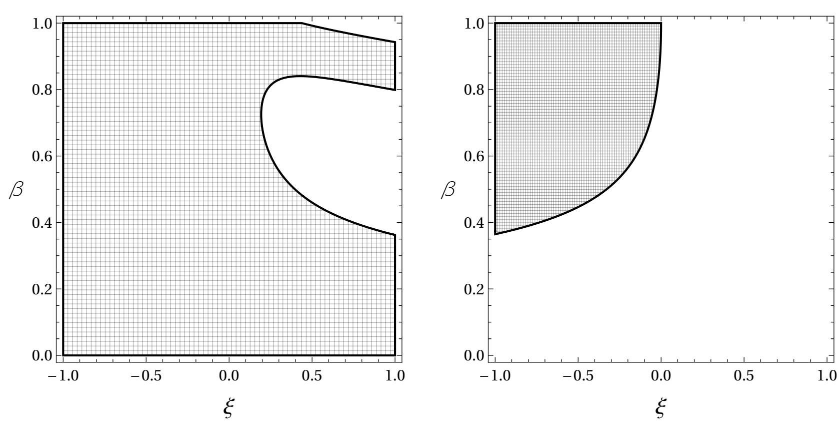

Fig. 2. Points within the shaded region represent values of the (ξ,β)-space for which the vacuum polarization is negative at the event horizon (left panel). The shaded region in the right panel represents points with the property $\langle\phi^2\rangle_6 \to 0^-$ as $r \to \infty$.

Now, let us analyse the results obtained for the 7-dimensional black hole. At the event horizon for the conformal and minimal coupling, one has

$$K\left\langle\phi^2\right\rangle_7 = \frac{f''}{192r_+^4} - \frac{7f''^2}{2304r_+^2} - \frac{653f''^3}{2903040} - \frac{f'}{32r_+^5} - \frac{13f'^2}{1728r_+^4} - \frac{221f'^3}{72576r_+^3} - \frac{23f^{(3)}f'^2}{4032r_+} + \frac{47f'f''}{1728r_+^3} - \frac{139f'^2f''}{16128r_+^2} - \frac{37f'f''^2}{96768r_+} + \frac{191f^{(3)}f'f''}{120960} - \frac{f^{(4)}f'^2}{5040} - \frac{f^{(3)}f'}{864r_+^2} - \frac{13}{1296r_+^6} \tag{43}$$

and

$$K\left\langle\phi^2\right\rangle_7 = -\frac{8f''}{9r_+^4} + \frac{f''^2}{18r_+^2} - \frac{1}{630}f''^3 - \frac{112f'}{9r_+^5} + \frac{199f'^2}{36r_+^4} + \frac{19f'^3}{126r_+^3} - \frac{f^{(3)}f'}{9r_+^2} - \frac{2f^{(3)}f'^2}{63r_+} - \frac{f'f''}{6r_+^3} + \frac{97f'^2f''}{336r_+^2} + \frac{f'f''^2}{126r_+} - \frac{1}{140}f^{(4)}f'^2 + \frac{1}{210}f^{(3)}f'f'' + \frac{16}{3r_+^6}\,, \tag{44}$$

respectively. On the other hand, the vacuum polarization for the extremal and ultraextremal black hole is given by

$$K\left\langle\phi^2\right\rangle_7 = -\frac{1}{630}f''^3 + \frac{16}{3r_\pm^6} + \frac{\xi^3\left(r_\pm^2f''-20\right)^3}{6r_\pm^6} - \frac{\xi^2\left(r_\pm^2f''-20\right)^3}{12r_\pm^6} + \frac{\xi\left(3r_\pm^4f''^2-100r_\pm^2f''+960\right)\left(r_\pm^2f''-20\right)}{180r_\pm^6} - \frac{8f''}{9r_\pm^4} + \frac{f''^2}{18r_\pm^2} \tag{45}$$

and

$$K\left\langle\phi^2\right\rangle_7 = -\frac{16(5\xi-1)^2(10\xi-1)}{3r_\pm^6}\,. \tag{46}$$

Finally, the result for the electrically charged Tangherlini black hole is

$$\begin{aligned}\langle\phi^2\rangle_7 \;=\; & \frac{1}{r_+^6}\left[\frac{1}{x^{14}}\left(\frac{360\rho}{7}-144\beta^4+1152\beta^4\xi+\frac{360}{7}\right)\right. \\ & +\frac{24\eta}{x^{18}}\left(\frac{109\beta^4}{7}-232\beta^4\xi-\frac{22}{7}\rho\right) \\ & +\frac{\beta^4}{x^{22}}\left(-\frac{940\rho}{7}-\frac{49688\beta^4}{63}+2640\beta^4\xi^2+5160\rho\xi+\frac{60944\beta^4\xi}{3}\right) \\ & +\frac{\eta\beta^8}{x^{26}}\left(-3600\xi^2-\frac{48400\xi}{3}+\frac{1240}{3}\right) \\ & \left.+\frac{\beta^{12}}{x^{30}}\left(288\xi^3+4416\xi^2+\frac{175072\xi}{15}-\frac{60544}{315}\right)\right], \end{aligned} \tag{47}$$

where $\eta = 1+\beta^4$ and $\rho = 1+\beta^8$. The sign of the vacuum polarization at the event horizon as well as its asymptotic behaviour as $r\to\infty$ is shown in Fig. 3. A closer examination shows that the vacuum polarization at the event horizon is always negative for the minimal coupling, whereas for the conformal coupling, it is positive for $0.697<\xi<0.825$. Note qualitative similarity of the results presented in Figs. 4 and 3.

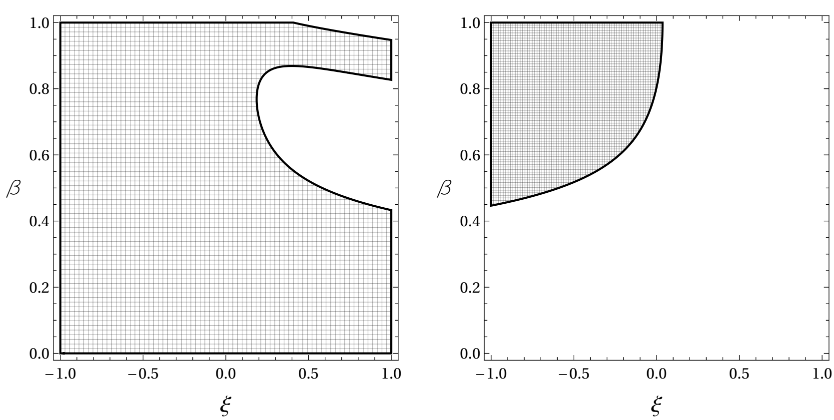

Fig. 3. Points within the shaded region represent values of the (ξ,β)-space for which the vacuum polarization is negative at the event horizon (left panel). The shaded region in the right panel represents points with the property $\langle\phi^2\rangle_7\to 0^-$ as $r\to\infty$.

In order to shed some light on this problem, we made analogous calculations in $D=8$ and $D=9$. In both cases, there is a qualitative similarity with the $(D=5)$-case. Moreover, at the event horizon, one can observe some sort of qualitative complementarity, *i.e.*, the shaded and unshaded regions

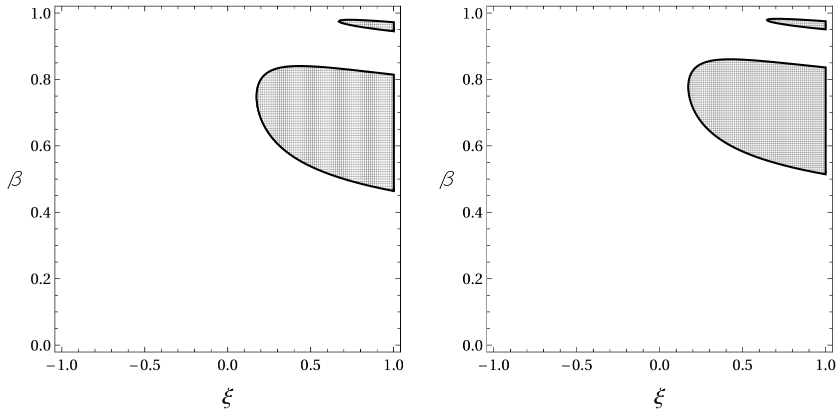

Fig. 4. Points within the shaded region represent values of the (ξ, β)-space for which the vacuum polarization is negative at the event horizon of the $(D = 8)$-dimensional (lef panel) and $(D = 9)$-dimensional (right panel) charged Tangherlini black hole.

in the (ξ, β)-space for the six- and seven-dimensional Tangherlini black holes become unshaded and shaded for $D = 8$ and 9 (see Fig. 4). Thus, roughly speaking, for $4 \leq D \leq 9$, the shape of the regions of (ξ, β)-space for which the vacuum polarization at the event horizon is negative (positive) depends on the coefficients a_i rather than on the dimension. On the other hand, the regions characterized by the condition $\lim_{r\to\infty}\langle\phi^2\rangle_D \to 0^-$ do not depend on the dimension of the Tangherlini black hole and look qualitatively the same. Finally, observe that treated as a function of the radial coordinate, the vacuum polarization rapidly falls to zero, possibly with a few local extrema located near the event horizon. In Fig. 5, we have plotted the vacuum polarization of the minimally coupled massive scalar field in the spacetime of the neutral Tangherlini black hole. A comparison with the results presented in Refs. [28, 29] reveals qualitative similarity in the behaviour of $\langle\phi^2\rangle$ for massive and massless fields.

We also remark that the vacuum polarization on the degenerate horizon of the extremal black hole can easily be calculated using the line element (7) describing the spacetimes with the maximally symmetric subspaces[2]. Because of the symmetries, this approach is especially useful for the higher-dimensional black holes.

[2] We have checked that these two methods give precisely the same results.

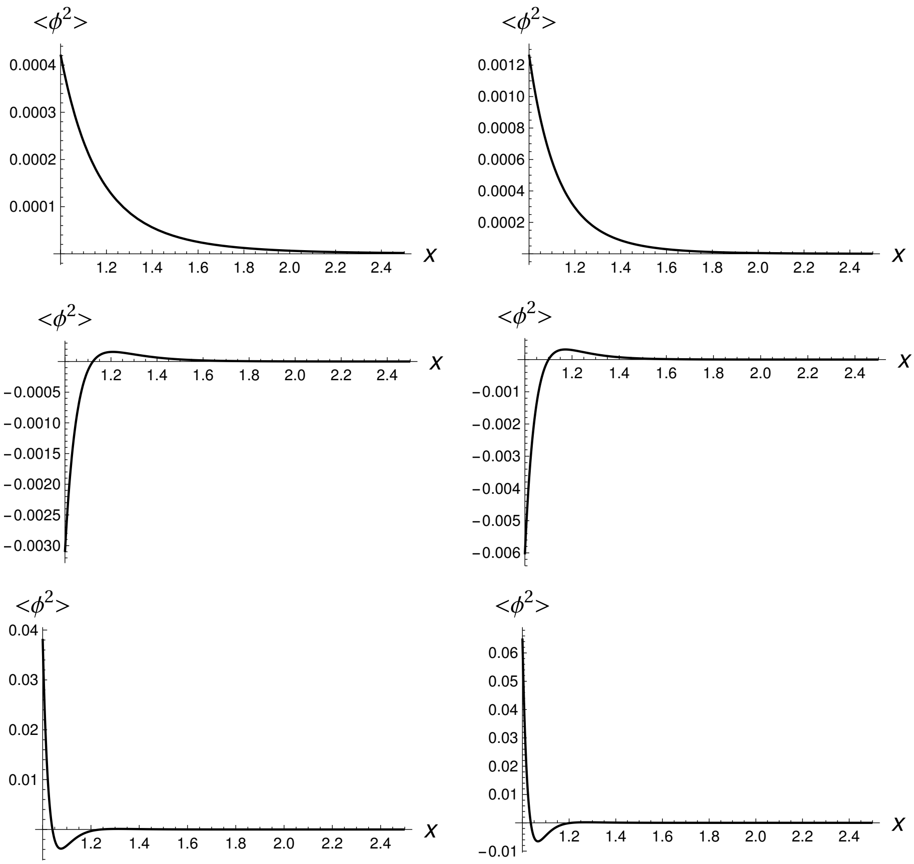

Fig. 5. This graph shows $\langle\phi^2\rangle$ of the massive scalar field ($\xi = 0$) as a function of $x = r/r_+$ in a spacetime of the neutral Tangherlini black holes for $D = 4, 6, 8$ (left column) and $D = 6, 7, 9$ (right column).

2.3. Trace of the stress-energy tensor of the conformally coupled massive fields

The one-loop effective action constructed from Green function (12) is given by

$$W^{(1)} = \int \mathrm{d}^D x (-g)^{1/2} \mathcal{L}\,, \tag{48}$$

where

$$\mathcal{L} = \frac{1}{2(4\pi)^{D/2}} \sum_{k=\lfloor \frac{D}{2} \rfloor + 1}^{N} \frac{a_k}{(m^2)^{k-D/2}} \Gamma\left(k - \frac{D}{2}\right)\,, \tag{49}$$

and the stress-energy tensor can be calculated using the standard formula

$$T^{ab} = \frac{2}{(-g)^{1/2}} \frac{\delta}{\delta g_{ab}} W^{(1)} \,. \tag{50}$$

It should be noted that the total divergences that are present in the effective action can be discarded. For example, when calculating the stress-energy tensor in $D = 4$ and $D = 5$, the number of the curvature invariants can be reduced from 28 to 10.

For the conformally coupled fields, one has interesting relations between the trace of the stress-energy tensor and the field fluctuation [5, 43]. Indeed, provided $\xi = (D-2)/(4D-4)$, we have

$$\langle T_a^a \rangle_D = \begin{cases} \mathfrak{C}_D - m^2 \langle \phi^2 \rangle_D & \text{for} \quad D\text{-even} \,, \\ \frac{1}{2} m \mathfrak{C}_D - m^2 \langle \phi^2 \rangle_D & \text{for} \quad D\text{-odd} \,, \end{cases} \tag{51}$$

where $\mathfrak{C}_D$ is given by

$$\mathfrak{C}_D = \frac{a_{\lfloor D/2 \rfloor}}{(4\pi)^{\lfloor D/2 \rfloor}} \,. \tag{52}$$

This relations can be explained as follows, for a conformally coupled classical massless fields, the trace of the stress-energy tensor is zero, whereas for the conformally coupled massive fields, the trace is $-m^2 \langle \phi^2 \rangle$. On the other hand, after quantization in even dimensions, the trace of the massles field acquires anomalous value, and thus combining these facts, one expects that some relations between the anomalous trace, the vacuum polarization and the trace of the stress-energy tensor should hold. For the Schwinger–DeWitt expansion, the first-order term cancels with the "anomalous term" and the next-to-leading term is precisely the first-order approximation to the trace. Similarly, the next-to-leading term of the trace is equal to the next-to-next-to-leading term of the vacuum polarization $\langle \phi^2 \rangle_D$. It should be noted that although the calculation of the trace of the stress energy with the aid of Eq. (51) requires prior knowledge of the next-to-leading terms of the field fluctuation (which are expressed in terms of the Hadamard–DeWitt coefficients), it is still much more simple than the computations of the functional derivatives of the action with respect to the metric tensor. Moreover, it can be regarded as a useful check of the calculations.

Using (50), we have constructed the stress-energy tensor of the massive quantized fields in $4 \le D \le 7$ for the general static and spherically-symmetric spacetime. Additionally, for $D = 4$ and 5, we have also calculated the next-to-leading terms. On the other hand, we have calculated the first three terms of the expansion of the field fluctuation in $D = 4$ and $D = 5$ and the first two terms in $D = 6$ and $D = 7$ (see Table I), and demonstrated the validity of Eq. (51) for the general static and spherically-symmetric metric. Since the final results are complicated and not very illuminating, here we present only the trace of the stress-energy tensor in the spacetime of the (uncharged) Tangherlini black hole. Regardless of the adapted method, after some algebra, one has

$$\langle T_a^a\rangle_4 = -\frac{r_+^2(81r-97r_+)}{4032\pi^2m^2r^9} - \frac{r_+^2\left(420r^2-1125rr_+ + 727r_+^2\right)}{2100\pi^2m^4\,r^{12}}, \tag{53}$$

$$\langle T_a^a\rangle_5 = -\frac{r_+^4\left(243r^2-323r_+^2\right)}{5040\pi^2mr^{12}} - \frac{r_+^4\left(1152r^4-3704r^2r_+^2+2727r_+^4\right)}{2240\pi^2m^3r^{16}}, \tag{54}$$

$$\langle T_a^a\rangle_6 = -\frac{r_+^6\left(22680r^6-81970r^3r_+^3+65041r_+^6\right)}{10080\pi^3m^2r^{20}} \tag{55}$$

and

$$\langle T_a^a\rangle_7 = -\frac{r_+^8\left(320r^8-1256r^4r_+^4+1047r_+^8\right)}{128\pi^3mr^{24}}. \tag{56}$$

The first term in the right-hand side of Eqs. (53) and (54) has been calculated from a_3, whereas the second-order terms from a_4. Similarly, the first order terms in Eqs. (55) and (56) have been calculated from a_4. To calculate the next-to-leading term, one needs the coefficient a_5.

Since for the spherically-symmetric black hole we have three algebraically-independent components of the stress-energy tensor, the covariant conservation equation and knowledge of the trace reduce their number to one. It suffices, therefore, to calculate one component of the stress-energy tensor to reconstruct the remaining ones easily.

We conclude this section with a few remarks on the significance of the presented results, especially in the context of the calculations of the stress-energy tensor. The most important are: *(i)* such calculations are surely possible to execute for ($4 \leq D \leq 7$), *(ii)* the components of T_a^b rapidly decrease with r, *(iii)* the oscillatory character of the components of the stress-energy tensor (caused by existence of local extrema) is confined to the close vicinity of the event horizon, *(iv)* the calculations can be extended to the topological black holes, *(v)* interesting back reaction effect near the event horizon are expected as we have strong evidence that the energy density ($\rho = -T_t^t$) of the quantum field may be negative.

3. Final remarks

The Schwinger–DeWitt method gives unique possibility to study the quantum effects in various dimensions. Moreover, as the sole criterion for its applicability is demanding that the Compton length associated with the field be small, with respect to the characteristic radius of the curvature of the background geometry, the Schwinger–DeWitt approach is also quite robust. In practice, it turns out that the reasonable results can be obtained for $Mm > 2$, where M is the mass of the black hole [44]. A comparison made in Refs. [24, 45] between the numerical and analytical results confirms the accuracy of the Schwinger–DeWitt method.

In this paper, using the generalized Schwinger–DeWitt approach, we have calculated the vacuum polarization effects of the quantized massive scalar field in the spacetime of the D-dimensional $(4 \leq D \leq 7)$ static and spherically-symmetric black hole. A special emphasis has been put on the charged Tangherlini solutions. Contrary to the simplest Reissner–Nordström case, the vacuum polarization for the charged Tangherlini black hole depends on ξ and a ratio r_-/r_+ in a quite complicated way, as expected. Our results can also be used to construct $\langle\phi^2\rangle$ when a cosmological constant is present, as, for example, in a spacetime of the lukewarm black hole. The calculations reported in this paper can also be generalized to the case of topological black holes.

Since the geometry of the closest vicinity of the extremal black hole is a direct product of the two maximally symmetric spaces $\mathrm{AdS}_2 \times S^{D-2}$, it is possible to calculate $\langle\phi^2\rangle_D$ at the degenerate horizon without referring to the black hole metric. Because of massive simplifications in the product space, the thus constructed result is relatively simple to obtain and may serve as an important check of the calculations.

For the conformally coupled field, we have investigated the relation between the trace of the stress-energy tensor and the vacuum polarization. It should be emphasized that the calculation of the trace from the vacuum polarization is far more efficient than the calculations of the trace from the stress-energy tensor, even though the next-to-leading terms of the approximation of $\langle\phi^2\rangle_D$ are needed.

Finally, we briefly describe our calculational strategy. First, we have constructed the Hadamard–DeWitt coefficients for a general D-dimensional metric (2). The hard part of the calculations has been carried out using FORM [46] (a well-known program in high-energy physics), whereas massive simplifications have been performed in Mathematica. Further, the functional derivatives of the general effective action (constructed from the complicated algebraic and differential curvature invariants) with respect to the metric tensor have been calculated using the fast FORM code. The results have been checked against the analogous results obtained from the time and radial Euler–Lagrange equations. The remaining independent component of the stress-energy tensor has been constructed with the aid of the covariant conservation equation.

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Appendix

The general results

In this appendix, we present in a tabular form our general results for the massive quantized field in the spacetime of the static and spherically-symmetric black holes. Making use of formulas (19) and (36) and the informations contained in the tables, the vacuum polarization $\langle\phi^2\rangle_D$ can be easily reconstructed.

TABLE II

The functions $F_k(r)$ and the coefficients α_k^i of the massive scalar field in $(D = 4)$ and $(D = 5)$-dimensional static, spherically-symmetric black hole.

		$D = 4$			$D = 5$		
k	$F_k(r)$	α_k^0	α_k^1	α_k^2	α_k^0	α_k^1	α_k^2
1	$\frac{1}{r^4}$	$1/15$	$-2/3$	2	$1/2$	-6	18
2	$\frac{f}{r^4}$	0	$2/3$	-4	-1	12	-36
3	$\frac{f^2}{r^4}$	$-1/15$	0	2	$1/2$	-6	18
4	$\frac{f'}{r^3}$	$-1/3$	$10/3$	-8	$-4/3$	14	-36
5	$\frac{ff'}{r^3}$	$7/15$	-4	8	$26/15$	-16	36
6	$\frac{f'^2}{r^2}$	$13/45$	-3	8	$59/120$	-6	18
7	$\frac{f''}{r^2}$	$-1/18$	$2/3$	-2	$-1/6$	2	-6
8	$\frac{ff''}{r^2}$	$-1/90$	$-1/3$	2	$-7/30$	0	6
9	$\frac{f'f''}{r}$	$-1/30$	$-2/3$	4	$-1/20$	-1	6
10	f''^2	$1/60$	$-1/6$	$1/2$	$1/60$	$-1/6$	$1/2$
11	$\frac{ff^{(3)}}{r}$	$-1/5$	1	0	$-3/10$	$3/2$	0
12	$f^{(3)}f'$	$-1/30$	$1/6$	0	$-1/30$	$1/6$	0
13	$ff^{(4)}$	$-1/30$	$1/6$	0	$-1/30$	$1/6$	0

TABLE III

The functions $F_k(r)$ of the quantized massive field in the spacetime of the static and spherically-symmetric black holes in $D = 6$ and 7.

k	$F_k(r)$	k	$F_k(r)$	k	$F_k(r)$
1	$\frac{1}{r^6}$	13	$\frac{f^2 f''}{r^4}$	25	$\frac{f^{(3)} f'^2}{r}$
2	$\frac{f}{r^6}$	14	$\frac{f' f''}{r^3}$	26	$\frac{f f^{(3)} f''}{r}$
3	$\frac{f^2}{r^6}$	15	$\frac{f f' f''}{r^3}$	27	$f^{(3)} f' f''$
4	$\frac{f^3}{r^6}$	16	$\frac{f'^2 f''}{r^2}$	28	$f f^{(3)2}$
5	$\frac{f'}{r^5}$	17	$\frac{f''^2}{r^2}$	29	$\frac{f f^{(4)}}{r^2}$
6	$\frac{f f'}{r^5}$	18	$\frac{f f''^2}{r^2}$	30	$\frac{f^2 f^{(4)}}{r^2}$
7	$\frac{f^2 f'}{r^5}$	19	$\frac{f' f''^2}{r}$	31	$\frac{f f^{(4)} f'}{r}$
8	$\frac{f'^2}{r^4}$	20	f''^3	32	$f^{(4)} f'^2$
9	$\frac{f f'^2}{r^4}$	21	$\frac{f f^{(3)}}{r^3}$	33	$f f^{(4)} f''$
10	$\frac{f'^3}{r^3}$	22	$\frac{f^2 f^{(3)}}{r^3}$	34	$\frac{f^2 f^{(5)}}{r}$
11	$\frac{f''}{r^4}$	23	$\frac{f^{(3)} f'}{r^2}$	35	$f f^{(5)} f'$
12	$\frac{f f''}{r^4}$	24	$\frac{f f^{(3)} f'}{r^2}$	36	$f^2 f^{(6)}$

TABLE IV

The coefficients α_k of the quantized massive field in the spacetime of the static and spherically-symmetric black holes in $D = 6$.

k	α_k^0	α_k^1	α_k^2	α_k^3
1	74/63	−116/5	144	−288
2	−58/15	356/5	−432	864
3	58/15	−356/5	432	−864
4	−74/63	116/5	−144	288
5	−58/15	952/15	−336	576
6	164/15	−2384/15	752	−1152
7	−778/105	1456/15	−416	576
8	116/45	−204/5	216	−384
9	−1103/315	2224/45	−236	384
10	−2/315	184/45	−112/3	256/3
11	−29/90	89/15	−36	72
12	−1/45	−68/15	52	−144
13	65/126	−11/5	−16	72
14	−1/15	−44/15	36	−96
15	683/315	−232/15	4	96
16	131/630	−64/45	−14/3	32
17	1/30	−8/15	3	−6
18	89/630	−109/45	7	6
19	2/315	−1/15	−2/3	4
20	−1/630	1/60	−1/12	1/6
21	−11/15	116/15	−20	0
22	92/105	−42/5	20	0
23	−1/15	11/15	−2	0
24	149/630	−229/45	56/3	0
25	−8/315	−8/45	4/3	0
26	−4/315	−29/45	10/3	0
27	1/210	−1/20	1/6	0
28	1/840	−1/60	1/12	0
29	−1/15	11/15	−2	0
30	−5/42	2/15	2	0
31	−44/315	16/45	4/3	0
32	−1/140	1/30	0	0
33	−1/420	−1/60	1/6	0
34	−2/35	4/15	0	0
35	−1/70	1/15	0	0
36	−1/280	1/60	0	0

TABLE V

The coefficients α_k of the quantized massive field in the spacetime of the static and spherically-symmetric black holes in $D = 7$.

k	α_k^0	α_k^1	α_k^2	α_k^3
1	16/3	−320/3	2000/3	−4000/3
2	−4/3	1120/3	−400/3	4000
3	80/3	−1280/3	6800/3	−4000
4	−32/3	160	−800	4000/3
5	−112/9	1880/9	−3400/3	2000
6	104/3	−4622/9	2500	−4000
7	−5843/252	1855/6	−4100/3	2000
8	199/36	−1697/18	1600/3	−1000
9	−2659/504	569/6	−1625/3	1000
10	19/126	71/12	−200/3	500/3
11	−8/9	148/9	−100	200
12	−4/3	2/9	340/3	−400
13	803/252	−127/6	−40/3	200
14	−1/6	−17/3	220/3	−200
15	1093/252	−209/6	25	200
16	97/336	−53/24	−20/3	50
17	1/18	−8/9	5	−10
18	95/504	−23/6	35/3	10
19	1/126	−1/12	−5/6	5
20	−1/630	1/60	−1/12	1/6
21	−14/9	149/9	−130/3	0
22	793/504	−599/36	130/3	0
23	−1/9	11/9	−10/3	0
24	61/252	−8/9	30	0
25	−2/63	−2/9	5/3	0
26	−1/63	−29/36	25/6	0
27	1/210	−1/20	1/6	0
28	1/840	−1/60	1/12	0
29	−1/9	11/9	−10/3	0
30	−115/504	13/36	10/3	0
31	−11/63	4/9	5/3	0
32	−1/140	1/30	0	0
33	−1/420	−1/60	1/6	0
34	−1/14	1/3	0	0
35	−1/70	1/15	0	0
36	−1/280	1/60	0	0

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