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ACROSS NEURONS AND SILICON: SOME EXPERIMENTS REGARDING THE PERVASIVENESS OF NONLINEAR PHENOMENA[*]

Ludovico Minati

Complex Systems Theory Department, Institute of Nuclear Physics
Polish Academy of Sciences (IFJ PAN), 31-342 Kraków, Poland
and
Tokyo Tech World Research Hub Initiative (WRHI)
Institute of Innovative Research, Tokyo Institute of Technology
226-8503 Yokohama, Japan
and
Center for Mind/Brain Sciences (CIMeC), University of Trento
38123 Trento, Italy

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The nonlinear dynamics of neurons can be viewed as the substrate through which the vastity of mental states and processes making up our subjective experience emerges from the brain as a physical object. While at present linear dynamical systems and devices may appear to have greater practical usefulness owing to their easier mathematical tractability, nonlinear phenomena pervade nature at all scales and harbor immense generative potential. Such phenomena have aspects of universality and, therefore, can be elicited, among other possible scenarios, also in analog electronic networks containing one or more nonlinear elements, and these are particularly convenient to realize and study experimentally. Here, a concise review of the author's work in this area is presented, without any attempt to comprehensively survey the field. Firstly, atypical circuits based on bipolar-junction transistors, inverter gates, and neon lamps are introduced; these recapitulate, at least phenomenologically, certain aspects of neural dynamics such as the generation of irregular spike trains. Secondly, the spontaneous emergence of synchronization patterns featuring modular organization, remote entrainment or implementing viable walking gaits is illustrated in networks constructed of those and other circuits. Some reflections on the potential relevance of comparing such profoundly different physical systems experimentally and possible directions for future work are given.

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1. Background and introduction

1.1. Physical nature of the brain

How can the mind emerge from such a small, wet, warm, fragile, messy lump of matter as is the human brain? Is there anything unique from the viewpoint of the underlying physical phenomena that firmly sets it apart from the other natural and artificial systems? After all, it is the only system we know of that is trying to study itself. If yes, what is it and what would be the cause of such a remarkable singularity? If not, what are the implications for our mental existence and impression of free will? One can argue that these are among the oldest and deepest questions that our species faces. Dauntingly, whichever direction one turns, there appear to be dilemmas, which are compounded by uncertainty around whether or not there is a distinct point along the phylogenetic tree past which we can consider that a mind begins to emerge from nervous system activity. Thus far, biophysics in all its vigorous ramifications has yielded no experimental evidence that from the viewpoint of matter, energy and their fundamental interactions at the level of the brain anything unique happens, which cannot and does not happen in other physical systems. Yet, the idea that there is nothing fundamentally different between our brain and other matter, living or not, even in comparison to a neuroelectrically active organ such as the heart, generally fills us with discomfort [1–6].

One could be tempted to treat the brain as a strictly deterministic system and the mind as an ontologically separate entity. Such a viewpoint implicitly or explicitly pervades modern neuroscience, in particular regrading the mechanistic, input–output architectural models that have been developed for particular networks, most notably the visual system. It is certainly fertile and has enabled formulating and verifying influential hypotheses about sensorimotor function and beyond. However, venturing towards higher aspects of cognition and emotion, the limitations of this approach readily appear, and eventually a paradox, or at least an apparent paradox, dawns: should we assume that mechanistic descriptions of the brain are sufficient to algorithmically capture the workings of the mind, then it would seem to be only a matter of time before we can run computer simulations at a scale large enough to reproduce the flow of our own unique identities. In that case, one could in principle even simulate a living person faster than real-time, thereby predicting future actions and thus negating free will. On the other hand, should we assume that this is not possible, it would imply that our account of brain function is fundamentally incomplete [5, 7–12].

In counterpoint to these issues, it has been repeatedly posited that stochastic dynamics are essential for brain function, and diverse viewpoints and models have been put forward wherein intrinsic unpredictability is conferred by entropy sources ranging from thermal noise to quantum decoherence. Even though we have ample experimental evidence that stochasticity is indeed present and important for brain dynamics, for example regarding stochastic resonance in sensory processing, to the author's knowledge no clear, falsifiable hypotheses have yet emerged regarding the way in which unpredictability in and of itself would help address the mind–brain duality. Asserting that the mind and free will are in some form residing within or expressed through inherently unpredictable processes may provide some apparent relief from computability dilemmas, but it seems to effectively only relocate, rather than address, the mystery of their emergence [2, 10, 13–15].

The author's speculative work surveyed in the present contribution has been ultimately inspired by the more recent view of the brain as a nonlinear dynamical system. Even under the assumption of complete determinism, the consequences of nonlinear dynamics appear to yield some level of liberation from the quagmire described above, in that they provide the substrate for emergence and chaoticity. These are distinct but related concepts, which together loosen the naive expectation that assuming the brain is deterministic would imply that we should concretely be able to simulate it. With emergence, we intend the ability of ensembles of interacting elements to display highly structured behaviors having qualitative features, such as the presence of scale-free phenomena in flocks of birds, which are not trivially predictable from the dynamical properties of the constituent elements considered in isolation, even when these are fully known such as in a numerical simulation. With chaoticity, we intend the *de facto* quantitative unpredictability of individual or collective trajectories due to extreme sensitivity to initial conditions, even when again the underlying dynamics are completely known and deterministic. This should not be mistaken as randomness, for the underlying dynamics are fully deterministic and geometrically-ordered, giving rise to self-similar structures known as strange attractors. The possibility for chaos to arise also confers to a system the freedom of expressing certain universal phenomena, known as critical phenomena, which have profound implications and preferentially occur in the vicinity of order-to-chaos transitions [16–23].

Any attempt to introduce the theory and phenomenology of chaos and emergence goes well beyond the narrow scope of the present review. Much less ambitiously, here the aim is only to introduce a limited number of experimental attempts to establish some hopefully fertile parallels and correspondences between the nonlinear dynamics observed in the brain and in much simpler and smaller networks of nonlinear electronic devices.

1.2. Connectivity, dynamics and synchronization

Since the dawn of neuroscience, histological studies performed using neuron-staining techniques as well as dissection of the major white matter bundles have revealed the essential nature of the brain as a network. While Camillo Golgi represented it as a continuous reticle, we modernly view it as a structured, distributed system whose collective activity reflects the orchestrated dynamics of billions of distinct constituent elements, namely neurons. Such a network delineates a form of structural connectivity, effectively reflecting a wiring architecture, which is realized over multiple spatial scales and whose global features are broadly set by genetic and epigenetic factors. At the microscopic scale, structural connectivity is determined by the density, type, and strength of synapses. At the mesoscopic scale, it indexes the arrangement of neurons into assemblies, layers, or partially segregated circuits such as the cortical columns which implement somatotopic and retinotopic tiling. At the macroscopic scale, it is reflected in multiple populations of axonal fibers, some of which project radially from the brain stem and basal ganglia, some of which bend sharply to connect adjacent gyri, and some of which interlink specific, distant cortical areas [5, 24–27].

In its sheer complexity, the structural connectivity of the human, and more generally, mammalian brain expresses some fundamental topological features that are highly conserved in other natural and artificial self-organized networks of the most diverse types, such as social relationships and transportation infrastructure. One such feature is self-similarity intended as the recurrence of similar motifs across scales; among other ways, this scale-free organization is manifest as a power-law distribution of node degrees, which entails the presence of a limited number of disproportionately strongly-connected regions, that are in turn known as the "cortical hubs" and are crucial for cognitive functioning. Another feature is small-worldness, intended as the instantiation of a minuscule proportion of long-distance connections, which greatly enhance the information-transfer efficiency of the network as a whole without incurring the prohibitive costs of a densely connected architecture. Statistical features such as these may be recapitulated, concurrently or separately, by several generative models, some of which are directly inspired by other physical processes. In the case of the brain, they are widely considered to have emerged under evolutionary pressure as optimal or near-optimal solutions maximizing, for example, intelligence or reaction speed while keeping the metabolic load and head volume down to acceptable levels. From a physical viewpoint, the non-trivial arrangement of structural connections acquires particular importance in consideration of the fact that the brain exists as a system permanently operating in a non-equilibrium condition [25–33].

It is similarly well-known that, based on electrochemical synaptic couplings, neurons collectively generate highly irregular and time-dependent signals, which one can attempt to capture from the perspectives of synchronization and causality. This effort is highly non-trivial, particularly in that it usually entails separating the intrinsic activity spontaneously generated while at rest, from the perturbations which are evoked by external stimuli or endogenous events. In the domain of neuroimaging and neurophysiology, these aspects of coordinated dynamics are considered with reference to functional and effective connectivity. Similarly to structural connectivity, the study of activity interdependence can be carried out across multiple spatiotemporal scales by probing single cells, microcircuits or the entire brain, separately or concurrently. At a minimum, one can search for statistically-significant entrainment between activity over different sites, for example in the form of phase locking, synchronization of amplitude (envelope) fluctuations, or more convoluted generalized relationships: this is broadly regarded as functional connectivity. In order to gain deeper insight, it is however necessary to establish directed causal effects, either purely on the basis of the intrinsic activity recorded in an observational framework, or predicated on the responses to external stimulation administered in an interventional framework: this delineates effective connectivity [5, 7, 8, 11, 25, 27, 34–38].

In light of the above, it appears fruitful to conceptualize the brain as a dynamical system described by at least two distinct networks: one representing the structural couplings between its nodes, and another representing the engagement attained between the same according to a chosen synchronization or information transfer measure (Fig. 1 (a)). A complex mutual relationship exists between these two aspects of brain architecture. One might be tempted to view structural connectivity as completely or largely fixed, however, the situation is considerably more nuanced due to a multitude of physiological mechanisms collectively referred to as plasticity, and encapsulated for instance in Hebb's rule. At the macroscopic scale, after ontogenetic development is complete, the layout of the large axonal bundles remains largely unchanged, and any degenerative, compensatory or adaptive changes require many orders of magnitude longer time compared to the timescale of neural events. However, at the microscopic level, synaptic strengths can be pervasively and permanently modulated by processes which unfold on the scale of milliseconds [5, 39–41].

One could, furthermore, be tempted to view synchronization and causal interdependence as a straightforward albeit perhaps imperfect reflection of the structural couplings. Without question, this is far from being the case because the nonlinear dynamics unfolding at each node imply that the relationship is weak and non-stationary, and the emergent spatiotemporal activity patterns express motifs absent in the structural connectivity. Such a

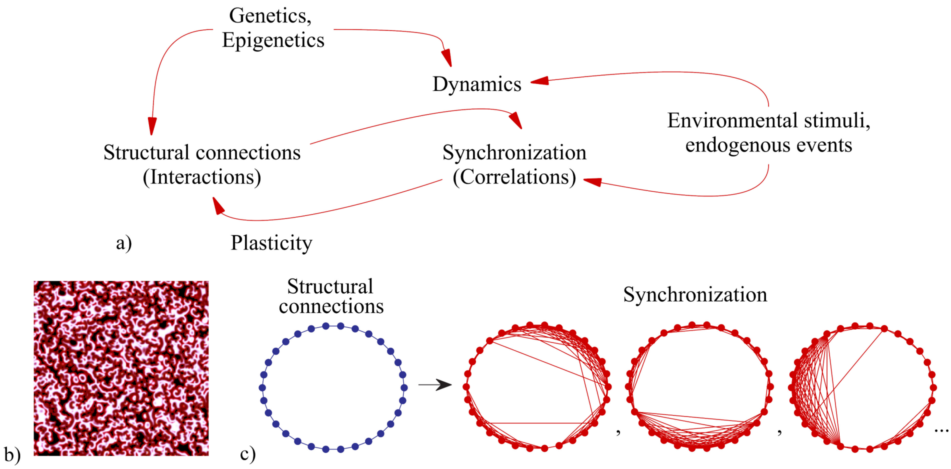

Fig. 1. Conceptual framework. (a) Relationship between structural connectivity and synchronization in the brain. (b) Example of morphogenesis in a simulation of the Belousov–Zhabotinsky reaction. (c) Diversified synchronization patterns emerging from a common underlying ring network due to electronic component parametric heterogeneities.

situation reflects a wide variety of dynamical phenomena, which altogether yield the very rich dynamical repertoire accessible to individual neurons and populations of neurons. It, furthermore, follows from the fact that neural systems preferentially dwell close to order–chaos transition, where critical phenomena confer the largest degree of freedom with respect to the spatiotemporal patterns that can be generated from a given structural network. At the critical point, the layout of interactions even becomes irrelevant as all correlations are long-range, which has fundamental implications even though we note that neural systems appear to dwell close to criticality, yet are not in an equilibrium state at the critical point. In the case of the human brain, the correspondence between structural connectivity and synchronization is strongest for the regions and signal components indexing the lowest-level sensorimotor processing, whereas at the global level, it is inversely related to the degree of preservation of cognitive activity and awareness, the closest overlap being observed during coma and under anesthesia [22, 25, 42–50].

As also regards these topics, any attempt to provide a detailed account falls well beyond the limited scope of the present contribution. We shall only content ourselves with introducing the notion that the observed decoupling between structural connectivity and synchronization, representing one of the key reflections of brain dynamics, could, at least to some limited extent, be recapitulated and thus indirectly explored also in other physical systems, such as networks of nonlinear electronic oscillators.

1.3. Relevance of experimental comparison to other physical systems

The approaches of computational neuroscience and artificial intelligence strive to provide accounts of brain function that are, one could argue in different senses, as detailed and realistic as possible. On the one hand, there is the aim of representing as closely as possible the physiological processes in terms of variables which can be experimentally measured in living systems, such as intra- and extra-cellular potentials, and of reproducing the associated anatomical interconnection architectures, as found for example in large-scale simulations of spiking networks. On the other hand, there is the aim of replicating as realistically and usefully as possible chosen aspects of high-level cognition, such as in the domains of language processing, motor planning, and image analysis, with more limited focus on the biological plausibility of the underlying infrastructure. The enormous contribution given by these methodologies is unquestionable, and they do provide frameworks entirely relevant to the approaches of chaos and synchronization theory, but at the same time, one could perhaps be misled into thinking that a high level of physiological, architectural or functional detail is necessary to meaningfully recapitulate any statistical aspect of brain dynamics [2, 51–54].

The rationale behind the present speculative line of work is that this seemingly turns out not to be the case, the reason ultimately being found in the universality of nonlinear phenomena, as exemplified by the well-known case of scaling period-doubling bifurcations. Even though a general theory of emergence remains lacking and, in this respect, one could argue that the field remains in a sort of pre-Galilean phase, for instance, it is beyond question that common synchronization phenomena occur across diverse systems and scales. In fact, emergence, at least in its simplest manifestations, does not in any way require complicated dynamics or large ensembles. Instead, there appears to be something deeply, fundamentally different between linear and nonlinear dynamical systems and it does not seem to matter much, if at all, what the nonlinearity precisely consists of, such as a threshold, a power or a more complicated function. The only additional requirement is set by the Poincaré–Bendixson theorem, and it is that the collective phase space must be at least three-dimensional in order for transition to chaotic dynamics to be possible [16, 18, 19, 36, 38, 55–57].

A particularly compelling case of emergence in a nonlinear dynamical system is the formation of geometrically-ordered patterns in homogeneous media, which is elegantly exemplified by the Belousov–Zhabotinsky chemical reaction (Fig. 1 (b)). A related phenomenon is the arising of non-trivial synchronization patterns, such as clusters and waves, in networks that have a high degree of topological symmetry and wherein the symmetry is broken via nonlinear dynamics influenced for example, but not necessarily, by the presence of parametric heterogeneities. As described below, in such cases small parametric changes are often sufficient to yield patterns that are topographically very different from one another, yet share consistent global features (Fig. 1 (c)). The potential for simple rules to generate complex shape is also strikingly exemplified by Wolfram's automata. These situations can altogether be viewed as akin to morphogenesis, which pervades biology and had already captured the attention of Alan Turing, whose pioneering work lead to realizing that structures such as the ordered arrangement of the tentacles of the hydra may be brought about by elementary nonlinear interactions. In a similar vein, one would view the highly dynamic synchronization patterns which are formed between neurons and brain regions as emergent features: structural connectivity "sets the initial conditions", but nonlinear dynamics have the liberating effect of enabling the formation of an enormous repertoire of patterns. Toy models have always been highly instrumental to understanding in physics, and assuming that the nonlinear phenomena at play are universal, one should be able to probe the underlying mechanisms also in other, vastly smaller and simpler systems. In this regard, ring structures have particular relevance given their low dimensionality and high degree of symmetry: set aside possible parametric heterogeneities, each node "sees" the same neighbors, unlike the case of chains, stars, *etc.* [16, 26, 38, 57–63].

Besides certain cases such as the Kuramoto model, the mathematics available to us at present does not allow extensive analytical treatment of nonlinear systems, and we dauntingly need to resort to numerical simulations. A solid body of knowledge on the control of nonlinear dynamical systems is nevertheless accumulating, including applications to chaotic control, information processing and transmission. While keeping well away from the question of whether a Turing machine may ever be able to simulate the emergence of a mind, which ultimately could even be inherently unanswerable by us, we note that numerical simulations have yielded a wealth of knowledge on the behavior of nonlinear systems across all branches of physics, chemistry, biology, *etc.* For example, the virtual totality of the experimentally-observed phase transition and pattern formation phenomena have been successfully simulated in some form. However, it is equally incontrovertible that, ever since the inception of computers, simulation and experiment have always been profoundly complementary, rather than adversarial; one should, therefore, not take the misleading view that numerical simulations are in themselves sufficient for gaining deep insight into the behavior of these systems and relegate experiments to a purely confirmatory role. On the contrary, serendipitous experimental discoveries illuminating the richness of physical reality are at a minimum indispensable to inspire numerical work. It furthermore appears truly remarkable that simulating numerically on a digital computer a small single-transistor system such as one of the oscillators described below requires many orders of magnitude more transistors and power compared to the analog physical object. The as yet latent disruptive potential of analog computation has, in fact, attracted attention since the early days of computer architecturing, and it appears fair to state that computing based on Turing machines has so far prevailed largely owing to their flexibility and to the ease of scalability of their modern realizations. From the point of view of the underlying architecture and dynamics, it is unquestionable that brains and nervous systems are totally unlike digital processors, foremost as their functioning is not hinged around designed features and algorithms, but largely reflects self-organization [18, 19, 58, 60, 63–73].

More practically, at least three issues potentially affecting numerical investigations in this area appear worthy of consideration. Firstly, the numerical systems encountered when modeling even small nonlinear dynamical networks can be highly stiff, particularly in the presence of parametric mismatches, and tend to be concerningly sensitive to issues such as solver choice, step size, and limited numerical precision. This is ultimately related to aspects of the continuous *versus* discrete representation of physical variables including time, as encapsulated in the notion of asymptotic consistency. Secondly, as mentioned above, the importance of noise and stochasticity, which can have a powerful generative role; crucially, here we refer to noise not only in the dynamical variables but in all system parameters, which complicates numerical solution even further. Thirdly, perfection does not belong to reality, only to our attempts to reduce and simplify it for aiding understanding: in all natural and artificial systems, there are parametric heterogeneities and all sorts of non-idealities, such as finite quality factor in reactive components, which are not always trivial to capture numerically. In canonical engineering practice, such imperfections are often the target of relentless minimization efforts, but in the context of nonlinear dynamics, they may be essential enablers of the emergence of complex global properties [12, 16–20, 36, 38, 57, 69].

In light of the above, it does not appear unreasonable to posit that useful insights could be gathered by comparing the human brain, alongside other nervous systems, with different natural and artificial nonlinear systems. The author furthermore posits that doing so not only in simulation but also via observation and experiment is fundamentally, inescapably important. Far from being an unusual situation, this approach, for example, maps closely onto the daily *modus operandi* of mechanical engineers, who inevitably make conjoint use of physical prototypes and simulations when determining the properties of complex structures. Predicated on the notion that an arbitrary nonlinearity may be sufficient to elicit emergent phenomena of interest, one could, in principle, equally well elect to experiment with any physical system. As we live in the era of electronics, electronic systems seem to be the most natural choice in terms of the ability to realize large experiments with maximum flexibility and minimal cost. *In vitro* preparations such as cultured neurons and dissected slices, while representing a powerful tool in exploring neural dynamics, are not without their issues: these are primarily related to the associated procedural difficulty and cost, and compounded by often limited reproducibility and even ethical issues. Mechanical systems would, in theory, be another equally valid option but entail generally lower integration, and optical systems are also being considered even though the associated experimental complexity and cost remain rather high. The present approach is, obviously, complementary and at no level antagonistic to those of computational neuroscience, artificial intelligence, and the affine areas: here, the focus is neither on physiological plausibility nor on functional effectiveness, but more agnostically and perhaps less ambitiously it is on replicating select observations of emergent statistical properties about network topology and dynamics. One way to view the potential contribution of the proposed approach is as a tool for exploring *in silico* and at the microscale, meaning at the level of few tens or hundreds of nodes, the same nonlinear phenomena that are commonly addressed by neuroscientists *in vitro* at the mesoscale and *in vivo* at the macroscale: an example is the spontaneous formation of modular and self-similar patterns (Fig. 2) [5, 17–20, 57, 60, 74–78].

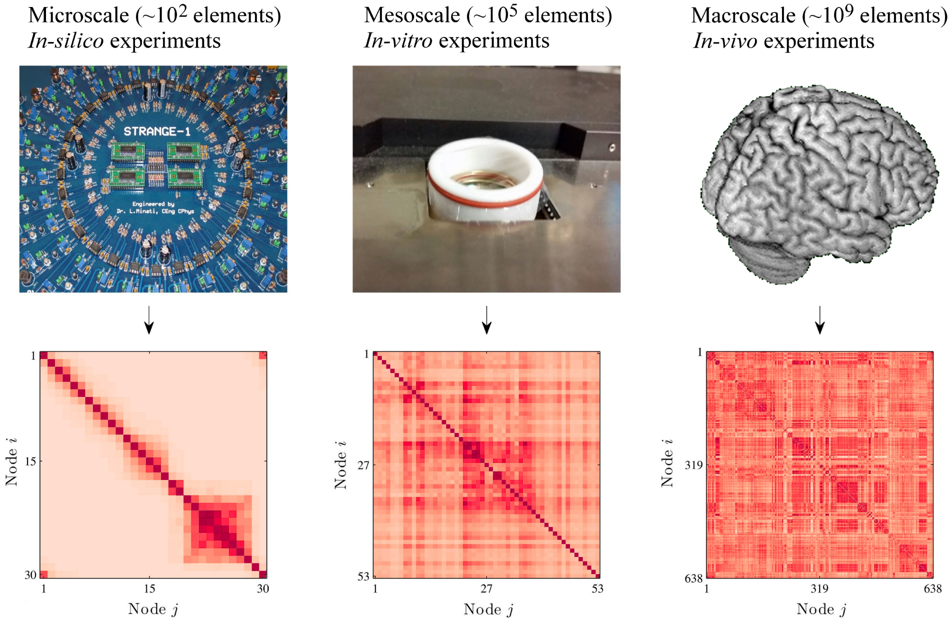

Fig. 2. Global features, *e.g.* modularity and self-similarity, may emerge analogously in synchronization matrices observed over different system sizes and measurement scales (micro, meso and macro) and across diverse experimental preparations (*in silico*, *in vitro* and *in vivo*). Further discussion in Ref. [78].

Having tentatively outlined the underlying approach and rationale, the remainder of the present contribution shall focus on surveying an initial set of findings obtained in the context of diverse, and in many aspects atypical, nonlinear electronic circuits.

2. Nonlinear oscillators and their dynamics

2.1. Rationale behind searching for atypical nonlinear oscillators

Two pervasive properties hallmarking nonlinear dynamics become apparent upon inspection of any neurophysiological recording, such as those provided by patch-clamp and multi-electrode array techniques. One aspect is that the generation of action potentials is quantized, in fact virtually binarized, at the presynaptic level. This all-or-nothing response plausibly reflects the physical requirements of electrochemical transmission, wherein neurotransmitter release cannot be controlled in a fully graded manner but is determined by the rupturing of synaptic vesicles having finite size. This mechanism provides significant transmission gain and implements a unidirectional, master–slave coupling scheme. At the same time, the postsynaptic processing taking place in dendritic trees is a graded, highly complex nonlinear spatiotemporal summation process, which can be summarized according to a variety of integrate-and-fire models. Even though they are considerably less prevalent, for completeness, we note that multiple types of electrical synapses also exist, which realize a rather different coupling scheme closer to a diffusive process. Whether there is any deeper theoretical reason for the all-or-nothing dynamics of action potential generation largely remains an open question, and is related to whether significant information is encoded in the precise times of occurrences of spikes or whether, in a more simplistic view, the corresponding point process can be locally approximated as a scalar, average firing rate [5, 20, 77, 79]. The other aspect is that neurons, and under appropriate circumstances even isolated axons, readily have access to a multitude of periodic, quasi-periodic and chaotic regimes which manifest themselves as the generation of isolated spikes, bursts and irregular sequences of spikes and bursts [80–83]. Rich repertoires of activity of these kinds have been consistently observed across the micro-, meso- and macroscopic scales, for example in the squid giant axon [84], in assemblies of cells [85, 86] and in entire brains [81, 87]. The specific dynamical properties of individual neurons are jointly determined by their phenotype, for instance as regards the expression of genes controlling receptor formation, and by electrical and biochemical variables such as intra- and extra-cellular ion concentrations [5, 79].

On another level, in the prevailing engineering approaches, besides specific situations exemplified by modulation circuits, digital gates and certain dynamical control systems, nonlinearity tends to be regarded as a hindrance, something to be avoided and minimized as far as possible since it often translates into distortion, poor signal transmission, instability, and possibly unreliability. Aside from the fact that for countless practical applications this is obviously the case, a more profound reason is that we have accumulated a powerful mathematical toolkit to deal with linear systems, even very complex ones, but from a theoretical viewpoint, we remain weaker when faced with nonlinear systems, and attempts to linearly approximate any nonlinear system unavoidably lead to incomplete accounts of its dynamics [18, 19, 72, 88, 89]. In fact, it could be argued that, as far as the physical world is concerned, referring to nonlinear systems is like referring to non-elephant animals: practically everything is nonlinear, and this is not a hindrance at all. Rather, on the contrary, it is the indispensable seed from which, across all scales, the incredible complexity of the world we experience sprouts. In a sense, this is reminiscent of the three-body problem, wherein simply adding one body generally prohibits closed-form solution and opens the way to scenarios inaccessible to two-body systems [16, 18, 19, 21, 22, 36].

At the time of writing, one has access to a wide range of well-understood electronic oscillator circuits and topologies having diverse desirable features such as low distortion, low power consumption, low noise, suitability for high-frequency operation, *etc.*: most of these have essentially linear dynamics and produce a sinusoidal output. Considering the particular case of simple circuits based on one or at most two transistors, topologies such as the Colpitts and Hartley oscillators are universally-known and have been investigated so extensively that they have effectively attained a special status of “canonical” circuits [88]. However, harmonic oscillators are inherently unsuitable for representing the nonlinear aspects of neural dynamics which ultimately give rise to emergence, in that they can show no transitions between different phases since they only have access to periodic dynamics; yet, transistors are nonlinear devices and this could, in principle, be harvested for realizing more complex systems.

The study of chaos in electronic circuits dates back to the first incidental observations of “irregular noise” by Balthasar van der Pol nearly a century ago, and was propelled by the much later discovery by Leon Chua of one of the first electronic circuits which intentionally behaves in a chaotic manner and generates an extraordinarily large repertoire of attractors. While a broad variety of transistor-based chaotic circuits have been described to date, many (though not all) represent adaptations of the above-mentioned periodic oscillators, rendered chaotic for instance through inserting additional nonlinear elements, resonant networks or otherwise disturbing integration. While such adaptations retain desirable features for practical applications, they also constrain exploration heavily. At the same time, a general methodology for synthesizing chaotic oscillators is presently lacking, and at most semi-systematic approaches are available [60, 75, 76, 90–92]. One might consequently be led to consider chaos as an infrequent occurrence in transistor-based circuits, but is this truly the case and is there anything fundamentally distinctive in those oscillators that we presently regard as canonical topologies?

The experiments reported herein could in certain aspects be viewed as an attempt to address this question; such an effort appears particularly pertinent to the present line of research because transistor-based oscillators are one of the smallest circuit scenarios wherein nonlinear phenomena can be realized, among other purposes for that of comparison with neural dynamics. Before delving into any description of some novel circuits, it seems appropriate to further underline that there is no intention of considering variables such as voltages and currents which would in any manner be physiologically meaningful. This is unlike certain neuromorphic circuits, which attempt to replicate neural architectures and, in particular, dynamics realistically, by means of associating chosen physiological variables such as ion currents and membrane potentials to corresponding electrical variables in an artificial circuit. Such spiking oscillators tend to be realizations or adaptations of well-known mathematical models of neurons such as the Hodgkin–Huxley and Fitzhugh–Nagumo systems. They retain physiological meaningfulness of the variables often at the price of circuit size, yielding devices that may include tens of transistors or even more [65, 93–95]. On the other hand, the present line of work mainly pertains to circuits which are as simple as possible and wherein the only nonlinearity is, for example, that provided by saturation or by the current-voltage curves of one or at most two transistors, as captured in the Ebers–Moll model.

2.2. Chaos and phase transitions in transistor-based circuits

To explore the possibility of synthesizing novel chaotic oscillators in a manner as unconstrained as possible, the author adopted an approach embodying a form of "synthetic serendipity", which does not tantamount to any systematic methodology and simply entails a large-scale numerical search. This is based on describing the topology and component values of a hypothetical circuit in the form of a bit-string, which could be viewed as a chromosome sequence: separate segments correspond not to amino acids but to distinct components and encode their connections as well as their characteristics, extracted from a predetermined catalog of discrete elements (Fig. 3 (a)). In an early study, the author deployed this approach within a genetic algorithm, numerically simulating the dynamics of individual circuits while aiming to maximize a fitness measure representing the entropy of the generated time-series. As customary in this area, an implementation of the *Simulation Program with Integrated Circuit Emphasis* (SPICE) was used. A set of atypical oscillators not intentionally related to known topologies was obtained, and subsequently selected ones were realized physically and found to be chaotic [96].

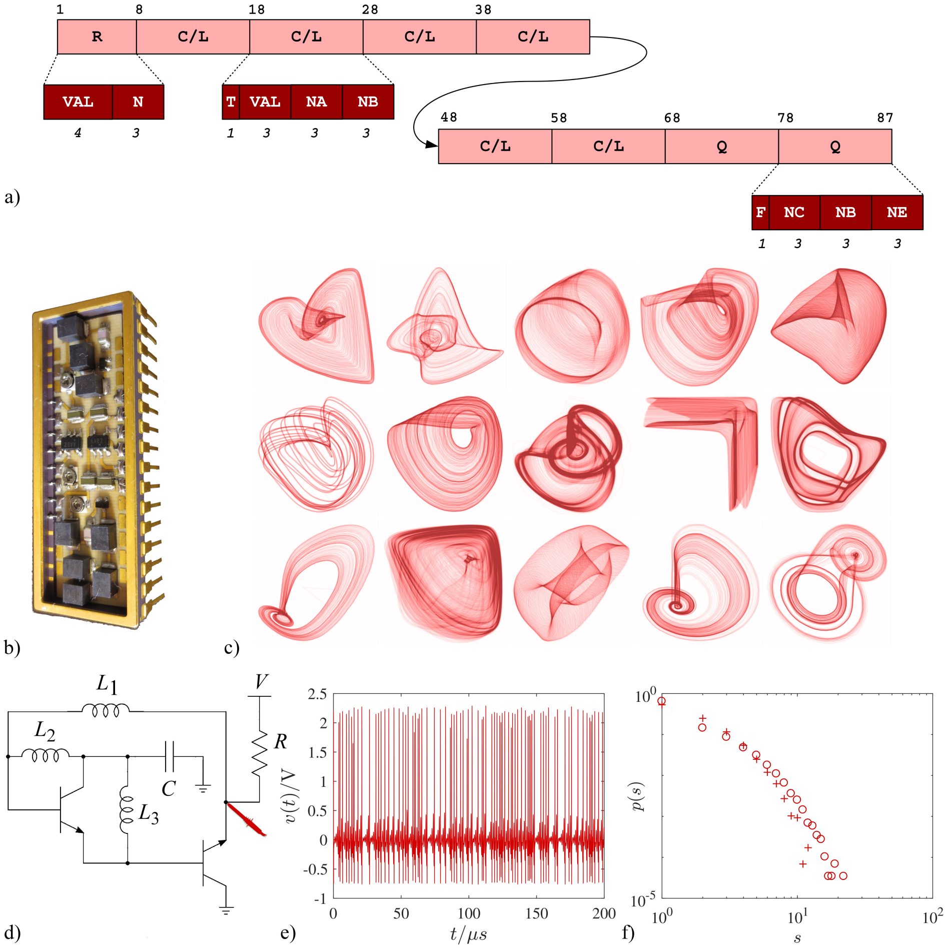

Fig. 3. Searching for atypical transistor-based chaotic oscillators. (a) Bit-string representing a circuit topology and the associated discrete component values. (b) Representative physical realization of two coupled oscillators on a hybrid module. (c) Gallery exemplifying the diversity of attractors that can be experimentally generated. Spiking oscillator with (d) circuit diagram ($L_1 = 15$ μH, $L_2 = 68$ μH, $L_3 = 150$ μH, $C = 470$ pF, $R \approx 200$ Ω, $V = 5$ V), (e) recorded waveform and (f) avalanche size distribution (circles: measurement, crosses: reshuffled data). Detailed description in Ref. [97].

Later on, five chosen circuits were subject to a detailed experimental characterization involving an automated laboratory test-bench, in preparation for their use as building blocks to realize networks of coupled oscillators. These circuits consist of only elementary components, namely one or two bipolar-junction transistors, inductors, capacitors, and a resistor. They are autonomous, in the sense that they oscillate without requiring any external excitation. One noteworthy aspect is that they contain several overlapping series and parallel LC networks, which make available multiple resonances at frequencies that are not trivially related by integer ratios; these are primarily determined by the possible combinations of discrete inductors and capacitors, but further enriched by the presence of parasitic effects and transistor junction capacitances, which can play a determinant role in the circuit dynamics. In these circuits, the transistor(s) often operate in large-signal mode, spanning forward and reverse active, cut-off and saturation regions, implying that there are significant nonlinearities which both determine and are in themselves influenced by the generated signals. Practically any combination of the available resonances can be excited, and nonlinear interactions between them may easily give rise to chaos via the quasi-periodicity route, as well as via period-doubling and intermittency effects, which could also be observed. Another important feature of all the dissipative circuits described in this section, is that energy is provided by a constant voltage source, in series to which a resistor is instanced: by setting the voltage/current slope at the input node, effectively the output resistance of the source, the value R of this resistor has a profound impact on the circuit dynamics, and as such serves as the only control parameter. From an experimental viewpoint, it is vastly easier to tune over a wide range compared to an inductance or a capacitance, and in the course of preliminary experiments, the value of this resistor was repeatedly found to have a stronger impact on the dynamics compared to the supply voltage [98].

A representative example of such a circuit comprises, in addition to the aforementioned resistor, just one transistor, two inductors and one capacitor (Fig. 4 (a)). While the author refrained from making this claim in the initial study, at least at the surface, it appears that the circuit is of a size similar to the simplest known transistor-based chaotic oscillators, such as the Lindberg–Murali–Tamasevicius circuit, yet its dynamics might be richer [75, 76, 99]. Its experimental realization is trivial and can be carried out using decades old components, incidentally exemplifying how, in this era of big data and huge projects, some interesting discoveries can perhaps still be made using highly inexpensive setups and apparently obsolete technology (Fig. 4 (b)). By varying R at high resolution via a stepper motor and experimentally obtaining the corresponding spectrogram and bifurcation diagram, presence of a multiplicity of phase transitions becomes well-evident; since such acquisitions are particularly demanding, in other studies they are often replaced with simulations. Most of the circuits of the present kind exhibit sharp transitions between periodicity or quasi-periodicity and chaos, whose occurrence is confirmed through estimating the largest Lyapunov exponent $\lambda_{\max}$ and the correlation dimension D_2 from the measured time-series [100, 101]. Besides the onset of chaos, which tends to occur prevalently at

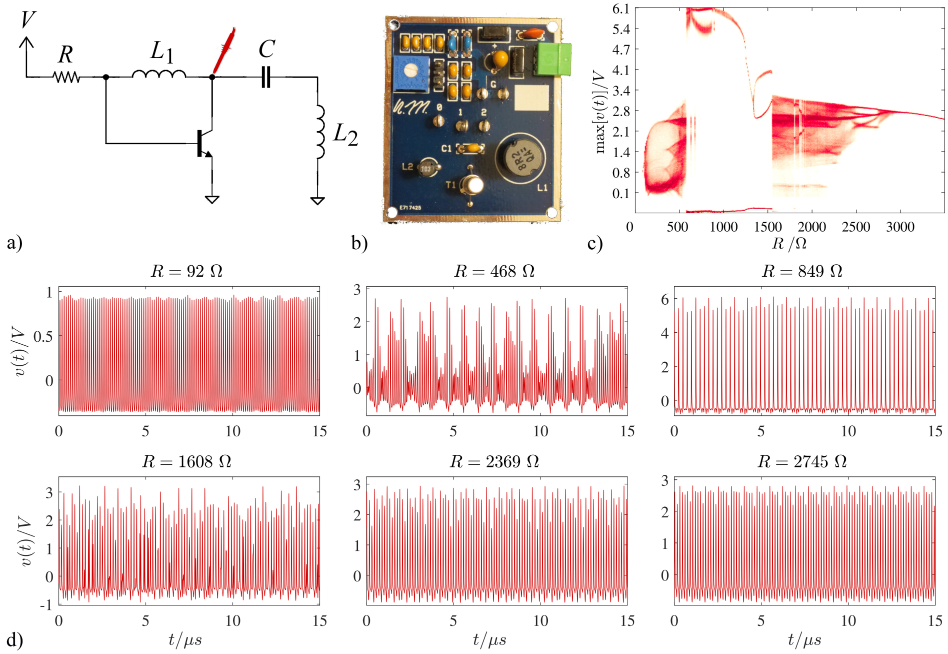

Fig. 4. Representative example of an atypical single-transistor chaotic oscillator. (a) Circuit diagram ($L_1 = 10$ μH, $L_2 = 8.2$ μH, $C = 30$ pF, $V = 5$ V). (b) Physical realization (top part: input voltage filters, bottom part: oscillator). (c) Bifurcation diagram measured experimentally as a function of R. (d) Voltage waveforms recorded in different regions (*e.g.*, periodic, quasi-periodic, chaotic with volleys, chaotic irregular). Detailed description in Ref. [98].

the interface between regions associated with oscillation modes at different frequencies, some circuits such as the one shown here also display a fine-grained bifurcation structure (Fig. 4 (c)). Visual inspection of the signals generated at different settings of R clearly illustrates the existence of diversified dynamical regimes, in that by sweeping the control parameter one elicits periodic and quasi-periodic oscillation at changing frequencies, chaos in the form of generation of irregular discharge volleys, and chaos evident as even more irregular cycle amplitude fluctuations (Fig. 4 (d)). In the case of this circuit and several others of the same kind obtained by the author and colleagues, periodicity is preserved to a detectable extent and chaos manifests itself prevalently in the form of cycle amplitude effects, which can appear at varying levels of asymmetry ranging from modulation of a sine-like wave through the generation of spike-like signals. While this unquestionably delineates a difference with respect to neural dynamics, we posit that the relevant common feature is the spontaneous emergence of multiple phases, associated with different dynamical features and degrees of order [98].

As said, the ability to seamlessly transition between generating regular spike trains, volleys or bursts, and irregular, highly disordered trains of individual spikes is a hallmark of neural dynamics, and is observed pervasively both in experimental recordings and in physiologically-realistic numerical simulations [5, 79–87]. The present observation thus hints that circuits of this kind may have relevance as building blocks for experimentally realizing networks capable of recapitulating certain emergent features of brain activity. Importantly, this assertion does not, in any way, imply that there is anything "special" about these circuits; quite on the contrary, observations of such similarity, even if taken only at face value, should inspire reflection that there is perhaps nothing inherently unique in the dynamics of neurons, since diversification of dynamics across regions of the control parameter space is a feature of virtually all nonlinear systems [18, 19].

Motivated by these results, the author and colleagues later re-considered in a broader sense the possibility of obtaining atypical transistor-based chaotic oscillators. An important issue with the initial approach was identified, namely that the cross-over operation inherent in the genetic algorithm nearly always produces an invalid circuit; consequently, given that only mutations and not inheritance matter, the process could effectively be simplified to a random search. Representing each hypothetical circuit as a string of 87 bits, which encodes an enormous number of potential candidates (Fig. 3 (a)), the search was implemented alongside basic heuristics allowing the outright rejection of circuits found to be topologically invalid or dynamically inactive. As before, the procedure entailed performing numerical simulations based on realistic circuit models while aiming to identify the onset of chaos. It was deployed on a supercomputer and, among millions of invalid and inactive circuits, hundreds of potentially-viable oscillator topologies were identified. Based on the correlation dimension D_2 and its associated uncertainty δD_2, two clearly distinct clusters were identified for the non-chaotic and chaotic individuals. Crucially, in this study, considerable attempts were made to maximize the agreement between simulation and experiment, through careful model choice and by placing particular emphasis on realizing the circuits with high-precision components, assembly techniques and packages minimizing parasitics (Fig. 3 (b)) [97]. Despite these efforts, and even though the experimental results were highly repeatable and reproducible, simulations were largely inaccurate in predicting chaoticity; they were meaningful only for the purpose of establishing the presence of sustained oscillatory activity and its predominant frequency. This finding is in line with discrepancies observed for other similar circuits and underlines the difficulties inherent in accurately simulating systems of this kind [98, 102, 103]. And yet, many of the circuits obtained through this partially-misguided search were nevertheless experimentally found to be chaotic for suitable values of the series resistor. The implication is striking and obvious:

among transistor-based circuits which are viable oscillators, chaotic oscillators do not in any way represent special or rare cases, hence they are quite easy to find. This is after all not surprising since transistors are nonlinear elements. In other words, it seems that we are led to deem chaotic oscillators as special occurrences only because of the pervasive legacy of canonical, periodic oscillator topologies. One concludes that there exists a potentially huge number of interesting and useful oscillator topologies which, despite their small size, have not yet been discovered. Many of them could be of interest not only for the present purpose of comparison to neural dynamics but also in practical engineering applications [97].

Among the one hundred circuits that were realized physically and carefully characterized, approximately half were found to be chaotic. A remarkable range of attractors could be generated by circuits of this small size, including spiral, phase-coherent attractors, attractors resembling the Rössler funnel attractor, attractors associated with a spiking behavior, attractors similar to the one of the Colpitts oscillator, attractors reminiscent of Shilnikov chaos, and attractors with a peculiar triple butterfly-like appearance (Fig. 3 (c)) [18, 19]. One seemingly previously unknown circuit topology having size and composition similar to that described above was recurrently identified by the random search, and appears to be a particularly versatile chaos generator; this is further considered in the next section. In addition, two circuits were deemed to be of particular interest. One generates a well-evident asymmetric double-scroll attractor, emerging from irregular switching between two unstable foci. This attractor is highly characteristic of Chua's circuit and has been elicited in diverse scenarios including operational amplifier-based nonlinearities, inductor-less designs, cellular neural network layouts, and monolithic designs, all of which translate into considerably more complex implementations [60, 75, 76, 97]. Besides being simpler, the present oscillator was discovered by serendipity rather than obtained as an implementation of Chua's circuit. The other notable circuit generates spikes of approximately quantized height at irregular intervals, much more closely resembling the qualitative features of neural action potentials than the first circuit described above, albeit still operating at different scales of time and voltage (Fig. 3 (d), (e)). While other circuits generating irregular spike trains have been reported, the present one is considerably simpler, and this again stems, at least partially, from the fact that it was discovered by serendipity and does not represent the realization of a predetermined mathematical model [95, 97, 104, 105]. Consideration of the distribution of avalanche size (sequences of spikes separated by time-intervals below a given threshold) in comparison to a randomly reshuffled series revealed the absence of power-law scaling, implying the lack of critical behavior and representing an important residual difference with respect to biological neural dynamics, which is further discussed in the next section (Fig. 3 (f)) [22, 42, 106, 107].

Before continuing, a brief reflection on this line of work appears appropriate. As said, the discovery of these circuits was predicated neither on a systematic method nor on any first principles. On the contrary, the work was based on experimental realizations and numerical simulations with realistic models, both of which actually include a spectrum of complex effects, such as the Early effect, voltage-variable junction capacitances, inductor self-resonances, *etc.* It should be acknowledged that, in this sense, the circuits are arguably less elementary than they may seem, since each transistor, inductor and so on represents an entity more complex than its equivalent in an idealized scenario. The impact that this has on the dynamics varies from case to case. One could argue that these circuits are therefore less relevant to progress than well-known, simple equation systems which have well-defined nonlinearities and often generate similar attractors [18, 19, 36, 38]. The author's viewpoint on this matter is that, again, there are two approaches which should be viewed as complementary and not adversarial. Serendipitous discoveries such as those reported here have an inherent value in any experimental realization, for which idealized components have limited relevance. Moreover, when attempting to realize a predetermined equation system electronically, one oftentimes makes recourse to integrated devices such as operational amplifiers and analog multipliers, which results in an incomparably larger transistor count [60, 75, 76]. In such context, one aims to minimize any non-idealities, in contrast to the fact that nature, on the contrary, seems to readily harvest any potential substrate for complexity, as in a sense was done in the present, small transistor-based circuits. Without question, this does not replace *ab initio* more rigorous theoretical and numerical work, which is needed to elucidate the route-to-chaos mechanisms at play, the minimum requirements in terms of what nonlinearities have to be present, the stability, *etc.* The fact that similar attractors are obtained between these oscillators and very different entities such as Chua's circuit and Röessler's system provides further illustration of the universality of the underlying nonlinear phenomena. Such considerations lead to a final comment regarding the relationship with memristor-based circuits, which are presently attracting enormous attention [75, 76, 108]. While symmetry considerations unquestionably support the fundamental importance of this device, to the author's knowledge as a nonlinear element in a chaotic oscillator, it does not appear to yield any unique dynamical phenomena which have not been obtained also in nonlinear circuits built of more conventional components. The importance of the memristor rather appears specifically related to the fact that it opens the way to activity permanently altering the circuit parameters through a self-contained mechanism. This has obvious importance in any attempt to represent neural plasticity, and stands in stark contrast to the present circuits, wherein all parameters are unchangeable by the emergent dynamics, and the only possible type of "memory" is in the form of hysteresis effects [109, 110].

2.3. Prime numbers, fractal structures and imperfections as sources of complexity

Constructing the oscillators described thus far unavoidably requires, in addition to transistors, also inductors and capacitors. While these circuits may have notable generative potential, the extensive usage of reactive components for their realization has undesirable practical consequences, in that it hinders the migration from implementation based on discrete components to the design of an integrated circuit. In turn, this effectively curtails their relevance to explorations at the deep microscale, intended as networks comprising at most a few tens of nodes. Even though nonlinear phenomena of potential interest are, as previously mentioned, to a certain extent similarly addressable across different scales, the ability to implement neural connectomes without excessive loss of resolution has inherent importance. For example, with reference to human functional connectivity, the author and colleagues have previously shown that parcellation granularity has a profound effect on the detectability of fundamental features such as scale-free node degree distribution, with a minimum of about 500 nodes required to properly represent the network topology [111]. On the other hand, the *Caenorhabditis elegans*, whose synaptic architecture is the only one that has been fully mapped, possesses approximately 300 neurons [112]. Even assuming a completeness of few percent links, these networks cannot be conveniently realized with discrete electronic components, due to size considerations and to the parasitic effects associated with implementing such a number of couplings at circuit-board level. On the other hand, because of their physical scaling inductors, capacitors and resistors are rather poorly suited for construction on integrated circuits. This implies that one has to accept either a high frequency of operation, which is exceptionally undesirable experimentally, or a very low area efficiency, meaning that the vast majority of silicon area is consumed in realizing these components as opposed to transistors and interconnection infrastructure [113]. This situation is, in fact, to a varying extent common to virtually all other chaotic oscillators realized with discrete transistors and operational amplifiers [75, 76].

As said, a rigorous route-to-chaos analysis has not yet been performed for the circuits described above, and inspection of the bifurcation diagrams, spectrograms and time-series suggests that period-doubling, quasi-periodicity, and intermittency effects are concomitantly present. Nevertheless, upon consideration of the self-evident fact that multiple overlapping resonant circuits yield distinct modes which can be simultaneously excited and “compete” with each other, the quasi-periodicity route to chaos acquires some appeal as a first-order way of conceptualizing the operation of these circuits [98]. In a mathematical sense, quasi-periodicity arises in the presence of oscillations linked by incommensurate ratios, however, for the purpose of causing transition to chaos, it is often sufficient to have frequencies linked by non-trivial frequency ratios, so that oscillation patterns do not repeat excessively closely and the nonlinear dynamics can be effective in preventing phase locking [18, 19, 114, 115].

One natural way of realizing non-trivial ratios is as fractions of prime numbers, a mechanism that evolution appears to have beautifully resorted to in enhancing the survival of some species of so-called periodic cicadas. These insects have developed prime-numbered life cycles of 13 and 17 years in order to reduce their vulnerability to geological effects and hybridization, and possibly to reduce resonance with the life cycles of predators [116, 117]. In a similar vein, one can consider the simplest and oldest known oscillator topology based on the inverter *Complementary Metal-Oxide-Silicon* (CMOS) logic gate, the ring oscillator, whose frequency is determined by the propagation delay and, therefore, inversely proportional to the number of stages [113, 118]. It is straightforward to realize rings having length equal, for example, to the three smallest odd prime numbers, namely 3, 5 and 7.

Predicated on these facts, the author developed a "pure" CMOS oscillator cell which comprises three or more such rings, cross-coupled in such a manner as to provide an intermediate level of energy exchange, sufficient for engendering transition to chaos but not such as to cause oscillation death. The cross-coupling can be realized in a variety of manners, however, a particularly convenient approach involves coupling each ring to all those of higher order, using CMOS-based diodes "stitching" together trice two arbitrary adjacent nodes on each ring and, in particular, linking the following: *(i)* the inputs of the chosen inverters, *(ii)* their outputs, and *(iii)* the input of the inverter in the shorter ring to the output of the inverter in the longer ring. In the present case, such scheme entails coupling the 3-ring to the 5-ring and to the 7-ring, and coupling the 5-ring to the 7-ring (Fig. 5 (a)). The resulting circuit is free of any lumped reactive element and as such facilitates the efficient use of silicon area and metal layers, yielding a compact topography even when including pass-gate switches allowing the dynamic connection and disconnection of the rings [119]. It has elements of similarity but is architecturally distinct from other CMOS-based chaos generators previously proposed by others [120, 121].

A prototype dubbed `CHARM-1` and consisting of a ring of 24 such cells, diffusively coupled via a mechanism akin to that described below for discrete bipolar transistor circuits, was realized on a standard 0.7 μm, 1-poly 2-metal CMOS process (Fig. 5 (b), (c)). Consideration of the activity of the individual, uncoupled cells reveals a particularly desirable property of this design, which is the digital controllability of chaos generation, akin to the effect of sweeping R in the previous section. As the 3-ring operates in isolation, a simple periodic waveform of stable amplitude is generated. If the 5-ring is connected, periodic envelope fluctuations appear. When the 7-ring is also enabled, transition to chaos occurs, and the dynamics can be further enriched by adding a 9-ring (despite this not being

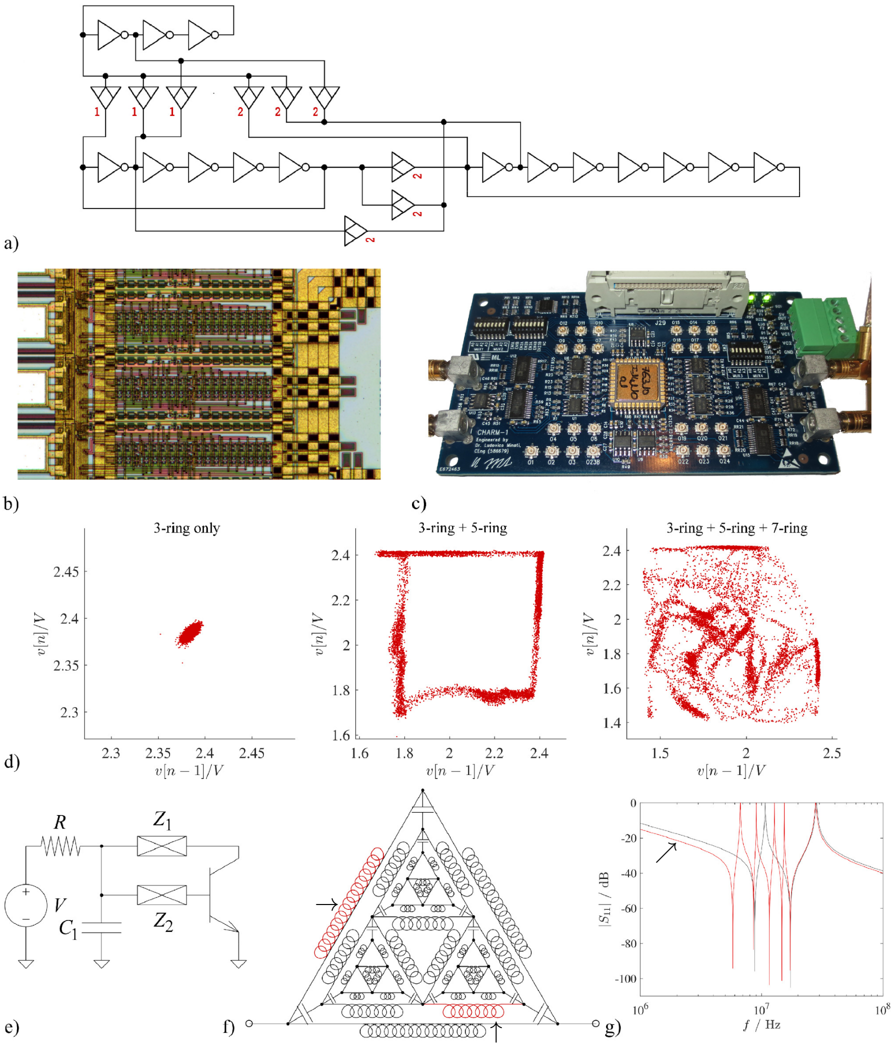

Fig. 5. (Color online) Alternative approaches to chaos generation. (a) Cross-coupled inverter rings having primal lengths, namely 3, 5 and 7. (b) Corresponding abutted cells realized on a CMOS integrated circuit. (c) Device assembled on test board. (d) Representative Poincaré sections demonstrating increasing complexity as longer rings are sequentially connected. (e) Elementary single-transistor oscillator circuit suitable for containing two complex reactive networks (Z_1, Z_2). (f) Feynman–Sierpiński fractal resonant network of depth 2, possibly containing imperfections (red/arrows: elements replaced by open or short circuits). (g) Corresponding resonances in the absence and presence of imperfections (black, red/arrow). Detailed description in Refs. [119, 122].

a prime number), or an 11-ring. Increased complexity is evident on the corresponding Poincaré sections, delineating a pattern loosely reminiscent of the folding steps for an origami (Fig. 5 (d)). This is confirmed by the corresponding correlation dimension estimated from the time-series, namely $D_2 = 1.1 \pm 0.1,\ 1.8 \pm 0.1,\ 2.2 \pm 0.3$ and 2.5 ± 0.6, and in this case, realistic simulations and measurements agree closely. In presently ongoing work precursory to the realization of complex networks, the author and colleagues are developing a revised version of this oscillator, wherein current starving is implemented independently for each ring; this enables additional, graded control over chaos generation, realizing a remarkable level of flexibility for a circuit of this size [113, 118, 119].

Even though inverters are elementary logic gates, in the analog domain, their transfer function is not trivial since the output voltage depends on the currents through the p- and n-channel MOS transistors, which visit subthreshold, linear and saturation modes as the input is swung between zero and the supply voltage. Similar considerations apply to the coupling diodes. Yet, the dynamics of this oscillator can to a certain extent be encapsulated in a much-simplified description containing exclusively piece-wise nonlinearities, which might even open the way to some level of analytical tractability. Let us represent each inverter as a capacitor of value C charged by a transconductance stage with $i_\mathrm{o} = v_\mathrm{a} G_\mathrm{i}$, wherein i_o represents the output current, v_a the input voltage and $G_\mathrm{i} < 0$ the transconductance. Since the supply voltage is finite, up to a maximum of V_s, practically one has

$$i_\mathrm{o} = G_\mathrm{i}[R(v_\mathrm{a})H(V_\mathrm{s} + v_\mathrm{o}) - R(-v_\mathrm{a})H(V_\mathrm{s} - v_\mathrm{o})]\,, \tag{1}$$

wherein $R(x)$ and $H(x)$ are, respectively, the ramp and Heaviside step functions. These are the only nonlinearities present in the model. Let us similarity approximate the diodes as piece-wise resistors having threshold voltage V_t between two nodes at voltages v_a and v_b, yielding

$$i_\mathrm{d} = G_\mathrm{c} R(v_\mathrm{a} - v_\mathrm{b} - V_\mathrm{t}) \tag{2}$$

then, taking as an example the output voltage of the first inverter in the 3-ring $v_{\mathrm{a}1}$, one can write

$$\begin{aligned}\frac{\mathrm{d}v_{\mathrm{a}1}}{\mathrm{d}t} &= \frac{G_\mathrm{i}}{C}\,[R(v_{\mathrm{a}3})H(V_\mathrm{s} + v_{\mathrm{a}1}) - R(-v_{\mathrm{a}3})H(V_\mathrm{s} - v_{\mathrm{a}1})] \\ &\quad - \frac{G_\mathrm{c}}{C}\,[R(v_{\mathrm{a}1} - v_{\mathrm{b}1} - V_\mathrm{t}) + R(v_{\mathrm{a}1} - v_{\mathrm{b}2} - V_\mathrm{t}) \\ &\quad + R(v_{\mathrm{a}1} - v_{\mathrm{c}1} - V_\mathrm{t}) + R(v_{\mathrm{a}1} - v_{\mathrm{c}2} - V_\mathrm{t})]\,,\end{aligned} \tag{3}$$

where $G_\mathrm{c} > 0$ represents a coupling conductance, $v_{\mathrm{b}1}, v_{\mathrm{b}2}$ are the voltages at the outputs of the first and second inverter in the 5-ring, and $v_{\mathrm{c}1}, v_{\mathrm{c}2}$ represent the same for the 7-ring [119].

In a different vein, we note that nature is pervaded by fractals, intended as abstract entities that are self-similar in topography, topology or over time. Surprisingly diverse objects including cloud outlines, coast lines, the vascular tree in the brain and lungs, the branching dendritic trees and axonal processes of neurons, the cortical gyri of the brain and even edible vegetables such as Romanesco broccoli have fractal shape [123–125]. Self-similarity, intended as the presence of a nested architecture of nodes and modules, or as more recently proposed by the author and colleagues intended as a relationship between edges and paths such as triangles, is readily detectable in the organization of countless networks, found for example in the domains of ecology, proteomics, and neuroscience, and also emerging from coupled chaotic oscillators [78, 126]. No less importantly, recurring dynamical features are recorded across diverse temporal scales in time-series produced by a wide range of nonlinear systems, for instance in finance and physiology, particularly regarding cardiac and brain activity, where coexistent scaling according to multiple exponents can also manifest itself in the form of multifractality [42, 127, 128].

Any attempt to survey the phenomenology and universality of fractals falls well beyond the narrow scope of the present work; however, we note that fractals, besides emerging pervasively in self-organized systems, also have geometric features that make them highly desirable for solving specific geometric problems in built devices [129, 130]. For instance, a fractal structure allows folding an unlimited area into a bounded volume, or, more practically assuming a finite number of iterations, a very large area into a small volume, and so on. Fractal-shaped elements have recently found commercial applications in engineering with regard to the construction and miniaturization of broadband antennas and resonators. In this context, the Sierpiński gasket has received particular attention as a means of realizing single- and multi-port devices whose frequency response depends on the number of iterations (or levels, depth) of the fractal. Recent work has also highlighted the apparently paradoxical properties of Feynman's inductor-capacitor ladder realized on the Sierpiński gasket, which in the limit of infinite depth is dissipative even though it only comprises purely reactive components [131, 132].

Within this framework, the author and colleagues have recently investigated the possibility of conferring high-dimensional dynamics, empirically defined as having an attractor dimension $D > 3$, to transistor-based chaotic oscillators through embedding such fractal resonators, which even when truncated have a considerably richer frequency response compared to the simpler LC networks considered so far. In particular, the smallest single-transistor oscillator previously identified as chaotic was selected as a starting point, and its two inductors were replaced with the Feynman–Sierpiński resonators, treated as two-port devices wherein the third vertex is left externally unconnected (Fig. 5 (e)). Numerical simulations and experiments concordantly indicate that increasing the fractal depth elevates the correlation dimension, even leading to hyperchaos. Interestingly, the effect is relatively insensitive to component parametric heterogeneities; however, the finite quality factor of the physical devices strongly hinders the generation of high-dimensional chaos: for the case of two fractal iterations, it lowers the maximum correlation dimension from $D_2 = 7.0 \pm 1.5$ in idealized simulations down to $D_2 = 2.7 \pm 0.1$ in measurements. Additional experiments demonstrated that this issue could be addressed through realizing more complex resonances, either by reshuffling all inductor values so that the fractal levels are no longer scaled according to a simple sequence, or more parsimoniously even only by introducing focal imperfections or defects. Replacing as few as 2 out of 39 inductors with open or short circuits can yield considerably richer resonances (Fig. 5 (f), (g)), whose exact features depend on the location of the imperfections and which aid the generation of high-dimensional chaos when instanced in a physically-built oscillator, resulting in $D_2 = 4.2 \pm 0.7$ [122].

As regards establishing tentative parallels with brain dynamics, the oscillators containing fractal resonators appear to have two noteworthy features. On the phenomenological level, they demonstrate that high-dimensional chaos can be generated even in the context of such simple circuits, wherein the only nonlinearity is represented by the transistor. This appears relevant because at the macroscopic scale, brain dynamics carry signatures of high-dimensional chaotic dynamics, which are posited to transiently collapse down to lower-dimensional dynamics in order to implement coding functions while responding to specific stimuli [81, 133]. The present, simple circuits might, therefore, also have relevance as experimental platforms to reproduce such kind of effects. On the architectural level, the results point to two speculative commonalities with the dendritic layout. Firstly, as mentioned above, the dendrites have fractal morphology and realize complex spatiotemporal summation processes; it appears that, despite the profoundly different nature of the system and fractal structure, also in the case of these oscillators, the self-similarity of a network serves to enrich its dynamics [134, 135]. It should, however, be underlined that in this study, it was not established whether similar effects could also be established, for example, in random topologies yielding arbitrary resonances. Secondly, natural objects are not mathematical fractals, not only in that they are truncated, by also in that they display pervasive deviation from regular recurrence: accordingly, the shape of neurites is fractal only in a statistical sense. The results obtained with these circuits motivate querying whether such situation may not only stem from a biological difficulty in realizing perfectly regular structures but may also play an adaptive role: evolution could have leveraged these imperfections as a means of supporting high-dimensional dynamics despite leakage, *etc.*, rhyming with the profound effect of even subtle defects on the crystallographic and macroscopic properties of solids [136].

It can easily be shown that in Feynman–Sierpiński resonators wherein the inductor values are halved at each iteration, representing edge lengths, and wherein the capacitances are kept constant since they do not have a geometric counterpart, the effect of increasing fractal depth is to add a pair of conjugate imaginary poles and zeros per iteration. This implies that a Foster equivalent network can be constructed, wherein each level of the fractal is condensed into a block comprising an inductor and a capacitor connected in parallel, considerably aiding the experimental realization. Furthermore, one may represent the transistor nonlinearity in a stylized manner through a hyperbolic tangent and a ramp function, the only additional requirement being to capture the junction capacitances in the form of one fixed capacitor C_2 between the collector and ground [122]. Following appropriate substitutions and normalizations, namely $x = v_{C_1}$, $y = v_{C_2}$, $z = Ri_{\hat{L}_0^{(1)}}$ and $w = Ri_{\hat{L}_0^{(2)}}$ (where $\hat{L}_0^{(1)}$ and $\hat{L}_0^{(2)}$ represent the inductors in the original circuit, now corresponding to the zeroth-iteration of the fractal), one obtains the following equation system, which is well-suited for numerical investigation:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{V_\mathrm{s} - x - z - w}{\tau_1}\,, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{w - \alpha(z, y)}{\tau_2}\,, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{1}{T_0^{(1)}}\left(x - \sum v_j^{(1)} - V_\mathrm{th}\right)\,, \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = \dfrac{1}{T_0^{(2)}}\left(x - \sum v_j^{(2)} - y\right)\,, \end{cases} \tag{4}$$

where, as said, the transistor nonlinearity can be represented by

$$\alpha\,(x, y) = \beta\varGamma(x)\tanh(ky)\,, \tag{5}$$

where k is an arbitrary scaling factor, the inductances and capacitances are represented by the time-constants $\tau_1 = RC_1$, $\tau_2 = RC_2$, $T_0^{(1)} = \hat{L}_0^{(1)}/R$ and $T_0^{(2)} = \hat{L}_0^{(2)}/R$, and where for each level of the fractal $j = 1, \ldots, n$

$$\begin{cases} \dfrac{\mathrm{d}v_j^{(1)}}{\mathrm{d}t} = \dfrac{z - u_j^{(1)}}{\tau_j^{(1)}}, \\ \dfrac{\mathrm{d}v_j^{(2)}}{\mathrm{d}t} = \dfrac{w - u_j^{(2)}}{\tau_j^{(2)}}, \\ \dfrac{\mathrm{d}u_j^{(1)}}{\mathrm{d}t} = \dfrac{v_j^{(1)}}{T_j^{(1)}}, \\ \dfrac{\mathrm{d}u_j^{(2)}}{\mathrm{d}t} = \dfrac{v_j^{(2)}}{T_j^{(2)}}, \end{cases} \tag{6}$$

wherein the state variables are similarly replaced by the voltages $v_j^{(k)} = v_{\hat{C}_j^{(k)}}$ and $u_j^{(k)} = Ri_{\hat{L}_j^{(k)}}$, and the inductances and capacitances are replaced by the time constants $\tau_j^{(k)} = R\hat{C}_j^{(k)}$ and $T_j^{(k)} = \hat{L}_j^{(k)}/R$, realizing, for n iterations of the fractal, a system of the order of $4 + 4n$ [122].

These studies exemplify two approaches to controlling the generation of chaos which have different practical and conceptual relevance for the present purpose. The former involves cross-coupled inverter rings having primal length, and appears practically well-suited for realizing large mesoscopic-scale networks on an integrated circuit. The latter entails replacing inductors with fractal resonators potentially containing imperfections, and has appeal as a means of obtaining high-dimensional chaos even in small transistor-based circuits.

2.4. Concomitant avalanching, hysteresis and metastability in an array of neon lamps

As exemplified throughout the following sections, the circuits described up to this point enable reproducing a multitude of synchronization phenomena; however, the author's research to date has failed to elicit in them any convincing signature of critical dynamics. Even for the transistor-based oscillator generating quantized spikes (Fig. 3 (d)), the distribution of inter-spike intervals is not over-dispersed compared to a Poissonian, and no power-law scaling is evident. The implication is that, thus far, these circuits can only provide a partial recapitulation of neural dynamics, excluding the critical phenomena which selectively occur close to the point of transition between ordered (periodic, laminar) and disordered (chaotic, turbulent) dynamics [42, 106, 137, 138]. The experiment summarized in this section represents an initial attempt to address this issue via recourse to a different nonlinear electronic device, which in virtue of its dynamics appears better suited for reproducing such effects.

It is perhaps remarkable that, upon a suitable choice of the physical variables and observation scale, much of the thermodynamical theory of phase transitions can be applied across diverse dynamical systems, yielding a robust theoretical framework within which one can formulate well-defined predictions about critical behaviors. In paradigmatic scenarios such as the Ising model, the correlation length and other observables diverge at the critical point, namely as $T \to T_\mathrm{C}$, $\xi \to \infty$, and in its proximity, such observables follow the relationship $A(T) \propto (T - T_\mathrm{C})^\alpha$, where α is a critical exponent. Diverse systems are found to possess identical critical exponents, revealing the existence of a limited number of universality classes reflecting the nature of the interactions, symmetry, *etc.* [139]. A broad class is that of branching processes, which are characterized by the fact that individual events generate descendants according, in the simplest case, to a fixed probability distribution; the reproduction of living organisms, nuclear chain reactions, the propagation of breakdown events in materials and the generation of neural action potentials all belong to this class. When the average probability of generating a descendant (branching parameter σ) is close to unity, the distributions of size (number of events) and duration of avalanches diverge according to critical exponents $\alpha_\mathrm{S} = 3/2$ and $\alpha_\mathrm{D} = 2$, yielding a power spectral density which decays similarly to $1/f^\beta$ noise. Experimental evidence suggestive of critical avalanching in neural systems has been coherently obtained across the micro-, meso- and macroscales, using *in vivo* techniques such as electroencephalography, magnetoencephalography and functional magnetic resonance imaging, and *in vitro* techniques such as recording the activity of spontaneously developing neural cultures or brain slices via multi-electrode arrays. As previously mentioned, computational studies further indicate that operation close to the point of criticality may confer advantages as regards, for example, to maximizing the dynamic range available for encoding sensory inputs and the ability of rapidly switching between activity patterns [12, 22, 42, 106, 107, 140–142].

The author and colleagues have realized a circuit possessing integrate-and-fire dynamics, which knowingly give rise to critical phenomena not only in neural systems but also in earthquakes and nuclear reactions, by exploiting the physical characteristics of glow lamps, which are small cold-cathode, gas discharge tubes typically filled with a neon–argon mixture [22, 140]. While presently mainly relegated to the function of inexpensive line-voltage indicators, these devices are physically complex and possess remarkably rich dynamics. Not only they were once relied upon for constructing active logic circuits, but electronic chaos was discovered by Van der Pol in an oscillator-based precisely on these devices, whose potential in creating physical analogues of neural dynamics has also been suggested before. Their main feature is possessing strongly hysteretic behavior, such that the breakdown (ignition) voltage V_b is appreciably larger than the recovery (extinction) voltage V_r, and the probability of transition between the "on" and "off" states, respectively corresponding to low and near-infinite resistance, markedly depends on the applied potential; for example, the corresponding rates can be approximated with $e^{(v-V_\mathrm{b})/\alpha}$ and $e^{(V_\mathrm{r}-v)/\beta}$ [91, 143, 144].

The circuit under consideration consists of a two-dimensional square lattice, wherein each cell comprises one neon lamp connected via a resistor to a globally-applied static voltage V_s acting as the only control parameter; each cell is furthermore capacitively coupled to its von Neumann neighbors of unitary distance (Fig. 6 (a)) [145]. An instance having size 34×34 was physically realized on a circuit board dubbed `VAN DER POL-1`, and instrumented with cameras and photodiodes accurately capturing the spatiotemporal evolution of activity (Fig. 6 (b)).

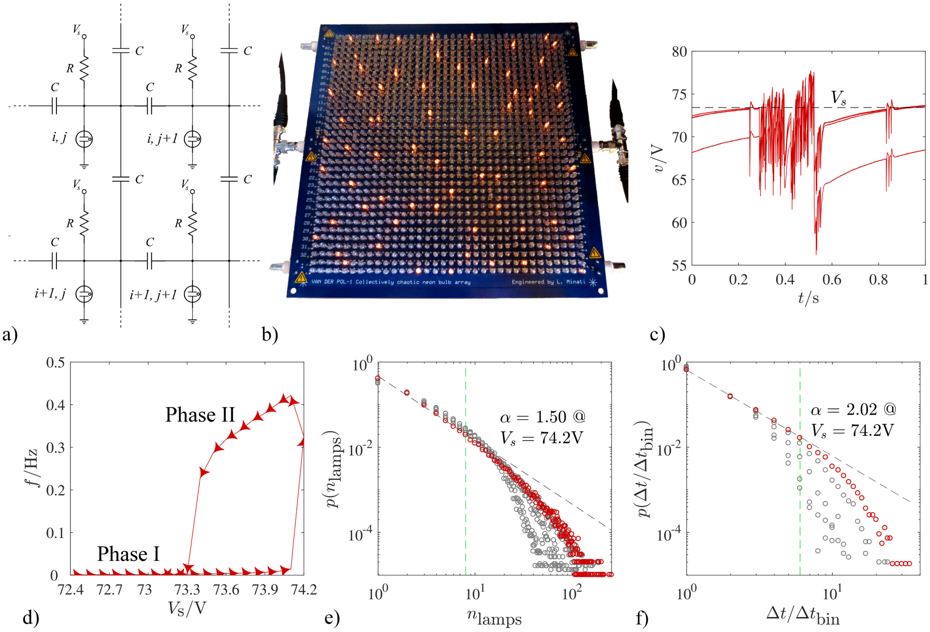

Fig. 6. Two-dimensional 34 × 34 lattice of capacitively-coupled neon lamps. (a) Circuit diagram (R = 2.2 MΩ, C = 220 nF). (b) Physical realization. (c) Measured voltage at an arbitrarily-chosen node, illustrating single breakdown (ignition) events and avalanches. (d) Hysteresis loop demonstrating first-order transition between a disordered, low-rate phase (Phase I) and a more ordered, high-rate phase (Phase II). (e) and (f) Avalanche size and duration distributions, confirming divergence and critical scaling as the globally-applied voltage approaches the spinodal (darker/red: highest voltage). Detailed description in Ref. [145].

The peculiar dynamics of this circuit are readily apparent upon considering a chain of three cells, whose corresponding voltages we denote as v_0, v_1 and v_2. Assuming that, following the breakdown (ignition) of the first lamp, $v_0 < V_\mathrm{s}$ and $v_1(0) = v_2(0) = v_0$, one has

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(v_2 - v_1) = \dfrac{V_\mathrm{s} - v_2}{CR}\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(v_1 - v_0) = \dfrac{2V_\mathrm{s} - v_1 - v_2}{CR} \end{cases} \tag{7}$$

whose solution is, assuming for convenience $V_\mathrm{s} = 0$,

$$\begin{cases} \dfrac{v_1(t)}{v_0} = \dfrac{1}{1+\varphi^2} e^{-t/\tau_1} + \dfrac{\varphi^2}{1+\varphi^2} e^{-t/\tau_2}\,, \\ \dfrac{v_2(t)}{v_0} = \dfrac{\varphi^3}{1+\varphi^2} e^{-t/\tau_2} - \dfrac{\varphi^{-1}}{1+\varphi^2} e^{-t/\tau_1}\,, \end{cases} \tag{8}$$

where $\tau_1 = RC\varphi^2$, $\tau_2 = RC\varphi^{-2}$ and φ is the golden ratio. This reveals without loss of generality that an overshoot exceeding the globally-applied control voltage is produced by each event, conferring to it the ability of generating descendants via the voltage dependence of the ignition probability. Accordingly, when the control voltage is sufficiently close to the average ignition potential, which is subject to considerable variation between lamps due to construction tolerances, the avalanching behavior is promptly observed both numerically and experimentally (Fig. 6 (c)). Moreover, it is easy to see that $v_2 > v_1$, in other words the overshoot is largest at the farthest site along the chain, which straightforwardly generalizes to more dimensions: as a consequence, despite the short-range structural couplings, activity propagates over distances comparable with lattice size [145]. This is reminiscent of synchronous-mode propagation of action potentials, and of observations in some other physical systems, wherein short-range coupling effectively realizes long-range interactions, leading to mean-field behavior [146, 147].

As the control voltage is swept by an external controller, a marked hysteresis effect is observed for the transition between two phases: one is glass-like and characterized by low event rate and low spatiotemporal order (Phase I), the other is crystal-like and characterized by higher event rate and higher spatiotemporal order (Phase II; Fig. 6 (d)). As the voltage is gradually increased approaching the Phase I → Phase II transition, avalanches hallmarking the expected divergence of size and duration according to the same critical exponents α_S and α_D regulating neural activity are readily generated (Fig. 6 (e), (f)) [145].

Critical phenomena exemplified by avalanche generation are generally associated with second-order phase transition, such as entailed in the framework of self-organized criticality, which posits that collective dynamics give rise to an attractor drawing a system towards the critical point [42, 106, 137, 138]. However, hysteresis in the present experiment unequivocally signals a first-order transition, which is further indicated by observing that, given the finite system size, the phases have finite lifetimes. This apparent incongruence is resolved through considering that critical phenomena can also appear in first-order transitions insofar as one enters the metastability region and approaches the spinodal curve, a situation similarly found in geophysical phenomena, fracture propagation, and network recovery; in all these scenarios, transition precursors diverge to infinity on the spinodal itself. In the present case, two spinodal voltages $V_s^{(1)} < V_s^{(2)}$ exist, at which the lifetimes of Phase II and Phase I respectively vanish [146–149]. Further confirmation of the transition order is derived from the fact that $\sigma > 1$ is never observed, namely the system cannot be super-critical, reflecting the discontinuous nature of the transition [140]. The present results therefore delineate a transition belonging to the same universality class as breakdown in disordered media, or the transition in the democratic fiber bundle model or in the long-range Ising model on the spinodal lines [138, 146, 147].

There exists, in fact, incontrovertible evidence of metastability and hysteresis in the brain, grounded in neurophysiological and even behavioral observations of its ability to sustain activity patterns and cognitive states which have long but finite lifetimes, past which sudden transition to another state occurs [5, 11, 150]. Accordingly, non-equilibrium first-order phase transition can be induced by an external field or additive noise, yielding situations wherein a coexistence region is found as a function of the control parameters. In this vein, it has recently been hypothesized that brain dynamics may not necessarily be drawn towards a critical point by a self-organized process as initially thought, but may operate close to a first-order transition, dwelling in the vicinity of bistability [151, 152]. Inspired by such a work and the present results, the author and colleagues later introduced a model based on leaky integrate-and-fire neurons, which concomitantly generates scale-free avalanches and metastable attractors reflecting stored patterns [153].

Altogether, the present results not only reaffirm the possibility of recapitulating chosen aspects of neural dynamics including criticality in other physical systems, in this case coupled gas-discharge tubes as opposed to solid-state devices, but also clearly exemplify how such comparisons may tangibly inspire a speculative re-evaluation of paradigmatic approaches to modeling neural dynamics.

3. Synchronization and pattern formation

3.1. Cluster synchronization in a transistor network

Having introduced a variety of nonlinear electronic circuits as possible generators of complex dynamics, the remainder of this paper will survey some initial attempts to couple them for realizing networks related in diverse manners to neural systems.

One prominent aspect of the organization of natural and artificial systems is their modularity. In the context of morphology, this refers to the fact that, in certain species, individuals comprise an indeterminate number of instances of modules, intended as well-identifiable and partially self-contained entities such as the leaves of a tree. The question of whether the brain and mind possess any degree of modular organization was among the earliest ones to be asked in the primordia of neuroscience, eventually yielding the hypothesis of a univocal association between mental skills and bumps of the skull. While nowadays entirely discredited, this viewpoint has been precursory to the modern theory of functional localization. It is presently recognized that the mind possesses a certain level of segregation between cognitive domains and skills, which is particularly evident in studies of optical illusions and autism. Similarly, structural and functional brain networks are also organized according to a modular structure, delineating bihemispheric as well as lateralized subnetworks, which are putatively related to sensorimotor processing, vision, cognition, and awareness. On the whole, modularity is frequently regarded as having emerged in response to environmental pressure, wherein specific domains would have evolved along distinct trajectories and remained partially segregated for diverse reasons. These include the need for operation over disparate time-scales, the fact that low-level skills such as visual and auditory processing are preferably implemented on tailored neural architectures, and the need to minimize writing cost and enhance overall resilience to damage. Notably, some degree of modular organization is detectable even in the simplest nervous systems such as that of the *Caenorhabditis elegans*, as well as in countless other natural and artificial networks [11, 26, 154–157].

In this section, an experiment exemplifying the spontaneous emergence of a synchronization pattern having modular features via a physical mechanism known as cluster synchronization is introduced. The phenomenon represents a type of partial synchronization, intended as a situation wherein there is, on average, detectable entrainment between the nodes of a network, but this is not sufficient to provide an energy exchange rate maintaining a globally-coherent trajectory: this is the region where pattern formation can occur. While the onset of complete synchronization can be addressed via consolidated tools such as the master stability function, the spatiotemporal patterns emerging under partial synchronization are, in general, considerably more difficult to predict. Cluster synchronization reflects the preferential entrainment among specific node subsets which become more synchronized between themselves than with the rest of the network. This effect can emerge in networks of identical or mismatched oscillators, either as a consequence of the topology of structural connections which may *ab initio* possess some degree of modular organization or as a consequence of the node dynamics. In some cases, the locations of the modules (or communities) are relatively easy to explain, for example owing to the action of a provincial hub which pulls together its neighbors, but frequently they follow a highly nontrivial interplay between structure, dynamics and parametric heterogeneities [19, 36, 38, 158–163].

The present experiment is based on a ring of single-transistor oscillators of the type introduced in Section 2.2, interconnected through diffusive coupling implemented by means of resistors having value R_C, attached to the collector nodes of all pairs of neighboring transistors and realizing an energy-exchange mechanism closely similar to an elastic band (Fig. 7 (a)). While experimental synchronization between pairs of single-transistor oscillators has been repeatedly demonstrated before, limited data were available on larger networks, wherein more compelling instances of morphogenesis can be found [75, 76, 164, 165]. A physical setup was devised by the author in the form of a circuit board, dubbed `STRANGE-1`, carrying 30 oscillators coupled as indicated alongside readout amplifiers and multiplexers allowing signal acquisition with minimal disturbance to the dynamics (Fig. 7 (b)). As expected, for very low and very high values of the coupling resistor R_C, complete synchronization and desynchronization are observed. In between these two extremes a partial synchronization region is found, wherein, for increasing coupling strength (lower resistor value), first phase locking occurs while amplitude fluctuations remain decoupled, then, as the energy transfer rate becomes sufficient, amplitudes also become synchronized. Such a scenario is typical of dynamical systems containing heterogeneities, which in this case were limited to the parametric mismatches introduced by the component tolerances [18, 19, 36, 38, 166].

While the detection and measurement of modularity are not trivial issues particularly as regards resolution considerations, in this case, the network was relatively small, hence a canonical measure termed Louvain modularity could be considered: it represents the average synchronization within as opposed to outside modules according to a hypothetical partitioning, which is greedily optimized aiming to maximize the modularity parameter

$$Q = \frac{1}{2m} \sum_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j)\,, \tag{9}$$

where A_{ij} is the matrix of edge weights (corresponding, in this case, to phase or amplitude synchronization after suitable rescaling), $2m = \sum_{ij} A_{ij}$ denotes the overall connectivity, $k_i = \sum_j A_{ij}$ the weighted node degrees (nodal strengths), c_i community membership and $\delta(i, j)$ is Kronecker's delta [26, 167].

Two observations were made. Firstly, $Q(R_\mathrm{C})$ follows a non-monotonic trend indicating that not only synchronization spontaneously develops significant modular features, but these are emphasized at an "optimal" intermediate coupling strength. Notably, this trend is observed similarly for phase and amplitude synchronization, however, the strongest modularity appears

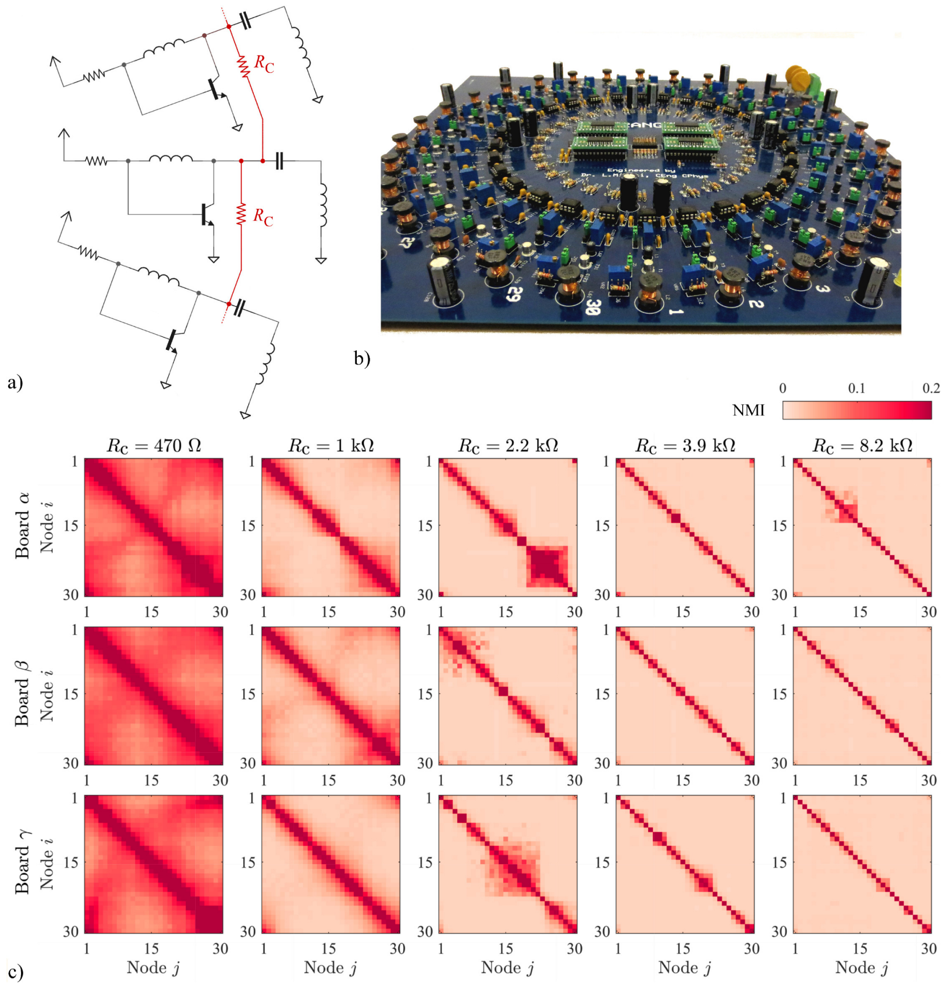

Fig. 7. Single-transistor chaotic oscillators diffusively coupled to form a 30-ring network. (a) Circuit diagram (excerpt). (b) Physical realization. (c) Synchronization matrices (normalized mutual information) recorded from three board specimens α, β and γ, demonstrating the formation of communities having a size varying with the coupling resistor value R_C and a topography reflecting the parametric heterogeneities. Detailed description in Ref. [166].

at different coupling strengths, respectively, $R_\mathrm{C} \approx 2.2$ kΩ and $R_\mathrm{C} \approx 120$ Ω. This indicates that cluster synchronization takes places concomitantly for phases and amplitudes and that its emergence requires an intermediate level of respective entrainment: as is known, a greater energy transfer rate is required to overlap the entire trajectories as opposed to merely maintaining phase locking between each oscillator pair. As the coupling strength was increased or decreased around these values, aggregation into larger communities and disaggregation into smaller ones were clearly observed (Fig. 7 (c)). Secondly, the trend of $Q(R_{\mathrm{C}})$ was closely repeatable over three realizations of the circuit board; however, the topography of the communities was uncorrelated between them, indicating that it was driven by the small parametric heterogeneities unavoidably associated with the individual physical components. Notably, the emergent organization was rather stable to perturbations, even though communities could be destroyed by adding links to distant regions, since this leads to large changes in the landscape of energy minima related to the synchronization manifolds. Due to the reasons described in the previous sections, the present effects could only partially be recapitulated in SPICE simulations [36, 38, 166].

Altogether, these results exemplify a situation wherein synchronization patterns having complex features are formed by an elementary structural network whose symmetry is broken through the presence of small mismatches: the emergence of communities and their size distribution are closely controlled by a single parameter, namely the coupling strength. This leads to three speculative parallels with neural dynamics. Firstly, upon application of suitable methods, a broad distribution of community sizes is readily observed in the synchronization of brain activity, which, furthermore, appears to respond to cognitive states dynamically: extrapolating from the present findings, one can hypothesize that this reflects fluctuations in the underlying coupling strengths, which in some situations are sufficient to "hold together" large modules and in others only entrain small groups of affine regions [168, 169]. Secondly, the local properties of the synchronization pattern, intended as the precise topography of the communities, were strongly driven by the individual parametric heterogeneities, yet the global effects, intended as the relationship between the coupling control parameter and the overall network topological properties, were highly repeatable. At least phenomenologically, the above recalls the situation for the localization of functions over the cortex, wherein inter-individual variability leads to significant topographical heterogeneity (leading, for example, to the need for careful presurgical planning before lesion resection), yet global features such as the presence of certain structure-function relationships are highly consistent, even between species [5, 11, 170]. In turn, this leads to speculating that the cortical organization of individual brains could be viewed as akin to morphogenesis in elementary networks similar to the present one: genetic and epigenetic factors reflected in the neural phenotypes play the same role of the parametric heterogeneities, then dynamics lead to the emergence of convergent, robust common features; clearly though, one fundamental difference is that in this simple electronic system there is no plasticity. Thirdly, even acknowledging the adaptive role of modular brain organization leaves open the question of what evolution would have "locked in" to begin tuning towards this feature: evolution in itself is selective, not generative. Results such as those presented here illustrate that cluster synchronization is a pervasive phenomenon, capable of yielding rich patterns out of elementary structures. Hence, one may speculate that, deep towards the bottom of the phylogenetic tree, this nonlinear phenomenon may have lead to the initial emergence of preferentially-synchronized communities in some "proto-brains". These features could then have become enhanced and eventually engraved also into the structural architecture of more recent nervous systems [26, 155–157].

3.2. Effect of non-trivial structural connectivity on dynamics: a toy model brain

As introduced above, brain structural and functional connectivity, alongside countless other self-organized networks, possess a considerable level of self-similarity, in other words, display scale-free features. One manifestation of this property is as a power-law node degree distribution, which implies the existence of hub nodes that are disproportionately strongly connected with the entirety of the network. Such hubs are topographically scattered over distant locations, primarily in the lateral parietal lobes, precuneus and medial frontal cortex, and are preferentially interconnected between themselves forming a so-called "rich-club". They have specific cytoarchitectonic features, serve to orchestrate high-level information integration, attention and awareness functions, and are accordingly implicated, causally or epiphenomenally, in a broad range of neurological disorders [5, 11, 24, 25, 31, 46, 171, 172].

The dynamics of the hub regions in the human and more generally mammalian brain is preferentially reflected by the so-called "resting-state networks", which are spatiotemporally-coherent patterns of activation that are spontaneously generated while at rest; these project into slow fluctuations of brain haemodynamic variables like the blood oxygen level-dependent signal and are detectable by means of techniques such as functional magnetic resonance imaging and independent component analysis. Notably, even though the neural dynamics unfold on the scale of milliseconds, these networks are commonly delineated from very slow fluctuations, occurring on the order of seconds, which reflect the heavy temporal smoothing imposed by neurovascular coupling. How it may be possible to establish meaningful inferences on neural activity at such a coarse temporal resolution remains puzzling, yet substantial empirical evidence is available confirming that the integrity of the resting-state networks is pervasively linked to cognitive states, traits, and disease conditions. The prevailing view is that this is an effect of the temporal scale-freeness of brain activity, which is apparent primarily in the form of a $1/f^{\beta}$-like spectrum and is deemed to be ultimately consequential to operation close to criticality; such perspective allows hypothesizing that similar information may be contained between the scale at which neural dynamics unfold, *i.e.* $\approx 10^2$ Hz, and the scale trackable by hemodynamics, *i.e.* < 1 Hz, indexing the tail of the power law [5, 28, 29, 173–178].

Predicated on the above, the author and colleagues have explicitly addressed the question of whether cortical hubs preferentially generate low-frequency activity. In order to avoid biasing the analyses, topographic maps of the node degree of synchronization were calculated at the voxel level, based on haemodynamic time-series acquired at a temporal resolution of ≈ 1.4 Hz using multiband techniques as part of the Human Connectome Project (Fig. 8 (a)) [179]. The corresponding operation is particularly computationally-intensive since it requires calculating an array of $\approx 10^9$ linear correlations; however, it could be accelerated, for example, through approximation with the ℓ_1-norm and deployment on a dedicated processor array, dubbed `HEBB-1` and based on a loosely neuromorphic architecture (Fig. 8 (b)) [180]. A remarkably strong statistical association is observed, whereby the amplitude of low-frequency activity steadily increases with the node degree [181]. It should be noted that in this case, the node degree of synchronization was considered as opposed to that of structural connectivity mainly because it could be computed from the same functional dataset and, therefore, allows the most direct comparison; assessment of brain structural connectivity requires different measurement techniques such as diffusion-tensor imaging [182].

Since large-amplitude low-frequency fluctuations could clearly emerge without a relationship to synchronization, such a relationship is not trivial and leads to hypothesizing that a nonlinear phenomenon may link these distinct local reflections of brain dynamics, namely synchronization and spectral shape. Since physiological time-series are highly noisy, this question was addressed by comparing the experimental recordings to surrogate data having identical value distribution and autocorrelation but devoid of any nonlinear effect. The chosen approach yields some level of confidence in detecting weak nonlinearity and is relevant especially in a purely observational framework such as the present one, wherein there are no accessible control parameters through which the dynamics can be causally investigated. For reasons related to its robustness, particularly on short time-series, the correlation dimension was selected for comparing the measured and surrogate data. In the presence of temporal self-similarity, the local slopes of the correlation integral converge to a plateau which spans an extended scaling range and provides an estimation of the possibly fractional dimension of the attractor; contrariwise, for a purely stochastic process, the curves do not exhibit a plateau, because the phase space is saturated irrespective of the embedding dimension. Accordingly, in this case, two signatures of nonlinear dynamics were considered, the difference in plateau level $\hat{D}_2$–D_2 and the tightness of convergence $\delta\hat{D}_2$–δD_2, identically and independently computed for the surrogate data ($\hat{D}_2$, $\delta\hat{D}_2$) and the experimental data (D_2, δD_2) [17–20, 34, 35, 37, 100, 101, 181]. According to both measures, a significant topographical correspondence is observed, whereby the most intensely synchronized regions, particularly in the lateral parietal and medial frontal cortex, also yield the strongest evidence of nonlinear dynamics (Fig. 8 (a)); qualitative differences are also well-evident between time-series recorded within and outside these regions (Fig. 8 (c)).

Such an association should be interpreted cautiously due to the limitations inherent in hemodynamic tracking of neural activity, to physiological contamination and to the only weak evidence of determinism that was obtained; moreover, since high-dimensional dynamics are expected in the brain, this may at best only be a distant reflection of the underlying attractors, also in light of the known difficulty in properly estimating the dimension in such cases, even more so based on scalar time-series [18, 19, 81, 133, 183]. Yet, associations between connectivity and the generation of low-frequency fluctuations cohesively transpire through the multitude of existing computational studies of the connectome, which are based on diverse approaches including realistic spiking models, mean-field models, two-state models, Kuramoto and Wilson–Cowan oscillators [175]. For example, it has been established that in the coupled Wilson–Cowan oscillators, slow fluctuations emerge for the degree of synchronization between communities, even in the absence of strong modulation of the activity of the individual populations [43, 184]. Similar effects are observed under partial synchronization in the Kuramoto oscillators with weak delayed coupling, and in mesoscale models of neural populations having critical excitatory/inhibitory balance and density of long-range connections [185, 186].

The author and colleagues allowed themselves to speculate that the observations could reflect a relationship between connectivity and dynamics which is at least to some extent generalizable. In order to address this hypothesis, a toy model network was prepared, based on a large ring of 90 oscillators analogous to that considered in the previous section. In this case, however, 9 long-distance connections were added, which inter-linked four ring segments elected to tentatively represent hub regions. One of them comprised 9 nodes, whereas the remaining ones were smaller, consisting of 3 nodes each. The smaller hubs were connected with the large one according to an intertwined pattern, reflecting an attempt to recapitulate the architecture of the default-mode network, wherein the lateral parietal and medial frontal regions hinge around the precuneus, which has the densest and most intricate axonal connectivity (Fig. 8 (d)). The setup was physically realized on three circuit boards, implementing the long-distance links with capacitive coupling so as not to change the working point of the oscillators spuriously; fixed values were set for the coupling resistors between ring neighbors and distant sites, respectively, $R_{\mathrm{C}} = 750\ \Omega$ and $R_{\mathrm{L}} = 40\ \Omega$, realizing an intermediate level of entrainment yielding partial synchronization (Fig. 8 (e)) [181].

As expected, elevated synchronization is elicited within the collective of the nodes interconnected to form the hubs, and the effect is consistent over a wide range of settings of the power supply series resistor, which, as said, controls the dynamics of each individual oscillator (Fig. 8 (f)). When these

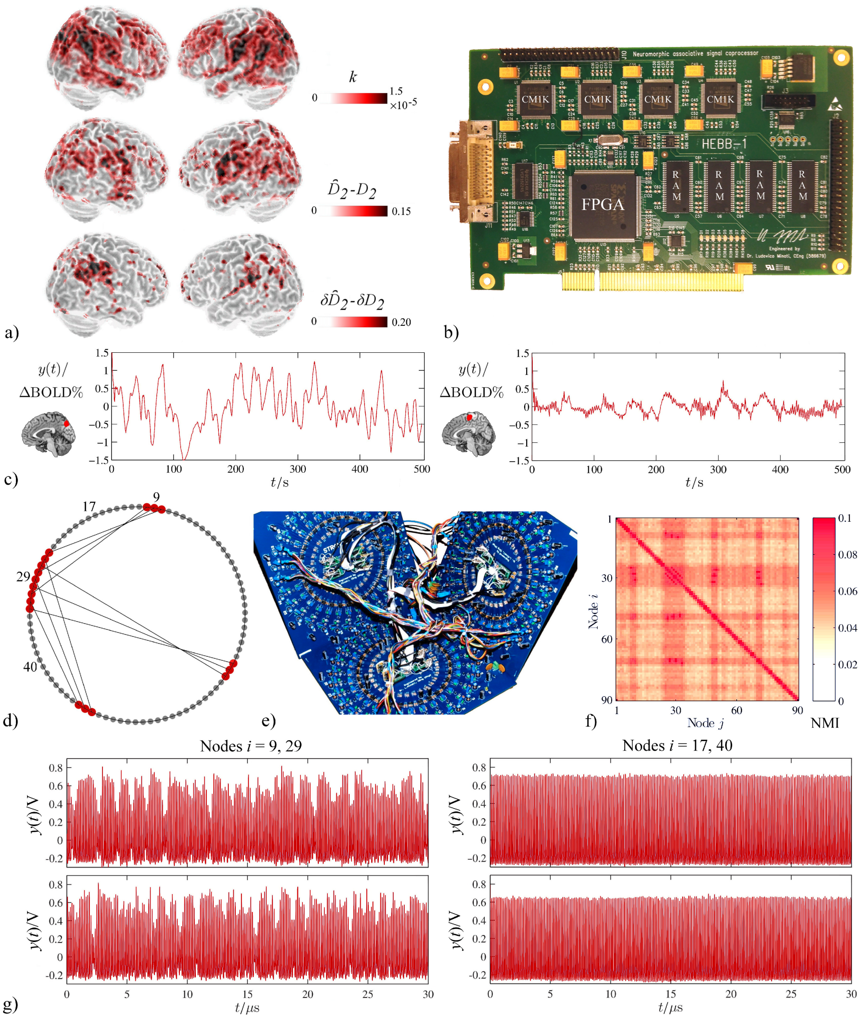

Fig. 8. Brain connectivity and nonlinear dynamics. (a) Co-localization of high synchronization node degree (k) and evidence of nonlinear dynamics w.r.t. surrogate data ($\hat{D}_2$–D_2, $\delta\hat{D}_2$–δD_2) in cortical hubs. (b) Co-processor board accelerating node degree calculation. (c) Haemodynamic time-series within and outside a hub region (left, precuneus; right, precentral gyrus), respectively demonstrating presence and absence of large-amplitude low-frequency fluctuations. (d) Toy model 90-ring network including four hardwired hub regions (bigger dots/red). (e) Physical realization. (f) Synchronization matrix confirming entrainment among the structurally-coupled nodes. (g) Representative time-series illustrating the selective transition to chaotic dynamics within the hub segments. Detailed description in Refs. [180, 181].

resistors are tuned for operation well into the chaotic or periodic regions, connectivity has no qualitative impact on the dynamics; however, when they are tuned close to the order-to-chaos transition (*e.g.*, $R = 1223 \pm 230\ \Omega$), a notable reorganization takes place. The network sharply splits into two, featuring periodic or quasi-periodic dynamics with small-amplitude fluctuations outside the segments wired as hub regions, and chaotic dynamics featuring large, low-frequency amplitude fluctuations inside them. These fluctuations emerge at a frequency considerably lower compared to that of the oscillators, recalling the observations in computational models and recapitulating, at least phenomenologically, the correspondence experimentally observed for the brain (Fig. 8 (g)). Similar results, highly reproducible over different network configurations, were obtained by estimating the correlation dimension. Nevertheless, a significant limitation of this initial, provocative work is that it is purely phenomenological; no attempts were made to understand the mechanisms underlying this effect, especially regarding the roles of node degree and topological distance of the nodes elected to construct the hubs, which are not discernible in this experiment [26, 98, 166, 181]. Yet, ongoing research by the author and colleagues suggests that a similar relationship between synchronization and nonlinear dynamics also arises in neurons cultured *in vitro*, which are initially dissociated and gradually regrow a structural network having small-world and scale-free features [187, 188]. Rigorous computational verification over multiple network topologies, sizes and node dynamics is now required.

Altogether, these results point to a relationship between nonlinear dynamics and the presence of hubs, intended as nodes or node subsets strongly structurally connected to the rest of the network including via long-range links and, therefore, highly synchronized with it. Namely, in the two scenarios under consideration, there appears to be a greater expression of nonlinear dynamics within the hubs, translating into the generation of large-amplitude low-frequency fluctuations, over a diffuse background of stochastic or periodic activity in the rest of the network. More generally, the present work exemplifies how a toy model electronic network can be leveraged in exploring whether an observed effect is generalizable beyond the macroscopic brain, and, in turn, illustrates how such models can at times provide insights despite the substantially different node dynamics, instantaneously diffusive coupling, incomparably smaller size, *etc.* This suggests that other toy model networks of this kind, for example capturing aspects of modularity, could be of use in establishing parallels between the brain and other physical systems, and perhaps also valuable for querying the phenomenology of network-related diseases such as some neurodegenerative dementias [172]. A crucial aspect of the importance of toy models in physics indeed lies in their ability to eventually generate new paradigms, and this possibly encompasses experimental comparisons of the present kind [58]. In this vein, the next section will introduce results obtained as regards another synchronization phenomenon which appears to be simultaneously detectable in the resting-state brain networks and replicable in a small electronic system.

3.3. Remote synchronization in a field-programmable analog array network

The statistical property of brain dynamics most closely associated with the emergence of cognitive processes and ultimately consciousness is unquestionably the generation of activity which is at the same time integrated and differentiated between cortical regions and circuits. This is in certain aspects akin to pattern formation in dynamical systems, which can only occur under partial synchronization because states of excessive differentiation or collective entrainment, occurring, for example, following diffuse structural damage and during generalized epileptic seizures, are inexorably manifest as severe dysfunction. Integration is primarily reflected in the generation of synchronized neural discharges, which arise in a highly time-dependent manner over cortical regions that may be quite distant from one another; such consonances are central to the brain's ability to bind together information from distinct sensory modalities and cognitive domains. Puzzlingly, these interdependencies not only seem to be rather weakly constrained by structural connectivity but often entail forms of apparently remote entrainment between regions that are not directly connected to each other in a structural sense. A similar situation may be evident even in the resting-state networks: while low-level aspects such as the sensorimotor component map closely onto the intra- and interhemispheric structural connections, in the default-mode network, the synchronization of the medial frontal cortex with the posterior nodes appears to bypass the layout of the large axonal bundles. It could be sustained by the basal ganglia, cingulate cortex or by a diffuse representation over the fronto-parietal cortex, yet no synchronized activity is clearly detected in these hypothetical intermediate stations [3, 5, 17, 27, 189–192].

The above readily brings to mind a phenomenon termed remote synchronization, which encompasses heterogeneous mechanisms and is receiving increasing attention in diverse fields; it consists of the preferential or even selective entrainment between nodes or subsets of nodes which are not directly connected in a structural sense, seemingly arising without involvement of the intermediate relay station(s). In the simplest scenario, it can be exemplified by a chain of three coupled nonlinear oscillators $A \leftrightarrow B \leftrightarrow C$, wherein A and C are selectively synchronized or more strongly synchronized with each other than with B. In a metaphorical sense, B acts as the courier of a message which it does not read. Similarly to cluster synchronization, remote synchronization is capable of giving rise to motifs which are not a trivial reflection of structural couplings; however, in this case, the constraint is considerably weaker, because the subsets of preferentially-entrained nodes do not need to "grow" along the structural connections since they can, at least to some extent, bypass them (obviously, remotely synchronized nodes need to belong to the same connected component). Remote synchronization can be elicited in diverse scenarios encompassing networks of identical and non-identical nodes. In the former case, it requires a mechanism preventing transition to complete synchronization such as links with time delays or phase frustrations, and it emerges as a reflection of topological symmetries in the structural connectivity. In the latter case, sufficient parametric mismatch can prevent the entrainment of a node while still allowing it to relay information between its neighbors, a situation which is well-evident in the paradigmatic case of a star network wherein the hub has a different natural frequency, yet, for intermediate coupling strengths, amplitude fluctuations enable it to establish and maintain synchronization between its leaves. This effect, which can also be elicited in complex networks, partially overlaps relay synchronization, wherein complete synchronization among a population of nodes is maintained by a station which is synchronized with them in a weaker sense, such as with a lag or a generalized relationship. Evidence suggestive of remote synchronization across diverse networks such as electrical power distribution grids, telecommunication infrastructure and meteorological phenomena is accumulating [38, 193–199].

Predicated on the above, the author and colleagues set out to apply an approach similar to that outlined in Section 3.1 for investigating the formation of patterns via remote synchronization, again in the context of a ring network. While existing research has focused mainly on networks with either identical or strongly mismatched nodes, here the focus was accordingly on small parametric mismatches, of the order of 1% but potentially much smaller, engendered by production tolerances between physical electronic components. The question was whether these would break the symmetry of the ring in a manner akin to the observations of cluster synchronization in the transistor network, such that remote synchronization could yield more complex synchronization patterns. Given the absence of a robust theoretical framework for predicting the emergence of remote entrainment in such a setting, recourse was made to an exhaustive search recalling that considered in Section 2.2 for discovering new transistor-based oscillators; even though inherently unrewarding from a theoretical perspective, this approach allows the serendipitous discovery of phenomena which can, subsequently, be analyzed rigorously. Transistor-based oscillators such as those considered thus far are poorly-suited for rapidly performing a high-dimensional search on an experimental system since the inductor and capacitor values cannot be tuned easily under software control. Addressing this issue, a novel oscillator was introduced, specifically optimized for deployment on a *Field-Programmable Analog Array* (FPAA) device; these are integrated circuits providing a fully analog pathway wherein amplification and filtering are performed through switched-capacitor banks that can be reconfigured dynamically according to a bit-string. It should be noted that, while signal processing in these devices is not digital, an undesirable yet unavoidable consequence of their implementation is operation in discrete-time mode, which diminishes the relevance of experimental study compared to numerical simulation; the situation is nevertheless fundamentally different from deployment on field-programmable gate arrays, which are entirely digital arrays: although, as recently shown, they can realize concurrent simulations of coupled nonlinear systems, there is no conceptual difference with respect to running the same calculations on a typical microprocessor [200–203].

The oscillator circuit was inspired by the canonical CMOS inverter ring topology considered in Section 2.3, but in this case, it comprises a ring of three composite analog stages, each of which realizes summation and low-pass filtering, overlaid to which two integrators are connected (Fig. 9 (a)). The node dynamics can be described by the following system:

$$\begin{cases} \dfrac{\mathrm{d}v_1}{\mathrm{d}t} = \mathit{\Gamma}\Big(2\pi F_1(G_4 v_4 + G_5 v_5 - v_1), v_1\Big)\,, \\ \dfrac{\mathrm{d}v_2}{\mathrm{d}t} = \mathit{\Gamma}\Big(2\pi F_2(G_1 v_6 - v_2), v_2\Big)\,, \\ \dfrac{\mathrm{d}v_3}{\mathrm{d}t} = \mathit{\Gamma}\Big(K_1 v_6, v_3\Big)\,, \\ \dfrac{\mathrm{d}v_4}{\mathrm{d}t} = \mathit{\Gamma}\Big(2\pi F_3(G_2 v_2 + G_3 v_3 - v_4), v_4\Big)\,, \\ \dfrac{\mathrm{d}v_5}{\mathrm{d}t} = \mathit{\Gamma}\Big(K_2 v_2, v_5\Big)\,, \\ \dfrac{\mathrm{d}v_6}{\mathrm{d}t} = \mathit{\Gamma}\Big(2\pi F_4(G_6 v_1 + G_\mathrm{i} v_\mathrm{i} + G_\mathrm{e} v_\mathrm{e} - v_6), v_6\Big)\,, \end{cases} \tag{10}$$

wherein the only nonlinearity is the function $\mathit{\Gamma}(x, y)$, which approximates saturation effects due to finite voltage swing V_s according to

$$\mathit{\Gamma}\,(x, y) = R\,(x)\,H(V_\mathrm{s} - y) - R\,(-x)\,H(V_\mathrm{s} + y)\,, \tag{11}$$

where the Heaviside step function $H(x) = 1$ for $x > 0$, 0 for $x \leq 0$ and the ramp function $R(x) = xH(x)$. As for the CMOS inverter rings, saturation in the physical device has more complex features, but the fundamental emergent properties can be recapitulated while assuming this elementary piece-wise nonlinearity. The node output voltage corresponds to $v_\mathrm{o} = G_\mathrm{o} v_6$, and each node receives as input the signal, denoted as v_i, from the preceding oscillator on a unidirectionally (*e.g.*, master–slave) coupled ring of 32 units (Fig. 9 (b)); v_e denotes an additional input used only when applying external perturbations in certain experiments. Even though remote synchronization has been elicited in networks of Stuart–Landau oscillators coupled diffusively in the same manner as the transistor networks considered above, in this specific case, the master–slave coupling scheme is essential for maintaining a spatial phase gradient, which supports an interference mechanism. The ring network was physically realized on a circuit board, dubbed `LYAPUNOV-1`, which provides the necessary infrastructure for rapid reconfiguration and data acquisition, and which conceptually realizes a sort of "chimera" in that it is effectively an analog computer embodied as a plug-in card for a digital one (Fig. 9 (c)) [201].

In this system, all parameters are held constant except three, which are treated as control parameters: the loop gain G_6, the coupling gain G_i and the integration constant K_1, in turn linked by a fixed ratio to K_2. As a function of these parameters three regions can be observed, two of which are characterized by the generation of fully-developed, broadband chaos and quasi-periodic activity; neither supports pattern formation, for they respectively feature gradual synchronization decay with distance and complete synchronization. At the interface between them, a more generative scenario ensues, characterized by a frequency spectrum containing distinct yet broad peaks, which hallmark a narrower-band manifestation of chaotic dynamics nearby the transition to quasi-periodicity (Fig. 9 (d)). In this region, the energy transfer rate is sufficient for maintaining global, albeit imperfect, phase synchronization (Fig. 9 (e)); however, the amplitudes remain partially synchronized and delineate structured patterns (Fig. 9 (f)). As in cluster synchronization, the precise topography of these patterns is driven by the parametric heterogeneities, however, there are consistent global properties, foremost the presence of a markedly non-monotonic effect of distance. Amplitude synchronization first decays, then peaks at an approximately-fixed distance, then eventually vanishes, delineating a pattern appearing in the form of bands parallel to the diagonal. The intensity of the effect is well-evident upon visual inspection of representative activity time-series, which appear almost entirely uncorrelated at a distance of ≈ 5 nodes, then seemingly mysteriously become correlated again at a distance of ≈ 8 nodes, heralding what might be interpreted as a mechanism of hidden information transfer (Fig. 9 (g)) [201, 203].

By virtue of the elementary underlying structural connectivity, the emergence of remote synchronization can be conveniently quantified according to the index

$$\eta\,[\boldsymbol{A}] = \frac{\sum_{ij} \Theta\,[H(\boldsymbol{A} - a')]_{ij}}{\sum_{ij} H(A_{ij} - a')}\,, \tag{12}$$

where one can empirically set the threshold a' to half the average synchronization between adjacent nodes and where $\Theta\,[\boldsymbol{A}]$ nulls all locations connected to the diagonal via non-zero entries on the resulting symmetric binary matrix, thus removing synchronization paths via chains of adjacent nodes. Upon sweeping the control parameters G_6, G_i and K_1, consistent observations of $\eta > 0$ are established for extended regions of narrow-band chaotic dynamics [201].

On a different note, small-world organization emerges in countless self-organized networks as a means of enhancing synchronizability while limiting the number of long-distance connections, which incur substantial realiza-

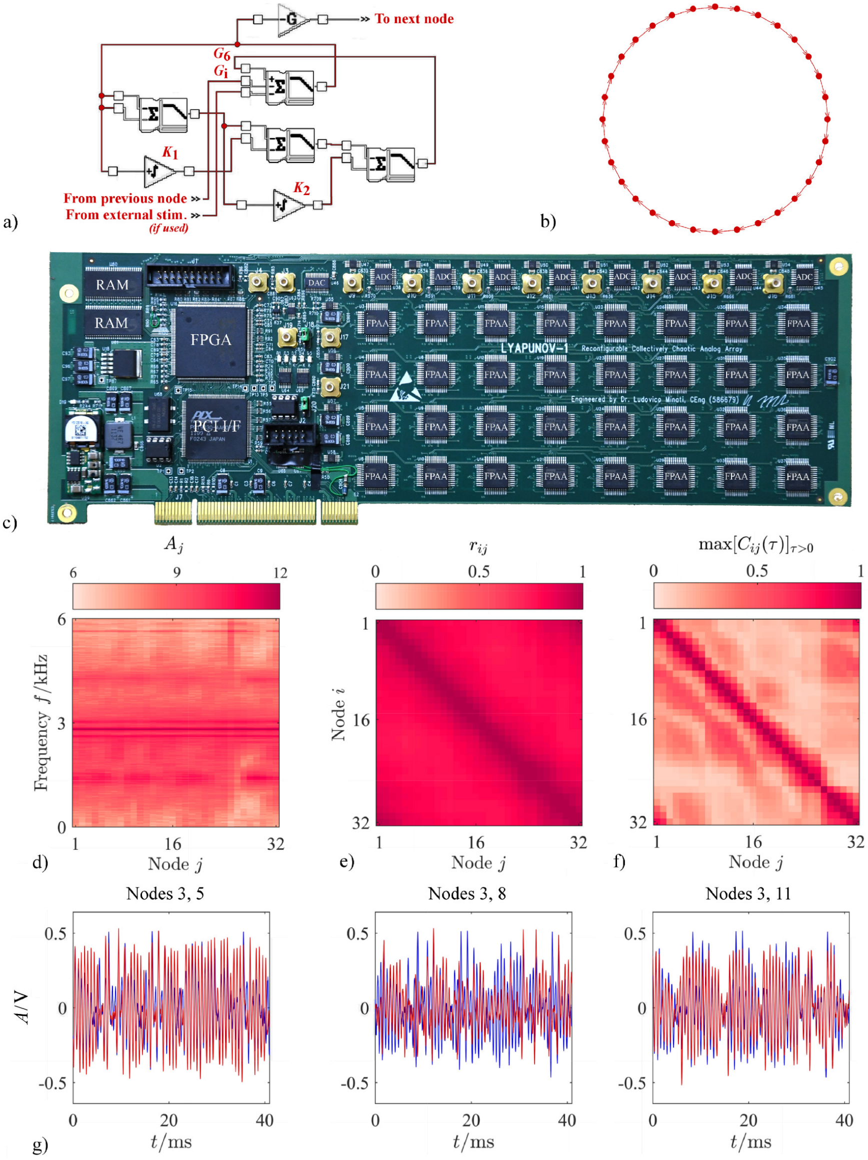

Fig. 9. Bespoke oscillator and network for attaining a form of remote synchronization. (a) Circuit diagram. (b) Unidirectionally-coupled ring connectivity. (c) Experimental realization based on field-programmable analog arrays. (d) Spectrogram across nodes. (e) Phase locking. (f) Maximum cross-correlation coefficient of amplitude fluctuations. (g) Apparent synchronization loss and subsequent recovery over sites at increasing distance. Detailed description in Refs. [201, 203].

tion and maintenance cost. Unlike in previous work by others involving the strong mismatching of nodes according to a predetermined scheme, the present form of remote synchronization entails breaking the symmetry of structural connectivity via small parametric heterogeneities. Given that these preferentially engender a non-monotonic decay of entrainment with distance, a certain degree of small-world organization is conferred to the emergent synchronization pattern. This could be captured by the canonical index

$$S^{\mathrm{WS}} = \gamma_{\mathrm{g}}^{\mathrm{WS}} \Big/ \lambda_{\mathrm{g}}^{\mathrm{WS}} = \left(C_{\mathrm{g}}^{\mathrm{WS}} L_{\mathrm{rand}}\right) \Big/ \left(C_{\mathrm{rand}}^{\mathrm{WS}} L_{\mathrm{g}}\right) , \tag{13}$$

where C^{WS} denotes the average clustering coefficient according to Watts and Strogatz, and L denotes the harmonic mean path length, both measured separately for the observed graph and a set of equivalent Erdös–Rényi random networks. In this experiment, one finds $\langle S^{\mathrm{WS}} \rangle \geq 1.7$, which is not irreconcilably distant from the values observed biologically for vastly larger neural systems and which is in sharp contrast to $S^{\mathrm{WS}} = C^{\mathrm{WS}} = 0$ in the underlying ring connectivity [26, 201, 204, 205].

Still further, synchronization patterns having diversified levels of nodal strength heterogeneity are obtained as a function of the control parameters. Each pattern delineates a weighted graph G expressing self-similarity in the form of varying correlation between the edge strengths and the sums of all triangles hinged around the corresponding node pairs; this heralds different degrees of quasi-idempotence of the corresponding matrix, intended as $\boldsymbol{A} \approx \boldsymbol{A}^2$. As introduced by the author and colleagues, representing as $\hat{G}(1,n)$ the graph associated with the matrix squared n times, this organization can be probed by calculating

$$\iota(n) = r\left(\mathit{\Xi}[G], \mathit{\Xi}\left[\hat{G}(1,n)\right]\right) , \tag{14}$$

where $r(\boldsymbol{x}, \boldsymbol{y})$ denotes the linear correlation between the elements of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\mathit{\Xi}[G]$ yields a vector $\boldsymbol{g}$ containing the edge strengths of G. A rich repertoire of synchronization patterns emerges within the chaotic regions of this system and, according to the proposed index, these patterns heterogeneously feature weak ($\iota(1) \gg \iota(\infty)$) as well as strong ($\iota(1) \approx \iota(\infty)$) forms of quasi-idempotence [78].

Closer evaluation of the present remote synchronization phenomenon indicates that the non-monotonic entrainment, which spontaneously arises in an interwoven pattern between the nodes, survives beyond the assumption that $y(t) \propto x(t)$; for example, it persists when replacing the cross-correlation with mutual information between the scalar time-series, which can capture an arbitrary relationship of the form of $y(t) = \phi(x(t))$. By contrast, measures capable of representing generalized synchronization, intended as a more convoluted time-dependent entrainment of the form of $\boldsymbol{y}(t) = \boldsymbol{\Phi}(\boldsymbol{x}(t))$, reveal a graded decay without remoteness: in other words, they fill the synchronization dip at intermediate distances. This initially puzzling dissociation is confirmed when comparing linear Granger causality with transfer entropy, which provides a model-free estimation of causal inter-dependence: such a result highlights that, in the present system, inferring the presence or absence of remote synchronization depends crucially on how entrainment is measured. Accordingly, the auxiliary system approach confirms the presence of an underlying generalized synchronization which decays monotonically: in turn, this leads one to question what mechanism could seemingly "hide" synchronization and causal inter-dependence over an intermediate range of distances [201, 203].

That question could be answered by considering a simplified model having the form of

$$\begin{cases} \dfrac{\mathrm{d}v_1}{\mathrm{d}t} = 2\pi F(G_4 v_4 - v_1)\,, \\ \dfrac{\mathrm{d}v_2}{\mathrm{d}t} = 2\pi F(G_1 v_\mathrm{o} - v_2)\,, \\ \dfrac{\mathrm{d}v_3}{\mathrm{d}t} = \mathit{\Gamma}(K v_\mathrm{o}, v_3)\,, \\ \dfrac{\mathrm{d}v_4}{\mathrm{d}t} = 2\pi F(G_2 v_2 + G_3 v_3 - v_4)\,, \end{cases} \tag{15}$$

where $v_\mathrm{o} = G_5 v_1 + G_\mathrm{i} v_\mathrm{i}$ and

$$\mathit{\Gamma}\,(x, y) = R\,(x) - R\,(-x)\,H(V_\mathrm{s} + y)\,; \tag{16}$$

for suitable parameter settings, in this system, each node still acts as a nonlinear relay for the input signal but, unlike in the full model, it does not generate an own chaotic signal, that is, it contributes no new information. While probing the dynamics of a chain of said units, it was found that remoteness becomes apparent via a mechanism of concomitant demodulation and interference: the ring resonator within each node filters and dephases the input signal, which furthermore sums to the integrator output, where envelope detection can take place due to the asymmetric nonlinearity. In the region where remote synchronization emerges, the spectrum is such that the predominant peaks are located at frequencies delineating an amplitude-modulation scheme wherein the baseband f_B and the lower sideband f_L coincide, *i.e.* $f_\mathrm{C} = f_\mathrm{L} + f_\mathrm{B} = f_\mathrm{H} - f_\mathrm{B}$ with $f_\mathrm{B} = f_\mathrm{L} = f_\mathrm{C}/2$, and therefore can interfere constructively or destructively. This accounts for the fluctuations observed in the spectrogram, which resemble a diffraction effect akin to those elicited by monochromatic light. At the distance where synchronization appears to fade transiently, the lower sideband f_L is almost completely suppressed and the signal is propagated in the other two frequency bands, namely by the carrier f_C and the higher sideband f_H, from which it is later recovered via demodulation. It appears truly remarkable that such a convoluted arrangement could emerge spontaneously, and even do so in an intertwined manner virtually across the entire ring, not just between a minority of sites [201, 203].

Two additional experiments were performed by means of perturbing the system and observing its responses in an interventional framework. This is conceptually important since a pertinent limitation of all the other experiments surveyed in the present contribution is that parallels with neural dynamics are established purely at an observational level, which is inherently limited in its explanatory value [34, 35, 37].

Firstly, the dynamics of individual nodes were obliterated through focally injecting high-intensity white noise. The results indicate a predominantly local desynchronizing effect, in turn suggesting that, in line with simulations of diversely-sized rings, the pattern-formation mechanism reflects short-range interactions rather than a collective behavior of the entire network; this situation stands in contrast with the long-range interactions realized in the lattice considered in Section 2.4. This type of intervention is reminiscent of repetitive transcranial magnetic stimulation, which can reversibly damage functional brain networks by regionally inhibiting activity, yielding effects that are similar to the present ones whenever peripheral nodes are targeted [203, 206]. Interestingly, such a disruptive perturbation could also elicit some situations of enhanced synchronization over sites distant from the injection point, hallmarking the presence of a complex landscape of energy minima, which are also visible through the emergence of non-stationary dynamics. Even though this aspect has not yet been addressed in any detail, the generation of a limited number of irregularly alternating discrete states, resembling brain microstates, was noted [19, 201, 207].

Secondly, the ability to propagate an external stimulus was probed via summatively injecting low-amplitude pulses, aiming not to excessively distort the intrinsic dynamics and recovering the response of each node through time-locked averaging. This approach is similar to recording the average electroencephalographic responses evoked by low-intensity transcranial magnetic stimulation pulses, as done for example while mapping the effective connectivity network to assess brain integrity after injury [189, 203]. It was concluded that remoteness is exclusively apparent for the intrinsic activity, in turn delineating, in the jargon of neuroimaging, an effective connectivity network which is clearly decoupled from functional connectivity and more closely related to structural connectivity, again resonating with similar results obtained in experimental neuroscience [7, 8]. In the author's opinion, these results point to fertile ground for future experimental and numerical investigation of the present system.

Altogether, these data delineate a scenario wherein a convoluted nonlinear interdependence emerges spontaneously and generates synchronization patterns considerably richer compared to those yielded by cluster synchronization in the transistor networks. The underlying mechanism could be primarily explained in terms of an interplay between three frequency bands, and for suitable control parameter settings engendered concomitant signal generation, nonlinear relay and demodulation occurring at each node. Besides prompting further reflection on measure choice and on the definition of remote synchronization and causality, with clear implications for neurophysiological studies, the present results illustrate the remarkable ability of nonlinear dynamics to reproduce, at least at a phenomenological level and even in such a small-scale network, some prominent aspects of brain organization [34, 35, 37]. In particular, as previously asserted concerning modularity, even acknowledging the adaptive role of small-world and self-similar topology leaves open the question of what evolution would have initially "locked in" to begin tuning towards these features. Results such as those presented here exemplify that phenomena such as remote synchronization, which in all its diverse manifestations as yet to be discovered is plausibly no less pervasive than cluster synchronization, can readily give rise to such properties. Again, one is allowed to speculate that, deep towards the bottom of the phylogenetic tree, this nonlinear phenomenon may have lead to the initial emergence of small-world synchronization patterns, which subsequently became enhanced and eventually engraved also into the structural architecture of nervous systems [26, 155–157].

3.4. Motor pattern generation for driving a hexapod robot

The results surveyed so far reflect instances of morphogenesis wherein the nonlinear electronic networks generate seemingly arbitrary synchronization patterns, which have statistical features of interest for comparison with those characterizing the brain and nervous systems. In this final section, a different situation is considered, wherein structural connectivity is no longer in the form of an elementary topology such as a ring but is *ab initio* arranged more complexly; this promotes collective oscillation having desired spatiotemporal features and aiming to serve a given application. In particular, the design of a nonlinear controller for a bioinspired insect-like robot is reported.

Most animals possess an innate ability to produce periodic movements supporting their locomotion, a feature that is well-evident across phyla and scales from micro-insects through large mammals. For evolutionary reasons plausibly related to functional segregation and cognitive economy, the generation of the corresponding neural activity is consistently realized via dedicated neural architectures, known as central pattern generators. These structures spontaneously produce sustained oscillations according to given phase relationships which realize viable walking gaits. In turn, they are controlled by afferences from other ganglia or higher brain regions, which influence movement initiation and inhibition, selecting a gait alongside its frequency and other parameters. Remarkably, the locomotion of insects and many arthropods hinges around a limited number of highly-conserved "canonical gaits", such as the alternating tripod gait, the metachronal or tetrapod gait, and the wave gait. These denote stereotyped orders according to which the power strokes are delivered by the legs; for example, in the case of a six-legged creature, they respectively correspond to the sequences (L1, R2, L3) → (R1, L2, R3), (L1, R3) → R2 → (L3, R1) → L2, and L1 → R2 → L3 → R1 → L2 → R3. Generation of these three gaits, which are generally associated with decreasing locomotion speed, is influenced by environmental factors such as surface inclination and roughness, as well as by endogenous variables related to exploring or fleeing behavior. Notably, these patterns are not produced in a rigid or discretized manner, and a virtually unlimited variety of intermediate gaits can be readily observed, manifesting even paradoxical phase relationships between the legs. Furthermore, as said, analogous gaits are found across species possessing vastly different biomechanical features; for example, in ants, the power stroke is delivered in an approximately similar manner over all six coxa-body axes, whereas in cockroaches, there is considerably greater differentiation between the front, middle and hind legs, yet the gaits are similar [208–212].

Central pattern generators have received considerable attention from the bioinspired robotics community owing to their pervasiveness in biology and because they can provide parsimonious solutions to otherwise demanding motion control problems [213, 214]. In this vein, the author and colleagues have realized a central pattern generator based on an adaptation of the reconfigurable oscillator initially introduced in the previous section as a means of eliciting remote synchronization (Fig. 9 (a), Eq. (10)). Its architecture reflects a purely connectionist approach wherein all node dynamical parameters are strictly fixed, and gait pattern generation is controlled solely by modulating the connection weights and signs. In order to realize the central pattern generator at the core of the proposed controller, an oscillator representing a mesoscale neural ensemble was instanced corresponding to each one of the six legs, and three overlapping sets of hardwired physical connections were provided to engender the phase relationships realizing the canonical gaits described above (Fig. 10 (a)) [215].

Building on the notion of a "generalized gait", the strengths of these three connection sets are jointly and continuously determined by a single gait parameter $P_1 \in [0, 1]$. Following a neurofuzzy approach, the following membership functions are considered and mapped to the corresponding input (coupling) gains of the oscillators:

$$\begin{cases} S_{\text{wave}}(P_1) = 1 - \dfrac{1}{1 + e^{-A_1(P_1 - C_1)}}\,, \\ S_{\text{metachr}}(P_1) = 1 - \dfrac{1}{1 + e^{-A_2(|P_1 - C_2| - C_2/2)}}\,, \\ S_{\text{tripod}}(P_1) = \dfrac{1}{1 + e^{-A_3(P_1 - C_3)}}\,, \end{cases} \tag{17}$$

where $A_i > 0$ and $C_i \in [0, 1]$ are scaling constants, and higher values of P_1 realize gaits generally associated with faster insect locomotion. Fuzzy

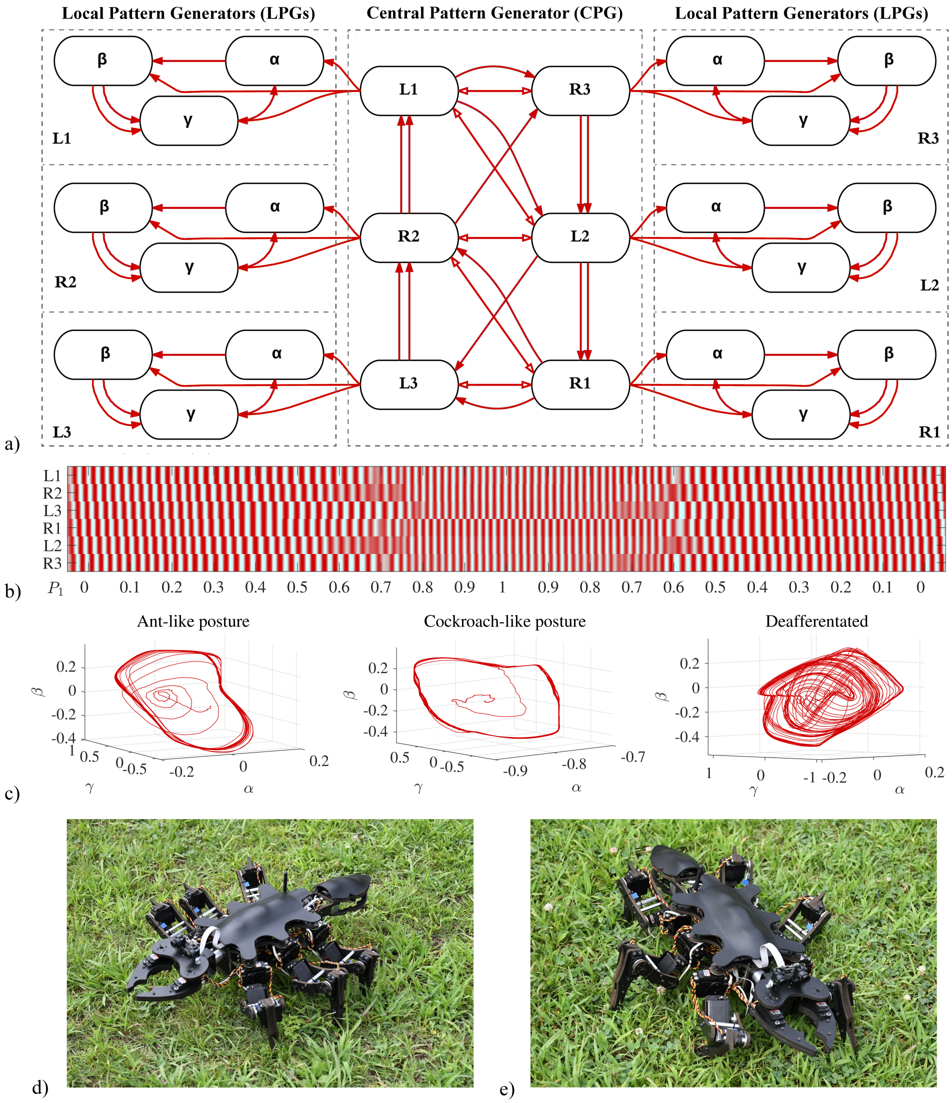

Fig. 10. Nonlinear controller for hexapod robot. (a) Hierarchical network comprising a common central (gait) pattern generator and one separate local (trajectory) pattern generator per each leg. (b) Gait control parameter P_1 sweep illustrating the generation of canonical gaits and intermediate ones. (c) Representative leg trajectories (joint angles α, β and γ) showing diversified limit cycles for the ant- and cockroach-like postures, and loss of limit cycle following deafferentation. (d) and (e) Walking in the ant- and cockroach-like postures. Detailed description in Ref. [215].

membership is inherently more biologically-plausible that discrete selection and allows a graded transition between the gaits, whose corresponding connection sets are combined summatively. Furthermore, since the couplings have heterogeneous signs and non-linearity is present, phenomena reflecting a "competition" between the gaits can be elicited [214–216].

Accordingly, while sweeping P_1 one observes not only reliable transitions between the canonical gaits but also the generation of a multitude of intermediate gaits; these are not trivially related to the canonical ones and feature phase-reversal effects, metastable patterns and hysteresis regions, which recall entomological observations (Fig. 10 (b)). The activation of the central pattern generator can furthermore be controlled by a second parameter $P_2 \in [-1, 1]$, which enables inhibiting, restarting and reversing the walking; notably, when this parameter is suddenly changed from zero, it is possible to observe the self-organization of the gait pattern, wherein disordered oscillations gradually build up and establishment of stable phase relationships follows. It should be noted that in this experiment, unlike the previous sections and other work on chaotic control, the nonlinear oscillators were operated in a periodic region. This choice may constrain but, importantly, does not negate the possibility of generating emergent synchronization phenomena, as exemplified by experiments on remote synchronization in Stuart–Landau systems [194, 214, 215, 217–220].

Having generated a gait pattern, there remains the problem of translating it into viable leg trajectories, which are constrained by the available degrees of freedom and need to take into account the specific mechanical features of the robot platform. Even when replicating biological central pattern generators, this step has at times been realized using conventional forward and inverse kinematic computations, which are conceptually convenient but computationally demanding and not biologically plausible. By contrast, the present robot controller was implemented exclusively with nonlinear analog computation, in a hierarchical form wherein each node of the central pattern generator drives a corresponding downstream local pattern generator, which is instanced independently for each leg and tasked with translating the phase information into joint trajectories (Fig. 10 (a)). As such, the local pattern generators effectively operate a degrees-of-freedom reduction operation akin to that posited for muscle synergies in humans and other animals [208, 212, 213, 215, 221]. Each one consists of a ring of three oscillators which drive corresponding joints and whose input weights are again controlled continuously by two high-level parameters, $P_3 \in [0, 1]$ and $P_4 \in [0, 1]$; the former sets the overall coupling strength to the central generator, the latter alters the amplitude and phase relationships between the joints, realizing the ant- and cockroach-like postures. As for P_1, these parameters are mapped onto individual amplifier gains through linear superposition equations, which involve a large number of fixed parameters that had to be determined empirically. For strong coupling between the two levels of the hierarchical controller, that is $P_3 \approx 1$, the emergence of distinct limit cycles specific to each posture and leg is readily observed; contrariwise, deafferentation of the local pattern generators, that is $P_3 \approx 0$, causes the loss of the limit cycle: this translates into a thrashing behavior reminiscent of that observed in living insects for example following insecticide poisoning (Fig. 10 (c)).

To evaluate the controller in a physical scenario, the same was deployed to drive a robot weighting ≈ 3 kg, instrumented with telemetry sensors and dubbed 非線形蟻-1 (`hisenkei ari-1`); this prototype represents a successor to an early design by the author, `GOLEM-1`, wherein attempts were made to realize a related locomotion control approach using a digital neural network processor (Fig. 10 (d)) [215, 222]. The robot was driven in real-time by the `LYAPUNOV-1` board, suitably rewired to implement the necessary structural connectivity, relying on digital processing only for reconfiguring the field-programmable analog arrays and relaying their rescaled outputs over a radio link. Notably, the telemetry data not only confirm the kinematic viability of the majority of emergent gaits but also highlight their heterogeneous features in terms of resulting robot attitude, body elevation, postural stability, structural strains and energy consumption; a similar kind of non-uniform performance under different conditions is, indeed, plausibly what drove evolution towards expressing a multitude of insect gaits as opposed to globally optimizing a single one [208–213].

The present results exemplify a practical application of one of the atypical oscillators introduced above, leading to a small-size hierarchical controller which realized a high level of versatility in gait, posture and coordination. In particular, they underline the generative potential of a connectionist approach to this task, which allows condensing substantial kinematic complexity into a minimal number of high-level parameters. These are suitable for purposeful setting by networks representing higher-level cognitive processes, or even for control via brain-machine interface systems. It is expected that the spectrum of biologically reminiscent behaviors available to the controller could be substantially expanded through going beyond the present purely open-loop implementation; for instance, further research needs to address the possibility of adaptively reconfiguring the local and central pattern generators based on environmental feedback.

4. Commentary and future directions

The experiments surveyed in this essay represent an initial and speculative attempt to establish some parallels, at least at a phenomenological level, between aspects of the dynamics of two profoundly different entities. On the one hand, the brain and nervous systems across the micro-, meso- and macroscopic scales, on the other hand, simple nonlinear electronic circuits and networks not corresponding to structurally- or physiologically-realistic neuronal models [5].

Namely, it was shown that rich phase transition structures spanning periodic and chaotic regions reminiscent of those elicited in neurons and axons, together with the generation of quantized spikes and bursts thereof, can be replicated even over diverse types of simple transistor-based oscillators [97, 98]. It was likewise demonstrated that avalanches scaling with the precise same critical exponents regulating spontaneous neural activity can be generated in a lattice network of gas-discharge tubes, and that this takes place concomitantly with metastability and hysteresis, in turn pointing to the possibility that alike mechanisms, related to first-order transition, may also be at play in biological systems [145, 153]. It was similarly exemplified how the generation of chaos could be influenced by structures such as cross-coupled inverter rings having primal lengths and transistor-based oscillators harboring fractal resonator networks; in this context, observations on the effect of irregularities and imperfections, which could have relevance for the interpretation of biological fractals such as dendritic trees, were established [119, 122]. It was furthermore illustrated how a rich community structure spanning multiple scales as a function of coupling strength could be readily elicited in a ring network of single-transistor oscillators, whose symmetry is broken by small parametric mismatches [166]. It was subsequently shown how synchronization patterns having apparent small-world, self-similar features could arise through remote synchronization in another ring network, wherein modulation and interference effects lead to non-trivial entrainment between the nodes [201, 203]. Besides highlighting some potential pitfalls in applying linear measures to complex systems such as the brain, altogether these results fuel speculating on how key features of brain network organization could find their distant roots in fundamental synchronization phenomena, which might have originated motifs and topological features subject to later adaptive selection and enhancement [25, 30, 36, 38]. Furthermore, it was demonstrated how the expression of nonlinear dynamics in the form of large-amplitude, low-frequency fluctuations such as those driving brain hemodynamic responses could be readily engendered by structural connectivity even in a drastically-simplified toy model network of cortical organization into hub regions [181]. Lastly, it was exemplified how nonlinear oscillators of the present kinds could offer a fertile substrate for realizing complex functions such as generating motor patterns and leg trajectories realistically driving an insect-like robot, simultaneously yielding biologically-plausible emergent behaviors beyond the designed features [215].

It is unquestionable that there are pervasive, profound differences between biological neural systems and the circuits and networks considered herein, which have so far only recapitulated in a fragmented manner few selected aspects of neural dynamics. The former operate on a completely different spatiotemporal scale, realize vastly more complex interactions and coupling schemes through synapses and multiple neurotransmitter systems, are highly adaptive via plasticity which is entirely absent in the latter, and plausibly operate in another regime wherein critical dynamics play a more prominent role, and wherein the level of synchronization between coupled units is lower [5, 17, 20, 22, 25, 27]. Yet, it seems equally incontrovertible that the emergent phenomena expressed by the present circuits place their physical dynamics closer to those of biological neural networks in comparison with digital computers and plausibly also in comparison with several analog neuromorphic systems, particularly in that both are generally based on different principles, centered foremost around explicitly designed features rather than around emergence and self-organization [17, 21, 38, 63, 65, 67, 71].

The main contribution of the present work is thus arguably to be found in the fact that it exemplifies experimentally how certain emergent phenomena, observed at diverse scales in biological neural systems, could tentatively be recapitulated through considerably smaller and simpler networks of analog electronic devices. These were selected among the realm of other possible nonlinear physical apparatus, *e.g.* mechanical and optical systems, mainly out of practical considerations related to their realization and manipulation. Besides offering a hopefully useful platform for performing experiments translatable across *in silico*, *in vitro* and *in vivo* situations, results such as the present ones will perhaps inspire reflections around the possible absence of any inherently unique dynamical phenomenon in the brain. In the author's view, nature, through evolution, might rather have efficiently harvested nonlinear phenomena that are pervasive in the physical world for realizing the infrastructure which eventually supports the emergence of cognitive activity. In an allusion to astrobiology as well as to Giordano Bruno, we could briefly allow ourselves to wonder whether the latter truly represents a singularity or there are other instances perhaps inaccessible to us. At least some emergent phenomena are sufficiently general that they could be readily elicited even in electronic circuits which are dynamically and topologically quite unrelated to neurons and brains, sharing virtually only the fact that they possess nonlinear dynamics. As such, the present findings appear to vindicate the enormous generative power of even elementary nonlinear circuits, and therefore should reinvigorate their study as potential building blocks also for practical engineering applications; at a minimum, they should point to a closer consideration of oscillator topologies beyond the canonical ones [2, 3, 18, 19, 60, 65, 75, 76].

The longest-term goal in this field undoubtedly consists of formulating a general theory of emergence translating into systematic tools for predicting and inducing collective behaviors. On the one hand, this would plausibly aid in understanding the biophysical origins of the mind, and on the other hand, it may enable harvesting the potentially enormous computational capability of analog nonlinear systems, which is expected to vastly exceed that of present-day digital computers per transistor and per power unit, as indirectly testified by the performance of the brain itself [2, 17, 20, 57, 59, 66, 67, 71].

Achieving such a goal would plausibly be aided by a deeper understanding of pattern-formation mechanisms, motivating further experiments considering larger, more complex structural connectivity. In that regard, future work should leverage the present oscillators for physically replicating nervous connectomes across different resolutions, for example capturing the full architecture of the *Caenorhabditis elegans*, or a mesoscopic model of an insect brain, or a macroscopic representation of a mammalian one. Furthermore, there is an obvious need to implement forms of plasticity allowing self-organization and memory beyond the minimal extent attainable in these thus-far fixed circuits: besides the preferable usage of self-contained devices such as memristors, an intermediate step may involve implementing digital analysis of the generated signals with real-time control of circuit parameters such as supply voltage or capacitor bank selection. Taking such a step appears relevant foremost for ascertaining whether any meaningful learning process can or cannot be replicated in this context [5, 17, 20, 25, 27, 46]. On another note, besides epistemological considerations around the inherent complementarity of simulation and physical measurement, there is a need to clarify the contribution and limitations of numerical models in this area, particularly as regards the prediction of global emergent properties which may be qualitatively affected by elemental heterogeneities, non-idealities and the alike: taking such a step will better delineate the scientific relevance of experiment-based explorations such as the present one [18, 19, 58, 68, 70, 74].

On another note still, it appears necessary to consider more closely the effects of non-stationarity and noise, whose impacts on the emergent properties of brain dynamics are empirically well-evident. With regards to the former, there is a need to evaluate the possibility of replicating aspects such as the spontaneous formation of microstates; these are discrete, irregularly alternating brain states which are detectable in neurophysiological recordings in the form of features that were only in passing juxtaposed to the dynamics of the present oscillators [19, 207]. As regards the latter, there is a need to address the generative potential of noise, diversely intended as its ability to enhance synchronizability, its ability to reveal hidden attractors for example related to bistable states, and even its ability to boost processing capabilities via phenomena such as stochastic resonance [14, 15]. Further research should also be performed applying notions of nonlinear and chaos control to these circuits, particularly with respect to the possibility of enriching their dynamical repertoire via external tuning, for instance aiming to maintain operation close to a transition point: as for plasticity, this might at first be conveniently implemented through an external digital controller [72]. The present work should, therefore, at best be considered as a precursory and undoubtedly incomplete account of the ability to recapitulate neural dynamics in nonlinear electronic oscillators.

Lastly but no less importantly, there appears to be ample opportunity for more explicitly integrating the notions which guided the present work with the domains of computational neuroscience, artificial intelligence and neuromorphic computing. For instance, nonlinear structures such as reentrant neural networks, whether deep or not, are clearly well-suited for generating emergent phenomena, as are multi-agent systems and many other learning automata, hence, there is an obvious motivation to approach them, perhaps more ambitiously, also from the perspectives of chaos and synchronization theory. A similar argument seems to apply to simulations and experimental realizations of neural circuit architectures not explicitly intended as substrates for emergence but as designed implementations of chosen functions [52–54, 59, 109, 110].

It is hoped that, despite all its limitations, the present preliminary, speculative and perhaps unusual attempt to draw together neuroscience and nonlinear electronics will possibly aid in inspiring others to undertake and continue interdisciplinary research across these areas.

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