ANTIFERROMAGNETIC ISING MODEL IN THE FRAMEWORK OF RIEMANNIAN GEOMETRY

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A metric is introduced on the three-dimensional space of two long-range sublattice order parameters and a short-range order parameter describing the Ising antiferromagnets in the Bethe approximation. The Riemannian geometry associated with this metric is investigated analytically. In terms of the equilibrium order parameters, thermodynamic curvature scalar ($\mathcal{R}$) is derived and its temperature (T) dependence near the Néel transition temperature ($T_{\rm N}$) is analysed. A divergence to infinity is observed for the curvature on both sides of the Néel temperature ($\mathcal{R} \to \infty$) which can be scaled as $\mathcal{R} \sim \epsilon^{\lambda}$ for $T < T_{\rm N}$, and $\mathcal{R} \sim (-\epsilon)^{\lambda'}$ for $T > T_{\rm N}$, with $\lambda = \lambda' = -2$ and $\epsilon = 1 - T/T_{\rm N}$. These observations fit well with those in the calculations of thermodynamic curvature in other spin models such as the spherical model and the ferromagnetic Ising model.

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1. Introduction

Riemannian geometric theory of thermodynamics and statistical mechanics is nowadays successfully applied in diverse research areas [1–9]. At the beginning, these applications are limited to simple and well-known spin models. Particularly, the authors started with the one-dimensional Ising model, for which it was possible to calculate the thermodynamic curvature expression exactly [9–16]. The main result was that divergence of the curvature occurs only at the zero-temperature and zero-field critical point of the model. Since this exact result is known, it is a good starting point for testing the geometry and curvature of the Ising-type mean-field model in the lowest order approximation. Indeed, the same singularity of curvature was observed at a nonzero Curie temperature [10–12]. Starting from these observations, the geometrical analysis has been extended to other Ising systems on various lattice structures [17–20].

Most of the analysis done in the past dealt with the thermodynamic curvature at or near the criticality of spin models with ferromagnetic interactions. Moreover, the role of thermodynamic curvature at characterizing the resulting magnetism in the antiferromagnetic phase has recently been shown [20]. However, almost nothing is known about the curvature concerned primarily with antiferromagnets having the Néel temperature (a temperature above which an antiferromagnetic material becomes paramagnetic). The simplest theory showing phase transitions and critical phenomena in the antiferromagnets is the spin-1/2 antiferromagnetic Ising model (AFIM). It has been widely used to understand the physical phenomena that occur in some antiferromagnetic compounds [21–23].

A simple solution of the spin-1/2 AFIM was first presented by Barry and Harrington [24]. Considering the magnetic Gibbs energy surface in the statistical Bethe (or pair) approximation and using the equilibrium conditions, they obtained two equations (also known as the Bethe equations) for the order parameters. In advanced years, the Bethe approximation (a useful tool for studying spin systems [25–27]) was also performed to calculate phase diagrams of AFIM in various lattice geometries [28–30].

Although both the Bethe approximation and the mean-field theory give the same critical behaviour near the transition temperatures, the key difference is that the former takes into account the short-range spin correlations which has the significant role in understanding the temperature dependence of the magnetic properties. Hence, in the AFIM, there exist three order parameters, two long-range sublattice order parameters and one short-range order parameter. In the present work, we firstly introduce a three-dimensional manifold with local coordinates using three order parameters for the antiferromagnetic Ising model and define a metric on this manifold. Using the metric components determined from the second derivatives of the Bethe-type magnetic Gibbs energy relation of Ref. [24], the curvature scalar of the order parameter manifold is determined and its behaviour near the Néel temperature is investigated analytically. A similar treatment has recently been done by us for the geometrical analysis of ferromagnetic Ising model in the pair approximation [31].

The rest of this article is organized as follows. In Section 2, the antiferromagnetic Ising model and its equilibrium (or static) properties in the Bethe approximation are reviewed. In Section 3, the Riemannian geometrical structure is introduced and the thermodynamic curvature $\mathcal{R}$ is calculated. The behaviour of $\mathcal{R}$ near the Néel temperature is shown. Finally, some concluding remarks are presented in Section 4.

2. Antiferromagnetic Ising model and equilibrium properties

This section is devoted to solution of the antiferromagnetic Ising model based on the Bethe approximation. In particular, we briefly review thermal behaviour of long- and short-range order parameters in the vicinity of the Néel transition temperature. These behaviours will be reflected in geometrical properties of the spin system in the next section.

2.1. Description of the model and magnetic Gibbs energy in the Bethe approximation

One of the most important models introduced for ferromagnetism is the Ising model [32]. It is an assembly of N spins which are localized on lattice points. Each spin has a magnetic moment of μ_0 and can only be up (positive) or down (negative) direction along the z-axis. The Hamiltonian of the system is described in terms of the energy coupling constant (J) as

$$\mathcal{H} = \frac{J}{2} \sum_{\langle ij \rangle} s_i s_j \,, \tag{1}$$

where $s_i = \pm 1$, $J < 0$ and $\langle ij \rangle$ indicates a sum over all $Nz/2$ nearest-neighbour pairs of lattice sites (z is the lattice coordination number, being the number of nearest-neighbour spins surrounding any spin). When $J > 0$, the model is called the AFIM and the interactions are between nearest neighbours on a bipartite lattice. A bipartite lattice (such as the square, honeycomb, and body centered cubic lattices) is one which can be divided into two sublattices, which we call A and B, such that an A site has only B neighbours, and a B site has only A neighbours.

In the Bethe approximation, the spin–spin configurational interaction energy and the total magnetic moment are defined by $E = -\frac{1}{4}NzJ\sigma$ and $M_{\mathrm{tot}} = \frac{1}{2}N\mu_0(r_1 - r_2)$, respectively. Here, $r_1 = \frac{2}{N}\sum_{i\in \boldsymbol{A}} s_i$ and $r_2 = -\frac{2}{N}\sum_{j\in \boldsymbol{B}} s_j$ are called the sublattice long-range order parameters, while $\sigma = -\frac{2}{Nz}\sum_{\langle ij\rangle} s_i s_j$ is called the short-range order parameter. The statistical expression for the entropy of the system is given by the formula $S = k_{\mathrm{B}} \ln \Omega(r_1, r_2, \sigma)$, where k_{B} is the Boltzmann constant and $\Omega(r_1, r_2, \sigma)$ is the number of ways of arranging the Ising spins consistent with the order parameters r_1, r_2, σ. The logarithm of this quantity is given by [33]

$$\ln \Omega(r_1, r_2, \sigma) = N \ln 2 + \frac{1}{4}N(z-1)\sum_{i=1}^{4} x_i \ln x_i - \frac{1}{8}Nz\sum_{j=1}^{4} y_j \ln y_j \,. \tag{2}$$

In order to simplify the notation and calculations, we have used the abbreviations $x_1 = 1 + r_1$, $x_2 = 1 - r_1$, $x_3 = 1 + r_2$, $x_4 = 1 - r_2$, $y_1 = 1 + r_1 - r_2 - \sigma$,

$y_2 = 1 - r_1 + r_2 - \sigma$, $y_3 = 1 + r_1 + r_2 + \sigma$, $y_4 = 1 - r_1 - r_2 + \sigma$. A good reference for the description of properties of this model is Barry and Harrington [24], whose notation is used here. Assuming the spin system to be placed in an external magnetic field (H) along the z-axis and to be in thermal contact with a heat bath having temperature T, the magnetic Gibbs energy ($\psi = E - TS - HM_{\mathrm{tot}}$) of the system may be written in the Bethe approximation as [24]

$$\psi(r_1, r_2, \sigma) = -\frac{1}{4}NzJ\sigma - \frac{1}{2}N\mu_0 H(r_1 - r_2) - k_{\mathrm{B}}T \ln \varOmega(r_1, r_2, \sigma)\,. \quad (3)$$

2.2. Magnetic Gibbs energy minimization and solutions at equilibrium

The equilibrium conditions $\frac{\partial\psi}{\partial r_1} = 0$, $\frac{\partial\psi}{\partial r_2} = 0$, $\frac{\partial\psi}{\partial\sigma} = 0$ result in the following coupled transcendental equations (also called the equations of state):

$$\begin{aligned} 2(z-1)\ln\left(\frac{x_1}{x_4}\right) &= z\ln\left(\frac{y_1 y_3}{y_2 y_4}\right) - \frac{4\mu_0 H}{k_{\mathrm{B}}T}\,, \\ 2(z-1)\ln\left(\frac{x_3}{x_4}\right) &= z\ln\left(\frac{y_2 y_3}{y_1 y_4}\right) + \frac{4\mu_0 H}{k_{\mathrm{B}}T}\,, \\ \ln\left(\frac{y_1 y_2}{y_3 y_4}\right) &= -\frac{2J}{k_{\mathrm{B}}T}\,. \end{aligned} \quad (4)$$

Setting $H = 0$, the equilibrium conditions offer the solutions $r_1 = r_2 = r_0$ and $\sigma = \sigma_0$, where r_0 and σ_0 are determined from the following equations:

$$\begin{aligned} 2(z-1)\ln\left(\frac{1+r_0}{1-r_0}\right) &= z\ln\left(\frac{1+2r_0+\sigma_0}{1-2r_0+\sigma_0}\right), \\ \ln\left[\frac{(1-\sigma_0)^2}{(1+2r_0+\sigma_0)(1-2r_0+\sigma_0)}\right] &= -\frac{2J}{k_{\mathrm{B}}T}\,, \end{aligned} \quad (5)$$

which are also called the Bethe equations. For temperatures below the Néel transition temperature T_{N}, these equilibrium values can be written conveniently in terms of the Bethe long-range order parameter δ through the relations [24]

$$\begin{aligned} r_0 &= \tanh z\delta\,, \\ \sigma_0 &= 1 - 2\frac{\sinh(z-2)\delta}{\sinh(2z-2)\delta\cosh z\delta}\,, \end{aligned} \quad (6)$$

where the temperature dependence of δ is given by

$$\exp\left(-J/k_{\mathrm{B}}T\right) = \frac{\sinh(z-2)\delta}{\sinh z\delta}\,, \quad (7)$$

and the relation between δ and the temperature slightly below the Néel temperature (T_{N}) is approximately written as [24]

$$\frac{2}{3}(z-1)\delta^2 = \frac{J\epsilon}{k_{\mathrm{B}}T_{\mathrm{N}}}\,, \tag{8}$$

where $\epsilon = \frac{T_{\mathrm{N}}-T}{T_{\mathrm{N}}}$ is the distance from the Néel temperature. On the other hand, for temperatures above T_{N}, the equilibrium order parameter solutions r_0 and σ_0 become, respectively,

$$\begin{aligned} r_0 &= 0\,, \\ \sigma_0 &= \tanh\left(\frac{J}{k_{\mathrm{B}}T}\right)\,. \end{aligned} \tag{9}$$

In order to analytically examine a physical quantity for temperatures just below T_{N}, one may use the following series expansions for (6):

$$\begin{aligned} r_0 &= z\delta + \mathcal{O}\left(\delta^3\right)\,, \\ \sigma_0 &= \frac{1}{z-1}\left[1 + \frac{1}{3}z(z-2)(3z-2)\delta^2 + \mathcal{O}\left(\delta^4\right)\right]\,, \end{aligned} \tag{10}$$

and for temperatures slightly above T_{N}, one may use the following series expansions for (9):

$$\begin{aligned} r_0 &= 0\,, \\ \sigma_0 &= \frac{1}{z-1}\left[1 - \frac{z^2-2z}{2z-2}\left(\ln\frac{z}{z-2}\right)\varepsilon \right. \\ &\quad \left. + \frac{z}{4}\ln\frac{z}{z-2}\left(2z-2-\ln\frac{z}{z-2}\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)\right]\,, \end{aligned} \tag{11}$$

where $\varepsilon = -\epsilon$. We see that the order at long distances reaches zero ($r_0 \to 0$), whereas for the order of neighbours, the short-range order remains finite, *i.e.*, $\sigma_0 = 1/(z-1)$, as $T \to T_{\mathrm{N}}$ on both sides. In Ref. [24], the above results for the order parameters were reflected in several dynamical properties of the same spin system. For example, using the ideas in Eqs. (10) and (11), the behaviours of relaxation times and dynamic susceptibility expressions near the Néel temperature were investigated analytically. Similarly, we have used the same analysis for understanding the main geometric characteristics of the system in the next section.

3. Riemannian geometrical structure and the thermodynamic curvature scalar

3.1. Riemannian geometry of a thermodynamic state space

Here, we briefly give some basics of the Riemannian geometry for the thermodynamic state space. Denoting by θ^i $(i = 1, 2, \ldots, n)$ various thermodynamic parameters, a metric (or a line element) in equilibrium thermodynamic state space is defined by Ruppeiner [11]

$$\mathrm{d}s^2 = G_{ij}\mathrm{d}\theta^i\mathrm{d}\theta^j \,, \tag{12}$$

where G_{ij} are the components of a covariant metric tensor given by

$$G_{ij} = -\beta\partial_i\partial_j\phi \,. \tag{13}$$

Here, $\beta = 1/k_{\mathrm{B}}T$, $\partial_i = \partial/\partial\theta^i$ and $\phi = \Phi/N$ (Φ is a thermodynamic potential). In terms of the metric elements G_{ij}, the Christoffel symbols are found by the formula

$$\Gamma^i_{jk} = \tfrac{1}{2}G^{il}\left(\partial_k G_{lj} + \partial_j G_{lk} - \partial_l G_{jk}\right) \,, \tag{14}$$

where G^{il} are the components of contravariant tensor. The curvature tensor may be written in terms of the Christoffel symbols as

$$R^i_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk} \,. \tag{15}$$

Then, the Ricci tensor is defined by

$$R_{ij} = R^n_{inj} \,, \tag{16}$$

and after another contraction of the Ricci tensor indexes, follows the Ricci curvature scalar

$$\mathcal{R} = G^{ij}R_{ij} \,. \tag{17}$$

This curvature scalar measures the complexity of a system and plays a central role in any attempt to look at phase transitions from geometrical perspective. The case $\mathcal{R} = 0$ corresponds to a flat geometry and a non-interacting model. When $\mathcal{R} > 0$ or $\mathcal{R} < 0$, the metric is not flat and the model is interacting.

3.2. Thermodynamic curvature scalar for the antiferromagnetic Ising model

For an Ising antiferromagnet in an external magnetic field, we parametrize a three-dimensional Riemann manifold by $(\theta^1, \theta^2, \theta^3) = (r_1, r_2, \sigma)$ and choose the thermodynamic potential as the magnetic Gibbs energy ($\Phi = \psi$). In this case, the components of the metric tensor are found from Eq. (3) as follows:

$$\begin{aligned}
G_{11} &= -\beta\frac{\partial^2\phi}{\partial r_1^2} = \frac{z-1}{4}\left(\frac{1}{x_1}+\frac{1}{x_2}\right) - \frac{z}{8}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right),\\
G_{12} &= G_{21} = -\beta\frac{\partial^2\phi}{\partial r_1\partial r_2} = -\frac{z}{8}\left(-\frac{1}{y_1}-\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right),\\
G_{13} &= G_{31} = -\beta\frac{\partial^2\phi}{\partial r_1\partial \sigma} = -\frac{z}{8}\left(-\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}-\frac{1}{y_4}\right),\\
G_{22} &= -\beta\frac{\partial^2\phi}{\partial r_2^2} = \frac{z-1}{4}\left(\frac{1}{x_3}+\frac{1}{x_4}\right) - \frac{z}{8}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right),\\
G_{23} &= G_{32} = -\beta\frac{\partial^2\phi}{\partial r_2\partial \sigma} = -\frac{z}{8}\left(\frac{1}{y_1}-\frac{1}{y_2}+\frac{1}{y_3}-\frac{1}{y_4}\right),\\
G_{33} &= -\beta\frac{\partial^2\phi}{\partial \sigma^2} = -\frac{z}{8}\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{1}{y_4}\right),
\end{aligned} \tag{18}$$

where $\phi = \psi/N$. With our metric tensor (18), due to Eqs. (14)–(17), after lengthy computations, we have reached the following simple expression for the curvature scalar $\mathcal{R}$ in terms of the known equilibrium values of $r_1 = r_2 = r_0$ and $\sigma = \sigma_0$:

$$\mathcal{R} = \frac{ar_0^8 + br_0^6 + cr_0^4 + dr_0^2 + e}{\left(fr_0^4 + gr_0^2 + h\right)^2}, \tag{19}$$

where the z- and σ_0-dependent coefficients are listed below:

$$\begin{aligned}
a &= -3z^3 + 6z^2,\\
b &= 10z^3\sigma_0 - 2z^3 - 20z^2\sigma_0 + 12z^2 + 10z\sigma_0 - 30z + 8,\\
c &= -11z^3\sigma_0^2 + 4z^3\sigma_0 + 22z^2\sigma_0^2 + z^3 - 24z^2\sigma_0 - 11z\sigma_0^2\\
&\quad -10z^2 + 36z\sigma_0 + 35z - 16\sigma_0 - 8,\\
d &= 4z^3\sigma_0^3 - 2z^3\sigma_0^2 - 8z^2\sigma_0^3 - 2z^3\sigma + 12z^2\sigma_0^2 + 4z\sigma_0^3\\
&\quad +20z^2\sigma_0 - 18z\sigma_0^2 - 34z\sigma_0 + 8\sigma_0^2 - 12z + 16\sigma_0,\\
e &= z^3\sigma_0^2 - 10z^2\sigma_0^2 + 17z\sigma_0^2 - 8\sigma_0^2 + 3z,\\
f &= z^2 - 2z,\\
g &= -2z^2\sigma_0 + 4z\sigma_0 - 2\sigma_0 + 2,\\
h &= z^2\sigma_0^2 - 2z\sigma_0^2 + \sigma_0^2 - 1.
\end{aligned} \tag{20}$$

In order to analytically investigate the behaviour of the curvature just below $T_{\rm N}$, we ignore the higher order terms in Eqs. (10) and insert the remaining parts into (19) to obtain

$$\mathcal{R}_- = \frac{3}{4}\left(\frac{p\delta^8 + q\delta^6 + r\delta^4 + s\delta^2 + t}{(u\delta^2 + v)z^2(w\delta^2 - 3)\delta^4}\right), \tag{21}$$

with the coefficients defined by

$$\begin{aligned} p &= 39z^9 - 116z^8 + 110z^7 - 32z^6\,, \\ q &= -141z^7 + 312z^6 - 228z^5 + 48z^4\,, \\ r &= 192z^5 - 339z^4 + 222z^3 - 48z^2\,, \\ s &= -117z^3 + 162z^2 - 72z\,, \\ t &= 27z - 27\,, \\ u &= 4z^4 - 14z^3 + 14z^2 - 4z\,, \\ v &= -3z^2 + 9z - 6\,, \\ w &= 4z^2 - 2z\,. \end{aligned} \tag{22}$$

Similarly, using the series expansions (11), (19) may be rewritten for temperatures slightly above $T_{\rm N}$ as

$$\begin{aligned} \mathcal{R}_+ = &\frac{1}{(y^2\varepsilon^3 - 2xy\varepsilon^2 + (x^2 + 2y)\varepsilon - 2x)^2\varepsilon^2} \\ &\times \left[(y^2z - 8y^2)\varepsilon^4 + (-2xyz + 16xy)\varepsilon^3 + (x^2z - 8x^2 + 2yz - 16y)\varepsilon^2\right. \\ &\left. + (-2xz + 16x)\varepsilon + 4z - 8\right], \end{aligned} \tag{23}$$

where the coefficients are as follows:

$$\begin{aligned} x &= \frac{z^2 - 2z}{2z - 2}\ln\frac{z}{z-2}\,, \\ y &= \frac{z}{4}\ln\frac{z}{z-2}\left[2z - 2 - \ln\frac{z}{z-2}\right]. \end{aligned} \tag{24}$$

From Eqs. (21) and (23), one can conclude that when z is a finite number, corresponding to a different lattice structure in solid state physics, $\mathcal{R}$ is always positive and tends to plus infinity ($\mathcal{R} \to \infty$) as $T \to T_{\rm N}$ from either below or above with only one exception, namely, that of linear chain ($z = 2$). The above curvature anomalies near the Néel transition temperature can be verified analytically via the critical-point exponents which are frequently found by determining the slopes of log–log plots of a calculated data [34]. Motivated by the similar calculations on the various physical properties presented in most references [35–37], we can also calculate the critical-point exponent for the thermodynamic curvature scalar in the antiferromagnetic phase (or below the Néel transition temperature) using

$$\lambda = \lim_{\delta \to 0} \frac{1}{2} \left(\frac{\ln \mathcal{R}_-}{\ln \delta} \right) , \tag{25}$$

and in the paramagnetic phase (or above T_{N}) from

$$\lambda' = \lim_{\epsilon \to 0} \frac{\ln \mathcal{R}_+}{\ln(-\epsilon)} . \tag{26}$$

These definitions are valid for all values of λ and λ' for which the negative values corresponding to the divergences of the variables $\mathcal{R}_-$ and $\mathcal{R}_+$ as δ and ϵ, respectively, go to zero. Using Eq. (21) in (25), it is found that for $z > 2$, the curvature in the antiferromagnetic phase ($\mathcal{R}_-$) follows an ϵ^{λ} law with $\lambda = -2$. Similarly, inserting Eq. (23) into (26), it is observed that the curvature in the paramagnetic phase just above the transition point scales as $\mathcal{R}_+ \sim (-\epsilon)^{\lambda'}$ with the same exponent value ($\lambda' = -2$).

It is of great interest to compare the above scaling results seen in the vicinity of T_{N} of the AFIM with the scaling law results of the curvature calculations in other spin models. Firstly, we have already shown the temperature dependence of $\mathcal{R}$ in the ferromagnetic Ising model near the Curie temperature using the Bethe approximation [31]. A similar calculation as in (25) yields the law $\mathcal{R} \sim \epsilon^{-2}$ around the Curie temperature, which is not included in Ref. [31]. From the standard scaling assumptions [4, 12–15, 18, 19], it is known that the curvature scalar can be scaled as $\mathcal{R} \sim \epsilon^{\alpha-1}$ when $\alpha < 0$ or $\mathcal{R} \sim \epsilon^{\alpha-2}$ with $\alpha \geqslant 0$, where α is the standard exponent characterizing the scaling of the specific heat. It should be noted from Ref. [30] that the specific heat displays no singular part but shows pronounced peaks corresponding to the Néel temperatures. This property is expressed as $\alpha = 0$ for the AFIM in pair approximation and the scaling behaviour of $\mathcal{R}$ is, therefore, $\sim \epsilon^{\alpha-2}$ which is the same as $\alpha > 0$. Hence, setting $\alpha = 0$, we reach a similar scaling of the curvature for the antiferromagnetic Ising model, $\mathcal{R} \sim \epsilon^{-2}$. These general results near the Néel temperature are also in a good agreement with the specific case of the spherical model associated with the fact that the specific heat exponent for $d \geqslant 4$ dimensions vanishes and the model attains the mean-field character [4]. However, the spherical model displays no transition for $d = 1$ and $d = 2$ and a transition for $d = 3$ with $\alpha = -1$. In this case, it has also be shown in Ref. [4] that $\mathcal{R} \sim \epsilon^{-2}$ rather than the expected $\mathcal{R} \sim \epsilon^{-3}$ as in the case of Ising model on planar random graphs [18].

For magnetic models, the scaling behaviour of the curvature in the vicinity of the critical point is also related to that of the correlation volume: $\mathcal{R} \sim \xi^d$, where $\xi = \epsilon^{-\nu}$ is the correlation length, ν is the correlation length exponent and d is the dimensionality of the system. The standard relation for the curvature exponent in terms of d is $\lambda = -d\nu = \alpha - 2$ which demonstrates the critical behaviour of $\mathcal{R}$ [13, 18, 19]. One simple example of this definition is the one-dimensional Ising model. It has been calculated for $d = 1$ that $\xi \sim e^{2\beta}$ ($\beta = 1/k_{\mathrm{B}}T$) near the zero temperature criticality with $\alpha = \nu = 1$, as expected [9]. This corresponds to $\lambda = -1$. Similarly, despite the absence of a phase transition in zero magnetic field, the calculated curvature expression of one-dimensional Potts model yields a divergence at zero temperature with the law $\mathcal{R} \sim \epsilon^{-1}$ (or $\xi \sim e^{\beta}$) [14]. The exponents are again $\alpha = \nu = 1$. However, for the Ising model in 2 dimensions $\alpha = 0$, while $\nu = 1$. As in 1-dimensional case, the same scaling form of curvature ($\sim \epsilon^{-1}$) was observed for a kagome Ising model in $d = 2$ [19]. This behaviour was expressed by $\mathcal{R} \sim \epsilon^{\alpha-1}$ which is similar to $\alpha < 0$ of the spherical model studied in [4] and $d = 2$ Ising model on planar random graphs studied in [18]. Here, we can define $\lambda = -d\nu + 1 = \alpha - 1$. In contrast, when $d = 4$ (also known as the upper critical dimension), the mean-field theory begins to take over and thereafter for all $d > 4$, the curvature exponent becomes $\lambda = -2$ with $\alpha = 0$ and $\nu = 1/2$ [10].

4. Conclusion

In this work, the curvature scalar $\mathcal{R}$ has been derived for the antiferromagnetic Ising model in the Bethe approximation using a three-dimensional order-parameter manifold. We have only focused on the behaviour in the vicinity of the Néel temperature. It is found that, similar to behaviour of the ferromagnetic Ising model near the Curie temperature [31], $\mathcal{R}$ has also a sigularity at the Néel temperature and increases rapidly with increasing temperature and diverges to infinity on both sides of the temperature T_{N}. From the critical-point exponents calculated for the cases $T < T_{\mathrm{N}}$ and $T > T_{\mathrm{N}}$, we see that the same scaling relation $\mathcal{R} \sim \epsilon^{-2}$ is valid, corresponding to the expected specific heat exponent of $\alpha = 0$, where the scaling behaviour $\mathcal{R} \sim \epsilon^{\alpha-2}$ is also verified. The calculated values of curvature exponent λ reported in this study coincide in other spin models with $d \geqslant 4$ dimensions although there is no physical relation between them. Hence, our results provide another example of statistical approximation in which the curvature of thermodynamic metric diverges at the critical point. As a final remark, we hope that given any thermodynamic potential which is written using any statistical approximate techniques, explicit calculations on geometrical properties of n-dimensional order parameter manifolds would be welcome, particularly in any spin model with a finite-temperature phase transitions.

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