SPECTRAL ANALYSIS ON SWANSON'S HAMILTONIAN

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In the original work of non-Hermitian Swanson's Hamiltonian and subsequent Hermitian counterpart of the same, it has been shown that the only condition $\omega - \alpha - \beta > 0$ reflects positive energy eigenvalues. However, we find that the Swanson Hamiltonian reflects both positive as well as negative energy under the same condition ($\omega - \alpha - \beta > 0$). In order to complete the work, we also discuss the wave function corresponding to negative energy.

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1. Introduction

In 2004, Swanson [1] has considered an interesting non-Hermitian quadratic Hamiltonian as

$$H = \omega\left(a^\dagger a + \tfrac{1}{2}\right) + \alpha a^2 + \beta\left(a^\dagger\right)^2 , \tag{1}$$

where α, β, ω are real constants and $\alpha \neq \beta$. Here, a and $a^\dagger$ are annihilation and creation operators of harmonic oscillator respectively. The author [1] proposed that the above Hamiltonian will produce real and positive eigenvalues provided that

$$\omega^2 - 4\alpha\beta \geq 0 . \tag{2}$$

Swanson's Hamiltonian has been used as a model to investigate the non-Hermitian systems by several authors [2–15]. Jones [3] showed that the Hamiltonian given in Eq. (1) with the condition

$$\omega > \alpha + \beta \tag{3}$$

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can be written as

$$H = \frac{p^2}{2}(\omega - \alpha - \beta) + \frac{\left(\omega^2 - 4\alpha\beta\right)}{(\omega - \alpha - \beta)}\frac{x^2}{2}\,. \tag{4}$$

Let us consider

$$h = \frac{2H}{(\omega - \alpha - \beta)} = p^2 + \frac{\left(\omega^2 - 4\alpha\beta\right)}{(\omega - \alpha - \beta)^2}x^2\,, \tag{5}$$

and re-write it as

$$h = p^2 + \omega_0^2 x^2 \tag{6}$$

with

$$\omega_0^2 = \frac{\left(\omega^2 - 4\alpha\beta\right)}{(\omega - \alpha - \beta)^2}\,, \tag{7}$$

whose energy eigenvalues and eigenfunctions are

$$E_n = (2n+1)\omega_0 \tag{8}$$

and

$$\psi_n(x) = \left[\frac{\sqrt{\omega_0}}{\sqrt{\pi}2^n n!}\right]^{\frac{1}{2}} e^{-\frac{\omega_0 x^2}{2}} H_n\left(\sqrt{\omega_0}x\right)\,, \tag{9}$$

where $H_n(\sqrt{\omega_0}x)$ is the Hermite polynomial. If one will consider the simultaneous non-Hermitian transformation of momentum and coordinate proposed by Rath and Mallick [16], then the energy eigenvalue of the transformed Hamiltonian of the Hamiltonian in Eq. (6) would lead to negative spectrum. The concept of negative energy in harmonic oscillator under simultaneous transformation of momentum and co-ordinate has been discussed by Fernandez [17] and Rath [18]. Further, the concept of negative energy in bosons has been discussed by Nielsen and Ninomiya [19]. Providencia *et al.* [20] have discussed Swanson's Hamiltonian considering the operator

$$H_\theta = \frac{p^2}{2}(1 - i\tan 2\theta) + \frac{x^2}{2}(1 + i\tan 2\theta)\,, \tag{10}$$

where θ is confined in the range of $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, and using the condition, $S^{-1}H_\theta S = H$, noticed that H can be transformed to the same Hermitian operator as obtained by Jones as

$$H = \omega\left(a^\dagger a + \tfrac{1}{2}\right)\,, \tag{11}$$

where $\omega > 0$. However, the authors of [20] have not discussed negative energy *i.e.* $\omega < 0$. On the other hand, authors have shown that another definite operator with negative potential energy,

$$H_\alpha = p^2 - \frac{\gamma(\gamma-1)}{\cosh^2(x - i\alpha)}\,, \tag{12}$$

can possess negative energy. We also noticed that Bagarello *et al.* [21] have discussed D-deformed harmonic oscillator following the relation

$$abf - baf = f\,, \tag{13}$$

where $f \in D$, and used the orthonormality relation

$$\langle \phi_n | \psi_m \rangle = \delta_{mn}\,, \tag{14}$$

where $|\phi_n\rangle$ is the basis of harmonic oscillator. One is able to construct D-deformed harmonic oscillator Hamiltonians using a transformation of the type $\psi = U\phi$ (see Eq. (3.11) in [21]). However, one is not able to construct D-deformed Hamiltonians if $\psi \neq \phi$ and $\psi \neq U\phi$. The paper motivates for constructing new operators satisfying the condition as stated earlier. In addition, we find that Bagarello *et al.* [22] have constructed non-self adjoint operator relating to position and momentum and hence discussed about the positive energy spectra.

The above works [17–22] inspire us to test its validity in Swanson's model. However, in this work, we apply the simultaneous non-Hermitian transformation of momentum and co-ordinate to Swanson's Hamiltonian and notice that the system can admit negative energy.

2. Non-Hermitian transformation of co-ordinate and momentum

Now, apply the non-Hermitian transformation proposed by Rath and Mallick [16] to define the co-ordinate, x, and momentum, p, as

$$x \to \frac{x + i\lambda p}{\sqrt{(1+\delta\lambda)}} \tag{15}$$

and

$$p \to \frac{p + i\delta x}{\sqrt{(1+\delta\lambda)}}\,. \tag{16}$$

With the above transformation, the Hamiltonian defined in Eq. (6) can be transformed to

$$h = \frac{1}{(1+\lambda\delta)}\left[p^2\left(1-\lambda^2\omega_0^2\right) + x^2\left(\omega_0^2 - \delta^2\right) + i\left(\lambda\omega_0^2 + \delta\right)(xp + px)\right]\,. \tag{17}$$

In order to solve the above Hamiltonian, we use the second quantization formalism [18] to define

$$p = i\sqrt{\frac{W}{2}}\left(a^\dagger - a\right) \tag{18}$$

and

$$x = \frac{\left(a^\dagger + a\right)}{\sqrt{2W}}\,, \tag{19}$$

where W is the unknown parameter. Here, $a^\dagger$ and a satisfy the commutation relation as

$$\left[a, a^\dagger\right] = 1\,. \tag{20}$$

Using Eqs. (18) and (19), we can rewrite Eq. (17) as

$$h = h_D + h_N\,, \tag{21}$$

where

$$h_D = \frac{\left(2a^\dagger a + 1\right)}{(1+\lambda\delta)}\left[\left(1 - \lambda^2\omega_0^2\right)\frac{W}{2} + \frac{\left(\omega_0^2 - \delta^2\right)}{2W}\right] \tag{22}$$

and

$$h_N = Ua^2 + V\left(a^\dagger\right)^2 \tag{23}$$

with

$$U = \frac{1}{(1+\lambda\delta)}\left[-\frac{W}{2}\left(1 - \lambda^2\omega_0^2\right) + \frac{\left(\omega_0^2 - \delta^2\right)}{2W} + \left(\delta + \lambda\omega_0^2\right)\right] \tag{24}$$

and

$$V = \frac{1}{(1+\lambda\delta)}\left[-\frac{W}{2}\left(1 - \lambda^2\omega_0^2\right) + \frac{\left(\omega_0^2 - \delta^2\right)}{2W} - \left(\delta + \lambda\omega_0^2\right)\right]. \tag{25}$$

Now, we consider the zero energy correction method introduced by Rath and Mallick [16] to solve the above Hamiltonian. In this formalism, one need to choose the coefficient of the appropriate operator *i.e.* either the coefficient of a^2 or $(a^\dagger)^2$ such that it is equal to zero and the non-diagonal terms of the Hamiltonian give zero contribution to the energy eigenvalues.

2.1. Case-I: For $V = 0$

If one considers the coefficient of $(a^\dagger)^2$ to be zero, then one will find that the value of W is

$$W_1 = \frac{-\left(\delta + \lambda\omega_0^2\right) \pm (\omega_0 + \lambda\omega_0\delta)}{\left(1 - \lambda^2\omega_0^2\right)}\,. \tag{26}$$

If we take the positive sign while calculating W_1, then we will get

$$W_{1+} = \frac{(\omega_0 - \delta)}{(1 + \lambda\omega_0)}\,. \tag{27}$$

Substituting the value of W_{1+} in Eq. (21) and using perturbation theory [16, 23–28], one can easily show that all orders of energy corrections, $E_n^{(m)}$, will be zero. Here, we find that the energy eigenvalues of the Hamiltonian (Eq. (21)) are the same as that of the original Hamiltonian (Eq. (6)) which is $E_n = (2n + 1)\omega_0$. If we consider the negative sign of Eq. (26) while calculating W_1, then we will get

$$W_{1-} = \frac{(\omega_0 + \delta)}{(\lambda\omega_0 - 1)}\,. \tag{28}$$

Substituting Eq. (28) into Eq. (21) and using perturbation theory, we find that the energy eigenvalues of the Hamiltonian (Eq. (21)) are

$$E_n = -(2n + 1)\omega_0 \tag{29}$$

and the wave function is

$$\psi_n(x) = \left[\frac{\sqrt{W_{1-}}}{\sqrt{\pi}2^n n!}\right]^{\frac{1}{2}} e^{-\frac{W_{1-}x^2}{2}} H_n\left(\sqrt{W_{1-}}x\right) \tag{30}$$

with

$$\langle\psi_n|\psi_n\rangle = 1\,. \tag{31}$$

The wave function of the Hamiltonian corresponding to the negative energy using perturbation theory will be of the form of

$$\Psi_n^k = \sum_{k=0}^{k}(-1)^k\left[\frac{(\lambda + \delta)}{(1 + \lambda\delta)}\right]^k\sqrt{\frac{(n + 2k)!}{n!}}|\psi_{n+2k}\rangle_{W_{1-}}\,. \tag{32}$$

The normalization condition and eigenvalue relation for the above case can be written as

$$\left\langle\psi_n|\Psi_n^k\right\rangle = 1 \tag{33}$$

and

$$\langle\psi_n|h|\Psi_n^k\rangle = E_n = -(2n + 1)\omega_0\,, \tag{34}$$

respectively.

2.2. Case-II: For $U = 0$

If one considers the coefficient of a^2 as zero, then one will find that the value of W is

$$W_2 = \frac{-\left(\delta + \lambda\omega_0^2\right) \pm \left(\omega_0 + \lambda\omega_0\delta\right)}{\left(\lambda^2\omega_0^2 - 1\right)} . \tag{35}$$

If we take the negative sign while calculating W_2, then we will get

$$W_{2-} = \frac{(\omega_0 + \delta)}{(1 - \lambda\omega_0)} . \tag{36}$$

Substituting Eq. (36) into Eq. (21) and using perturbation theory, we find that the energy eigenvalues of the Hamiltonian (Eq. (21)) is

$$E_n = (2n + 1)\omega_0 . \tag{37}$$

If we take the positive sign while calculating W_2, then we will get

$$W_{2+} = \frac{(\delta - \omega_0)}{(1 + \lambda\omega_0)} . \tag{38}$$

Substituting the value of W_{2+} (Eq. (38)) in Eq. (21) and using perturbation theory, we find that the energy eigenvalues of the Hamiltonian (Eq. (21)) is

$$E_n = -(2n + 1)\omega_0$$

which is the same as Eq. (29). For this case, the wave function corresponding to the negative energy will be

$$\phi_n(x) = \left[\frac{\sqrt{W_{2+}}}{\sqrt{\pi}2^n n!}\right]^{\frac{1}{2}} e^{-\frac{W_{2+}x^2}{2}} H_n\left(\sqrt{W_{2+}}x\right) \tag{39}$$

with

$$\langle\phi_n|\phi_n\rangle = 1 . \tag{40}$$

The wave function of the Hamiltonian corresponding to negative energy using perturbation theory will be of the form of

$$\Phi_n^k = \sum_{k=0}^{k} \left[\frac{(\lambda + \delta)}{(1 + \lambda\delta)}\right]^k \sqrt{\frac{n!}{2^k k!(n - 2k)!}} |\phi_{n-2k}\rangle_{W_{2+}} . \tag{41}$$

The normalization condition and eigenvalue relation for the above case can be written as

$$\left\langle \phi_n | \Phi_n^k \right\rangle = 1 \tag{42}$$

and

$$\langle \phi_n | h | \Phi_n^k \rangle = E_n = -(2n+1)\omega_0 \,, \tag{43}$$

respectively.

In both the cases, we have seen the appearance of negative energy eigenvalues of the transformed Hamiltonian. The values of negative energy for the specific values of α, β and ω satisfying the condition in Eq. (2) are reflected in Tables I and II.

TABLE I

The values of α, β, ω, ω_0 and corresponding negative energy eigenvalues with the condition, $\omega - \alpha - \beta = 1$.

α	β	ω	$\omega_0 = \sqrt{\frac{(\omega^2 - 4\alpha\beta)}{(\omega-\alpha-\beta)^2}}$	$E_n = -(2n+1)\omega_0$
4	3	8	4	$-4\,(2n+1)$
16	15	32	8	$-8\,(2n+1)$
25	24	50	10	$-10\,(2n+1)$

TABLE II

The values of α, β, ω, ω_0 and corresponding negative energy eigenvalues with the condition, $\omega - \alpha - \beta = 2$.

α	β	ω	$\omega_0 = \sqrt{\frac{(\omega^2 - 4\alpha\beta)}{(\omega-\alpha-\beta)^2}}$	$E_n = -(2n+1)\omega_0$
3	2	7	$\frac{5}{2}$	$-\frac{5}{2}\,(2n+1)$
10	9	21	$\frac{9}{2}$	$-\frac{9}{2}\,(2n+1)$
21	20	43	$\frac{13}{2}$	$-\frac{13}{2}\,(2n+1)$

3. Discussion and conclusion

The observation of negative energy by applying simultaneous transformation on harmonic oscillator has been reported by Fernandez [17] and Rath [18]. Rath [18] reported the occurrence of negative energy with convergent wave function for the case of positive frequency of oscillation, whereas Fernandez [17] showed the occurrence of negative energy with diverse wave function for the case of negative frequency of oscillation. However, here we apply simultaneous transformation to Swanson's Hamiltonian and our study indicated that if we consider either $V = 0$ or $U = 0$, then we find that the energy eigenvalues of the transformed Hamiltonian can be negative even though the condition (Eq. (2)) for getting positive energy proposed by

Swanson [1] is satisfied. Our study clearly indicated the violation of only positive energy condition of Swanson's Hamiltonian. The only constraint in our study is that one needs to choose the appropriate parameter in such a way that the values of W_1 or W_2 will be positive for each case which gives the corresponding well behaved wave function. It is worth mentioning here that one can get a complex energy value if and only if V and/or U are non-zero. This would lead to spontaneous breakdown of PT-symmetry. Similar type of work has already been reported [29].

Interesting point in the findings is that the normalized condition [23]: $\langle\psi_n|\Psi_n^k\rangle = 1 = \langle\phi_n|\Phi_n^k\rangle$ allows one to express $\Psi_n^k \neq S_\psi\psi_n$ or $\Phi_n^k \neq S_\phi\phi_n$. As S_ψ and S_ϕ cannot be written explicitly, hence, one will be restricted to constructing pseudo-bosons following the work of Bagarello and others [21, 22, 30].

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