2-CONNECTIONS, A LATTICE POINT OF VIEW*

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We show that the transition laws for a 2-connection can be recovered by discretizing the base 2-space of a 2-bundle into an Euclidean hypercubic lattice. The aim of this work is to serve as an example of how important results in higher gauge theory, which have been derived in a continuous setting, can also be derived in the lattice scheme.

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1. Introduction

In the ordinary gauge theory built upon a G-principal bundle $E \xrightarrow{p} B$, a connection A describes parallel transport of point particles along paths. This connection can be locally seen as a Lie(G)-valued 1-form on the base space B, hence it associates a group element $\mathsf{hol}_A(\gamma) \in G$ to each path γ of the space, called the holonomy of A along γ. In this way, group elements become associated to paths of the space. We call a configuration or coloring of this space a given choice of such associations. In the lattice gauge theory, the base space is discretized into an Euclidean hypercubic lattice with lattice spacing a, physical laws are recovered by taking the limit $a \to 0$ (see *e.g.* [1] and references therein).

On the other hand, transformations of extended objects like strings cannot be described using such a connection, since strings move along surfaces, whereas point particles move along paths, gauge theories need to be extended to include connections that can describe parallel transport of both point particles and strings along paths and surfaces, respectively. There

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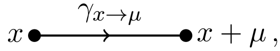
will certainly be some interplay between these two kinds of transformations, and this should be handled by the extended theory. This theory is called higher gauge theory [2] and is the extension of gauge theory in the language of higher category theory, which is well-suited to deal with such problems. The higher gauge theory is based on generalizations of spaces, groups, bundles and connections to, respectively, 2-spaces, 2-groups, 2-bundles and 2-connections using the so-called enrichment and internalization processes.

Our main goal is to give a simple example of application of lattice technics to higher gauge theory in order to recover the transformation laws for a 2-connection. Although these laws have already been derived in a continuous setting [3, 4], formulating higher gauge theory on a lattice has its own benefit: it may be applicable to computer-based numerical simulations (see *e.g.* [5] and references therein). Let us recall that passing to the lattice formalism has proven to be a crucial step for the computer-based numerical simulations of gauge theories in the past (see *e.g.* [6] and references therein).

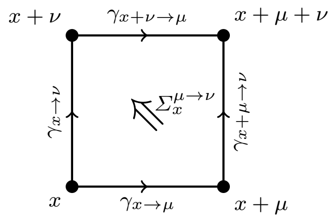
The higher lattice gauge theory involves associating not only group elements to links of the lattice, but also to its plaquettes. Then, using relations inherited from the higher category theory, important results that lie at the heart of higher gauge theory can be recovered. In the present paper, we use the same technics as in [7] with, however, a slight modification. First, we note that point-wise transformations have been represented by the authors as if they were propagating over the lattice. We remark that this choice leads to a truncated form of the transformation laws. This is mainly due to the higher order ε terms that vanish, where ε is what they called the "height" of the point-wise transformation. Instead, we describe transformations propagating on the space (such as the holonomies) by assigning group elements to links and plaquettes, whereas point-wise transformations are described by the assignment of group elements to vertices.

2. Definitions and notations

We descretize the trivial smooth base 2-space B into an Euclidean lattice $B = a\mathbb{Z} \times a\mathbb{Z} \times \cdots \times a\mathbb{Z}$. Let e_μ denotes the unit vector in the μ direction, the vector ae_μ will be written μ for short. A link $\gamma_{x\to\mu}$ stands for the oriented path between the ordered pair of points $(x,\, x+\mu)$



whereas a plaquette $\Sigma_x^{\mu\to\nu}$ stands for the oriented surface with boundary the ordered quadruple $(x,\, x+\mu,\, x+\mu+\nu,\, x+\nu)$



All quantities in the lattice written without argument will be understood as being evaluated at the origin $x = 0$. All groups considered in this paper are matrix groups.

The holonomy of the 2-connection for the patch U_i along a link at the origin propagating in the μ direction $\gamma_{0\to\mu}$ will be denoted

$$\mathsf{hol}_{i\mu} = e^{\int_{\gamma_{0\to\mu}} A_i}\,.$$

The holonomy of the 2-connection for the patch U_i on a plaquette at the origin propagating in the μ and ν directions $\Sigma_0^{\mu\to\nu}$ will be denoted

$$\mathsf{hol}_{i\mu\nu} = e^{\int_{\Sigma_0^{\mu\to\nu}} B_i}\,.$$

The 2-group $\mathcal{G}$ will be seen as a 2-category with a single object denoted $\star$. On each patch U_i, there is a 2-groupoid $\mathcal{P}_2(U_i)$ of thin homotopy classes of smooth lazy paths and surfaces [2], the holonomy is then a 2-functor

$$\mathsf{hol}_i : \mathcal{P}_2(U_i) \to \mathcal{G}$$

that takes each point of U_i to the single object $\star$ of $\mathcal{G}$

$$\mathsf{hol}_i^{(0)}(x) = \star\,.$$

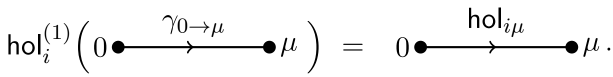
The 1-morphism map of the holonomy functor

$$\mathsf{hol}_i^{(1)} : \mathsf{1Mor}(\mathcal{P}_2(U_i)) \to \mathcal{G}^{(1)}$$

acts on origin-based links of the lattice as

$$\mathsf{hol}_i^{(1)}\left(0 \bullet\!\xrightarrow{\;\gamma_{0\to\mu}\;}\!\bullet\, \mu\right) = \star \xrightarrow{\;\mathsf{hol}_{i\mu}\;} \star\,.$$

Although the images of all lattice vertices are always $\star$, for the sake of clarity, we will write $\mathsf{hol}_i^{(0)}(x) = x$ for each vertex x

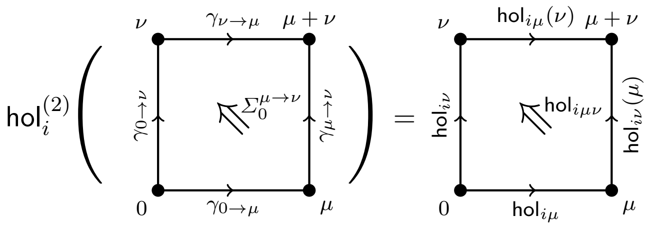


However, keeping in mind that, if seen as living in $\mathcal{G}$, all vertices of this diagram are the single object $\star$, while if it is seen as living in $\mathcal{P}_2(U_i)$, labels on links are the coloring of the lattice.

In the same spirit, the 2-morphism map of the holonomy functor

$$\mathsf{hol}_i^{(2)} : \mathsf{2Mor}(\mathcal{P}_2(U_i)) \to \mathcal{G}^{(2)}$$

acts on origin-based plaquettes of the lattice as



3. 2-connections

Let $E \xrightarrow{p} B$ be a $\mathcal{G}$-2-bundle, where $\mathcal{G}$ is some (strict) smooth 2-group corresponding to the Lie crossed module (G, H, t, α) and $(\mathfrak{g}, \mathfrak{h}, \mathrm{d}t, \mathrm{d}\alpha)$ be its differential crossed module [3, 8, 9]. We choose B to be a trivial smooth 2-space equipped with an ordinary cover $\{U_i\}_{i\in I}$ which is hypercubic-wise, *i.e.*, the open U_i are open hypercubes. This will ensure that no links and no plaquettes are partially included in some open U_i, except perhaps for their boundaries. We, however, remark that if an endpoint of a link does not belong to the patch U_i that contains the rest of the link, the integration is not going to differ much from that using the whole link. The transition functions on the cover $\{U_i\}_{i\in I}$ are g_{ij}, h_{ijk} and k_i. We will also restrict ourselves to the case of $k_i = 1$. On each patch U_i, the local holonomy 2-functor hol_i is specified by two differential forms

$$\begin{aligned} A_i &\in \Omega^1(U_i, \mathfrak{g}), \\ B_i &\in \Omega^2(U_i, \mathfrak{h}), \end{aligned}$$

such that the fake curvature vanishes

$$F_{A_i} + \mathrm{d}t(B_i) = 0,$$

where F_{A_i} is the curvature 2-form of A_i.

The transition pseudonatural isomorphism $g_{ij} : \mathsf{hol}_i \Rightarrow \mathsf{hol}_j$ is specified by the transition functions g_{ij} together with differential forms $a_{ij} \in \Omega^1(U_{ij}, \mathfrak{h})$, whereas the modification $h_{ijk} : g_{ij}g_{jk} \Rrightarrow g_{ik}$ is specified by the transition functions h_{ijk}, such that on every double overlap U_{ij}, the following transformation laws hold:

$$A_i = g_{ij}A_jg_{ij}^{-1} + g_{ij}\mathrm{d}g_{ij}^{-1} - \mathrm{d}t(a_{ij})\,, \tag{3.1}$$

$$B_i = \alpha(g_{ij})(B_j) + \mathrm{d}a_{ij} + a_{ij} \wedge a_{ij} + \mathrm{d}\alpha(A_i) \wedge a_{ij} \tag{3.2}$$

and on every triple overlap U_{ijk}, the following transformation law holds:

$$a_{ij} + \alpha(g_{ij})a_{jk} = h_{ijk}^{-1}a_{ik}h_{ijk} + h_{ijk}^{-1}\mathrm{d}h_{ijk} + h_{ijk}^{-1}\mathrm{d}\alpha(A_i)h_{ijk}\,. \tag{3.3}$$

Let us recall that these transformation laws have already been derived in [3, 4] using a continuous setting. Here, we use a different approach which has been inspired by [7]. There, the authors used lattice calculus to recover the fake curvature connection (which has also been previously derived in a continuous setting) as well as other important results.

4. Transition laws for the 2-connection

In the higher lattice gauge theory, transition functions are represented not only by 1-morphisms

$$\forall x \in \mathsf{Ob}(\mathcal{P}_2(U_{ij})) : g_{ij}(x) \in \mathcal{G}^{(1)}$$

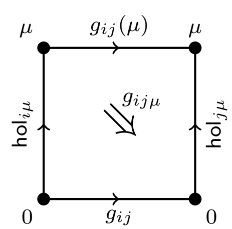
but also by 2-morphisms

$$\forall \gamma \in \mathsf{1Mor}(\mathcal{P}_2(U_{ij})) : g_{ij}(\gamma) \in \mathcal{G}^{(2)}$$

such that if $\gamma : x \to y$, we have

$$g_{ij}(\gamma) : \mathsf{hol}_i(\gamma)\ g_{ij}(y) \Rightarrow g_{ij}(x)\ \mathsf{hol}_j(\gamma)\,.$$

To derive equation (3.1), we take a link $\gamma_{0\to\mu} \in U_{ij}$, its images in $\mathcal{G}$ via the 2-connections (A_i, B_i) and (A_j, B_j) are related by the transition functions as follows:



The 2-morphism $g_{ij\mu}$ has $\mathsf{hol}_{i\mu} g_{ij}(\mu)$ as a source 1-morphism and $g_{ij}\mathsf{hol}_{j\mu}$ as a target 1-morphism, we have then the following relation which stems directly from the higher category theory [2]:

$$g_{ij}\mathsf{hol}_{j\mu} = t(g_{ij\mu})\ \mathsf{hol}_{i\mu} g_{ij}(\mu)$$

thus, $\mathsf{hol}_{i\mu}$ is

$$\begin{aligned} \mathsf{hol}_{i\mu} &= t(g_{ij\mu})^{-1}\ g_{ij}\ \mathsf{hol}_{j\mu}\ g_{ij}(\mu)^{-1}\,, \\ e^{\int_{\gamma_{0\to\mu}} A_i} &= t\left(g_{ij\mu}^{-1}\right)\ g_{ij}\ e^{\int_{\gamma_{0\to\mu}} A_j}\ g_{ij}(\mu)^{-1}\,. \end{aligned}$$

The differential forms a_{ij} describe the transition pseudonatural isomorphisms g_{ij} at the plaquette level, thus

$$g_{ij\mu} = e^{a a_{ij\mu}}\,.$$

Hereafter, as $a \to 0$, the symbol $\approx$ means that we approximate the equalities by neglecting terms of order higher than the dimension we are working on.

As $a \to 0$, the connection can be considered constant along each link, so that we get

$$e^{\int_{\gamma_{0\to\mu}} A_i} \approx e^{aA_{i\mu}}\,.$$

On the other hand, using a Taylor expansion in $g_{ij}(\mu)$ and the derivative of t, we finally get

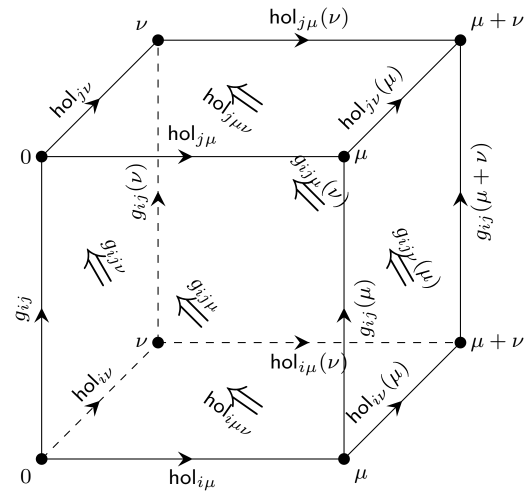
$$e^{aA_{i\mu}} \approx e^{-a\mathrm{d}t(a_{ij\mu})}\ g_{ij}\ e^{aA_{j\mu}} \left(g_{ij}^{-1} + a\partial_\mu g_{ij}^{-1}\right)\,.$$

Again, using Taylor expansions of exponentials, we get

$$\begin{aligned} 1 + aA_{i\mu} &\approx (1 - a\mathrm{d}t(a_{ij\mu}))g_{ij}(1 + aA_{j\mu})\left(g_{ij}^{-1} + a\partial_\mu g_{ij}^{-1}\right) \\ &\approx 1 + a\left(g_{ij}\partial_\mu g_{ij}^{-1} + g_{ij}A_{j\mu}g_{ij}^{-1} - \mathrm{d}t(a_{ij\mu})\right)\,. \end{aligned}$$

We thus recover (3.1).

Now, to derive equation (3.2), we take a plaquette at the origin, its images in $\mathcal{G}$ via the 2-connections (A_i, B_i) and (A_j, B_j) are related by the transition functions as follows:



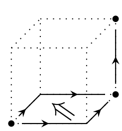
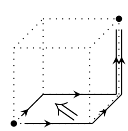
The coloring 2-morphism $\mathsf{hol}_{i\mu\nu}$ has a source $\mathsf{hol}_{i\mu}\mathsf{hol}_{i\nu}(\mu)$ and a target $\mathsf{hol}_{i\nu}\mathsf{hol}_{i\mu}(\nu)$, it sweeps the bottom side of the cube. Alternatively, it can also be seen as sweeping the remaining sides of the cube since the diagram commutes in $\mathcal{G}$.

Let us denote the horizontal (vertical) composition of 2-morphisms by $\circ_h$ ($\circ_v$).

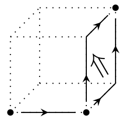
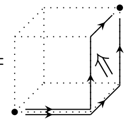
We note that the horizontal composition of a 2-morphism with a 1-morphism is a shortcut of the horizontal composition of this 2-morphism with identity 2-morphism of the 1-morphism, that is, for example for $g \in G$ and $h \in H$

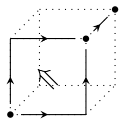
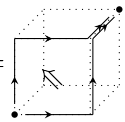
$$h \circ_h g := h \circ_h 1_g \,.$$

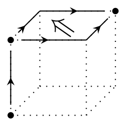
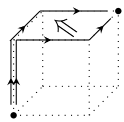
We remark first that

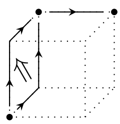
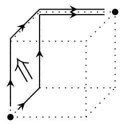
$\mathsf{hol}_{i\mu\nu} \circ_h g_{ij}(\mu+\nu) =$  $=$  $= \mathsf{hol}_{i\mu\nu}\,.$

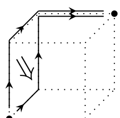
Now, to write down the other expression of that 2-morphism (*i.e.* when it sweeps the remaining sides), we will need the following pieces of the cube:

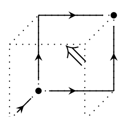
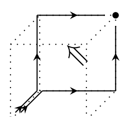
$\mathsf{hol}_{i\mu} \circ_h g_{ij\nu}(\mu) =$  $=$  $= \mathsf{hol}_{i\mu} \rhd g_{ij\nu}(\mu)\,,$

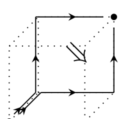
$g_{ij\mu} \circ_h \mathsf{hol}_{j\nu}(\mu) =$  $=$  $= g_{ij\mu}\,,$

$g_{ij} \circ_h \mathsf{hol}_{j\mu\nu} =$  $=$  $= g_{ij} \rhd \mathsf{hol}_{j\mu\nu}\,,$

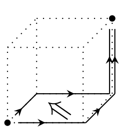
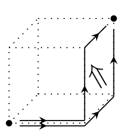
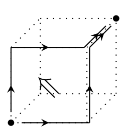
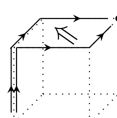
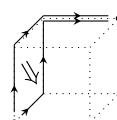
$g_{ij\nu} \circ_h \mathsf{hol}_{j\mu}(\nu) =$  $=$  $= g_{ij\nu}$

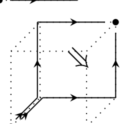
$\overset{(\bullet)^{-1}}{\longmapsto}$  $= g_{ij\nu}^{-1}\,,$

$\mathsf{hol}_{i\nu} \circ_h g_{ij\mu}(\nu) =$  $=$  $= \mathsf{hol}_{i\nu} \rhd g_{ij\mu}(\nu)$

$\overset{(\bullet)^{-1}}{\longmapsto}$  $= \mathsf{hol}_{i\nu} \rhd g_{ij\mu}^{-1}(\nu)\,.$

Using the following equality,

 $=$  $\circ_v$  $\circ_v$  $\circ_v$ 

$\circ_v$ 

we get

$$\begin{aligned}\mathsf{hol}_{i\mu\nu} &= \left[\mathsf{hol}_{i\nu} \triangleright g^{-1}_{ij\mu}(\nu)\right] g^{-1}_{ij\nu}\left[g_{ij} \triangleright \mathsf{hol}_{j\mu\nu}\right] g_{ij\mu}\left[\mathsf{hol}_{\mathrm{i}\mu} \triangleright g_{ij\nu}(\mu)\right] ,\\ e^{\int_{\Sigma_0^{\mu\to\nu}} B_i} &= \alpha\left(e^{\int_{\gamma_{0\to\nu}} A_i}\right)\left(g^{-1}_{ij\mu}(\nu)\right) g^{-1}_{ij\nu}\alpha\left(g_{ij}\right)\left(e^{\int_{\Sigma_0^{\mu\to\nu}} B_j}\right)\\ &\quad \times g_{ij\mu}\alpha\left(e^{\int_{\gamma_{0\to\mu}} A_i}\right)\left(g_{ij\nu}(\mu)\right) .\end{aligned}$$

Again, as $a \to 0$, the 2-connection can be considered constant on each plaquette, so that

$$e^{\int_{\Sigma_0^{\mu\to\nu}} B_i} \approx e^{a^2 B_{i\mu\nu}} .$$

Using the derivative of α and a Taylor expansion on $g_{ij\mu}(\nu)$, we get

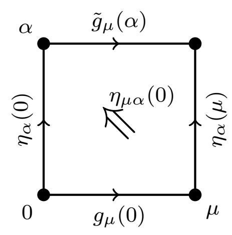
$$\begin{aligned}e^{a^2 B_{i\mu\nu}} &\approx e^{a\mathrm{d}\alpha(A_{i\nu})}\left(e^{-aa_{ij\mu}-a^2\partial_\nu a_{ij\mu}}\right) e^{-aa_{ij\nu}}\alpha(g_{ij})\left(e^{a^2 B_{j\mu\nu}}\right)\\ &\quad \times e^{aa_{ij\mu}} e^{a\mathrm{d}\alpha(A_{i\mu})}\left(e^{aa_{ij\nu}+a^2\partial_\mu a_{ij\nu}}\right) .\end{aligned}$$

By expanding exponentials and after some calculations, we get

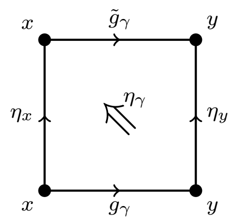
$$\begin{aligned}1 + a^2 B_{i\mu\nu} &\approx (1 + a\mathrm{d}\alpha(A_{i\nu}))\left(1 - aa_{ij\mu} - a^2\partial_\nu a_{ij\mu}\right)(1 - aa_{ij\nu})\\ &\quad \times\alpha(g_{ij})\left(1 + a^2 B_{j\mu\nu}\right)(1 + aa_{ij\mu})(1 + a\mathrm{d}\alpha(A_{i\mu}))\\ &\quad \times\left(1 + aa_{ij\nu} + a^2\partial_\mu a_{ij\nu}\right)\\ &\approx 1 + a^2\left[\alpha(g_{ij})(B_{j\mu\nu}) + \partial_\mu a_{ij\nu} - \partial_\nu a_{ij\mu} + a_{ij\mu}a_{ij\nu} - a_{ij\nu}a_{ij\mu}\right.\\ &\quad \left. + \mathrm{d}\alpha(A_{i\mu})(a_{ij\nu}) - \mathrm{d}\alpha(A_{i\nu})(a_{ij\mu})\right] ,\end{aligned}$$

where we have dropped out the symmetric terms. So we recover (3.2).

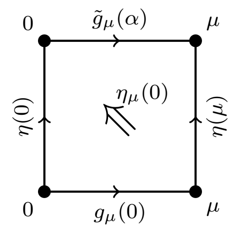
It is worth noting that in [7], the authors describe pointwise gauge transformations by the following diagram:



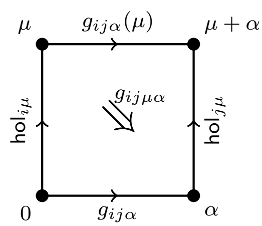
which they consider as the discrete analogous of



This is clearly inconsistent, since the correct analogous of the forementioned diagram is



It turns out that with such a choice, our diagram describing transition laws would have looked like

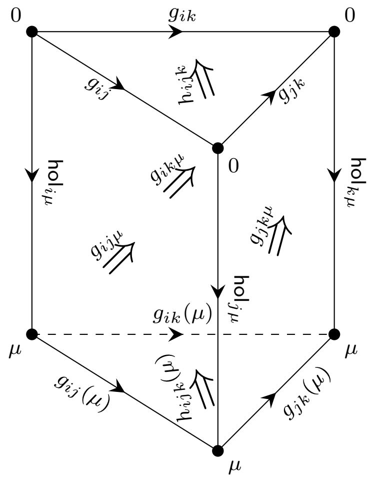


and transformations like $g_{ij\mu}(\nu)$ would have been

$$g_{ij\mu\alpha}(\nu) \approx e^{\varepsilon a a_{ij\mu} + \varepsilon a^2 \partial_\nu a_{ij\mu}}$$

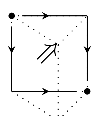
acquiring an extra index α, and then the $a_{ij} \wedge a_{ij}$ term would have disappeared from the transformation laws because of their $a^2\varepsilon^2$ order.

Finally, to derive equation (3.3), we take the triangle that represents the action of the modification h_{ijk} on transition functions, its images in $\mathcal{G}$ via the 2-connections (A_i, B_i) and (A_j, B_j) are related by the transition functions as follows:

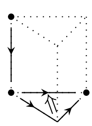
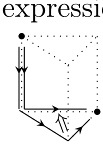


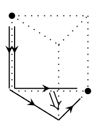
The coloring 2-morphism $g_{ik\mu}$ has a source $\mathsf{hol}_{i\mu}\ g_{ik}(\mu)$ and a target $g_{ik}\ \mathsf{hol}_{k\mu}$, it sweeps the backside face of the prism. Alternatively, it can be seen as sweeping the remaining sides of the prism since the diagram commutes in $\mathcal{G}$. Let us repeat the previous steps for this diagram.

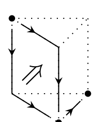
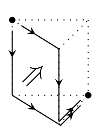
We remark first that

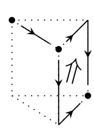
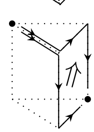
$$g_{ik\mu} =$$ 

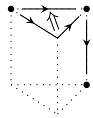
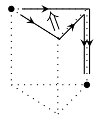
Now, we write down the other expression of that 2-morphism

$\mathsf{hol}_{i\mu} \circ_h h_{ijk}(\mu) =$  $=$  $= \mathsf{hol}_{i\mu} \rhd h_{ijk}(\mu)$

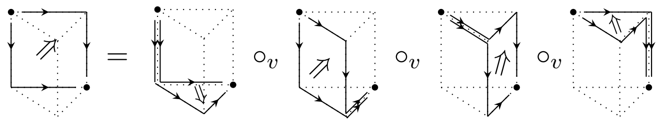
$\overset{(\bullet)^{-1}}{\longmapsto}$  $= \mathsf{hol}_{i\mu} \rhd h^{-1}_{ijk}(\mu)\,,$

$g_{ij\mu} \circ_h g_{jk}(\mu) =$  $=$  $= g_{ij\mu}\,,$

$g_{ij} \circ_h g_{jk\mu} =$  $=$  $= g_{ij} \rhd g_{jk\mu}\,,$

$h_{ijk} \circ_h \mathsf{hol}_{k\mu} =$  $=$  $= h_{ijk}\,.$

Using the following equality,



we get

$$\begin{aligned} g_{ik\mu} &= h_{ijk}\left[g_{ij} \triangleright g_{jk\mu}\right] g_{ij\mu}\left[\mathsf{hol}_{i\mu} \triangleright h^{-1}_{ijk}(\mu)\right]\,, \\ h^{-1}_{ijk} g_{ik\mu}\left[\mathsf{hol}_{i\mu} \triangleright h_{ijk}(\mu)\right] &= \left[g_{ij} \triangleright g_{jk\mu}\right] g_{ij\mu}\,. \end{aligned}$$

The R.H.S is

$$\begin{aligned} \left[g_{ij} \triangleright g_{jk\mu}\right] g_{ij\mu} &= \left(\alpha(g_{ij}) g_{jk\mu}\right) g_{ij\mu} \\ &= \left(\alpha(g_{ij}) e^{a a_{jk\mu}}\right) e^{a a_{ij\mu}}\,. \end{aligned}$$

Using Taylor expansions of exponentials, we get

$$\begin{aligned} \left[g_{ij} \triangleright g_{jk\mu}\right] g_{ij\mu} &\approx \left(\alpha(g_{ij})\left(1 + a a_{jk\mu}\right)\right)\left(1 + a a_{ij\mu}\right) \\ &\approx 1 + a\left(a_{ij\mu} + \alpha(g_{ij}) a_{jk\mu}\right)\,. \end{aligned}$$

The L.H.S is

$$h^{-1}_{ijk} g_{ik\mu}\left[\mathsf{hol}_{i\mu} \triangleright h_{ijk}(\mu)\right] = h^{-1}_{ijk} g_{ik\mu}\left[\alpha\left(e^{aA_{i\mu}}\right) h_{ijk}(\mu)\right]\,.$$

Using Taylor expansions of exponentials and of $h_{ijk}(\mu)$, as well as the derivative of α, we get

$$\begin{aligned} & h^{-1}_{ijk} g_{ik\mu}\left[\mathsf{hol}_{i\mu} \triangleright h_{ijk}(\mu)\right] \approx h^{-1}_{ijk}\left(1 + a a_{ik\mu}\right)\left[\left(1 + a\mathrm{d}\alpha(A_{i\mu})\right)\left(h_{ijk} + a\partial_\mu h_{ijk}\right)\right] \\ & \approx 1 + a\left(h^{-1}_{ijk} a_{ik\mu} h_{ijk} + h^{-1}_{ijk}\partial_\mu h_{ijk} + h^{-1}_{ijk}\mathrm{d}\alpha(A_{i\mu}) h_{ijk}\right)\,. \qquad (4.1) \end{aligned}$$

This leads us to obtain (3.3).

5. Conclusion and outlook

We have shown that calculus on the lattice can systematically be used to derive important results of the higher gauge theory. This is achieved by coloring plaquettes in addition to links without any other assumption.

The physical laws can be recovered from the $a \to 0$ limit. However, some coherence relations between group elements coloring links and group elements coloring plaquettes are pivotal for this construction. These relations prove to be at the heart of the higher category theory: the first group elements are morphisms (or 1-morphisms) of some 2-group, while the latter are 2-morphisms between these morphisms. This is reminiscent of what happens in some gauge theories where symmetries between symmetries appear due to the presence of second class constraints. It is worth noting that in BF theory, which is known to have this kind of metasymmetries, and which has close relations to quantum gravity [10], the gauge fields defining the theory form a 2-connection [11].

All this tends to prove that the higher category theory is a fertile ground where theories can be enriched, by systematically extending basic structures underlying them using the two main tools of higher category theory: internalization and enrichment [9].

Finally, let us point out that what we have called 2-bundles is also known under the name of *gerbes*, and that a whole theory of differential gerbes already exists [12]. It can be interesting to approach some aspects of this theory using calculus on lattice defined in the present article, for example, to found the more general transformation laws when no $k_i = 1$ restriction is made.

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