A NEW THEOREM ON THE REPRESENTATION STRUCTURE OF THE SL$(2,\mathbb{C})$ GROUP ACTING IN THE HILBERT SPACE OF THE QUANTUM COULOMB FIELD

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Using the results obtained by Staruszkiewicz in *Acta Phys. Pol. B* **23**, 591 (1992) and in *Acta Phys. Pol. B* **23**, 927 (1992), we show that the representations acting in the eigenspaces of the total charge operator corresponding to the eigenvalues n_1, n_2 whose absolute values are less than or equal $\sqrt{\pi/e^2}$ are inequivalent if $|n_1| \neq |n_2|$ and contain the supplementary series component acting as a discrete component. On the other hand, the representations acting in the eigenspaces corresponding to eigenvalues whose absolute values are greater than $\sqrt{\pi/e^2}$ are all unitarily equivalent and do not contain any supplementary series component.

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1. Introduction

In this paper, we prove a new theorem within the Quantum Theory of the Coulomb Field [1, 2]. This paper can be regarded as an immediate continuation of the series of Staruszkiewicz's papers [3–5] on the structure of the unitary representation of SL$(2,\mathbb{C})$ acting in the Hilbert space of the quantum Coulomb field and the quantum phase field $S(x)$ of his theory, and its connection to the fine structure constant. We use the notation of these papers. Basing on the results of these papers, we give here a proof of the following

Theorem 1.1. *Let $U|_{\mathcal{H}_m}$ be the restriction of the unitary representation U of SL$(2,\mathbb{C})$ in the Hilbert space of the quantum phase field S to the invariant eigenspace $\mathcal{H}_m$ of the total charge operator Q corresponding to the eigenvalue me for some integer m. Then for all m such that*

$$|m| > \textit{Integer part } \left(\sqrt{\frac{\pi}{e^2}}\right),$$

the representations $U|_{\mathcal{H}_m}$ *are unitarily equivalent*

$$U|_{\mathcal{H}_m} \cong_U U|_{\mathcal{H}_{m'}}$$

whenever

$$|m| > \textit{Integer part}\left(\sqrt{\frac{\pi}{e^2}}\right), \qquad |m'| > \textit{Integer part}\left(\sqrt{\frac{\pi}{e^2}}\right).$$

On the other hand, if the two integers m, m' *have different absolute values* $|m| \neq |m'|$ *and are such that*

$$|m| < \sqrt{\frac{\pi}{e^2}}, \qquad |m'| < \sqrt{\frac{\pi}{e^2}},$$

then the representations $U|_{\mathcal{H}_m}$ *and* $U|_{\mathcal{H}_{m'}}$ *are inequivalent. Each representation* $U|_{\mathcal{H}_m}$ *contains a unique discrete supplementary component if*

$$|m| < \sqrt{\frac{\pi}{e^2}},$$

and the supplementary components contained in $U|_{\mathcal{H}_m}$ *with different values of* $|m|$ *fulfilling the last inequality are inequivalent. If*

$$|m| > \textit{Integer part}\left(\sqrt{\frac{\pi}{e^2}}\right),$$

then the representation $U|_{\mathcal{H}_m}$ *does not contain in its decomposition any supplementary components.* □

This remarkable result can be compared to the well-known and curious coincidence concerning self-adjointness of the Hamiltonian of the bounded system composed of a heavy source (say nucleus) of the classical Coulomb field and a relativistic charged particle in this field. Namely, it is a well-known phenomenon in relativistic wave mechanics that whenever the charge of the nuclei is of the order of magnitude comparable to the inverse of the fine structure constant or greater, then the Hamiltonian loses the self-adjointness property (which sometimes is interpreted as an indication that the system when passing to the quantum field level becomes unstable). On the other hand (and this is a coincidence which no one understands), the nuclei of real atoms are unstable whenever the charge of the nuclei reaches the value of the same order (inverse of the fine structure constant). The mentioned breakdown of self-adjointness cannot explain, of course, this phenomenon because there are mostly the strong (and not electromagnetic) forces which govern the stability of nuclei. To this coincidence we add another coming from the quantum theory of infrared photons of the quantized Coulomb field. Although we should emphasize that the mentioned three phenomena come from three different regimes and so far we are not able to answer the question if these coincidences are merely accidental or not.

2. Proof of the theorem

Let us concentrate our attention on the specific state $|u\rangle$ in the eigenspace $\mathcal{H}_{m=1}$ corresponding to the eigenvalue e of the charge operator Q. For any time-like unit vector u, we can form the following unit vector (compare [4] or [5]):

$$|u\rangle = e^{-iS(u)}|0\rangle \tag{1}$$

in the Hilbert space of the quantum field S. It has the following properties:

(1) $|u\rangle$ is an eigenstate of the total charge Q: $Q|u\rangle = e|u\rangle$.

(2) $|u\rangle$ is spherically symmetric in the rest frame of u: $\epsilon^{\alpha\beta\mu\nu}u_\beta M_{\mu\nu}|u\rangle = 0$, where $M_{\mu\nu}$ are the generators of the $\mathrm{SL}(2,\mathbb{C})$ group.

(3) $|u\rangle$ does not contain the (infrared) transversal photons: $N(u)|u\rangle = 0$, where $N(u)$ is the operator of the number of transversal photons in the rest frame of u. If u is the four-velocity of the reference frame in which the partial waves $f_{lm}^{(+)}$ are computed, then in this reference frame

$$N(u) = \left(4\pi e^2\right)^{-1}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} c_{lm}^{+}c_{lm}$$

and (up to an irrelevant phase factor)

$$|u\rangle = e^{-iS_0}|0\rangle\,.$$

These three conditions determine the state vector $|u\rangle$ up to a phase factor.

Now, let us consider the subspace $\mathcal{H}_{|u\rangle} \subset \mathcal{H}_{m=1}$ as spanned by the vectors of the form of $U_\alpha|u\rangle$, $\alpha \in \mathrm{SL}(2,\mathbb{C})$.

Note that the above conditions (1) and (2) determine $|u\rangle$ as the "maximal" vector in $\mathcal{H}_{|u\rangle}$ which preserves conditions (1), (2), *i.e.* any state vector in the Hilbert subspace $\mathcal{H}_{|u\rangle}$ of the quantum phase field S which preserves (1) and (2), and which is orthogonal to $|u\rangle$ is equal zero.

First: in [2], it was computed that the inner product

$$\langle u|v\rangle = \exp\Big\{ -\frac{e^2}{\pi}(\lambda\coth\lambda - 1)\Big\},$$

where $u \cdot v = g_{\mu\nu}u^\mu v^\mu = \cosh\lambda$, so that λ is the hyperbolic angle between u and v; compare also [6].

Second: it was proved in [4] (compare also [5, 7]) that the state $|u\rangle$, lying in the subspace $Q = e\mathbf{1}$ of the Hilbert space of the field S, when decomposed into components corresponding to the decomposition of U into irreducible sub-representations contains

— only the principal series if $\frac{e^2}{\pi} > 1$,

— the principal series and a discrete component from the supplementary series with

$$-\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = z(2-z)\mathbf{1}\,, \qquad z = \frac{e^2}{\pi}\,, \qquad \text{if } 0 < \frac{e^2}{\pi} < 1\,,$$

in the units in which $\hbar = c = 1$. In other units, one should read $\frac{e^2}{\pi\hbar c}$ for $\frac{e^2}{\pi}$.

In particular from the result of [4], it follows that for the restriction $U|_{\mathcal{H}_{|u\rangle}}$ of the representation U of $\mathrm{SL}(2,\mathbb{C})$ acting in the Hilbert space of the quantum "phase" field S to the invariant subspace $\mathcal{H}_{|u\rangle}$, we have the decomposition

$$U|_{\mathcal{H}_{|u\rangle}} = \begin{cases} \mathfrak{D}(\rho_0) \bigoplus \int\limits_{\rho>0} \mathfrak{S}(n=0,\rho)\,\mathrm{d}\rho\,, \quad \rho_0 = 1 - z_0\,,\ z_0 = \frac{e^2}{\pi}\,, & \text{if } 0 < \frac{e^2}{\pi} < 1\,, \\ \int\limits_{\rho>0} \mathfrak{S}(n=0,\rho)\,\mathrm{d}\rho\,, & \text{if } 1 < \frac{e^2}{\pi} \end{cases} \tag{2}$$

into the direct integral of the unitary irreducible representations of the principal series representations $\mathfrak{S}(n=0,\rho)$, with real $\rho > 0$ and $n = 0$, and a discrete direct summand of the supplementary series $\mathfrak{D}(\rho_0)$ corresponding to the value of the parameter

$$\rho_0 = 1 - z_0\,, \qquad z_0 = \frac{e^2}{\pi}\,;$$

and where $\mathrm{d}\rho$ is the ordinary Lebesgue measure on $\mathbb{R}_+$.

Note that the irreducible unitary representations $\mathfrak{S}(n,\rho)$ of the principal series correspond to the representations $(l_0 = \frac{n}{2}, l_1 = \frac{i\rho}{2})$, with $n \in \mathbb{Z}$ and $\rho \in \mathbb{R}$ in the notation of the book [8], and correspond to the character $\chi = (n_1, n_2) = \big(\frac{n}{2} + \frac{i\rho}{2}, -\frac{n}{2} + \frac{i\rho}{2}\big)$ in the notation of the book [9], and finally to the irreducible unitary representations

$$U^{\chi_{n,\rho}} = \mathfrak{S}(n,\rho)$$

induced by the unitary representations of the diagonal subgroup corresponding to the unitary character $\chi_{n,\rho}$ of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{C})$ within the Mackey theory of induced representations.

Recall also that the irreducible unitary representations $\mathfrak{D}(\rho)$ of $\mathrm{SL}(2,\mathbb{C})$ of the supplementary series are numbered by the real parameter $0 < \rho < 1$, and correspond to the representations $(l_0 = 0, l_1 = \rho)$ in the notation of [8]. They also correspond to the character $\chi = (n_1, n_2) = \big(\rho, \rho\big)$ in the notation of [9], and finally to the irreducible unitary representations

$$U^{\chi_\rho} = \mathfrak{D}(\rho)$$

induced by the (non-unitary) representations of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{C})$ corresponding to the non-unitary character χ_ρ of the diagonal subgroup of $\mathrm{SL}(2,\mathbb{C})$ within the Mackey theory of induced representations.

Next, for each integer $m \in \mathbb{Z}$ and a point u in the Lobachevsky space, we consider spherically symmetric unit state vector $|m,u\rangle \in \mathcal{H}_m$

$$|m,u\rangle = e^{-imS(u)}|0\rangle$$

in the Hilbert space of the quantum field S. If u is the four-velocity of the reference frame in which the partial waves $f_{lm}^{(+)}$ are computed, then in this reference frame

$$|m,u\rangle = e^{-imS_0}|0\rangle$$

up to an irrelevant phase factor. The unit vector $|m,u\rangle$ has the following properties:

(1m) $|m,u\rangle$ is an eigenstate of the total charge Q: $Q|u\rangle = em|m,u\rangle$.

(2m) $|m,u\rangle$ is spherically symmetric in the rest frame of u: $\epsilon^{\alpha\beta\mu\nu}u_\beta M_{\mu\nu}$ $|m,u\rangle = 0$, where $M_{\mu\nu}$ are the generators of the $\mathrm{SL}(2,\mathbb{C})$ group.

(3m) $|m,u\rangle$ does not contain the (infrared) transversal photons: $N(u)|m,u\rangle$ $= 0$.

Proceeding exactly as Staruszkiewicz in [2] (compare also [6]), we show that for any two points u, v in the Lobachevsky space of unit time-like four-vectors

$$\langle u, m|m, v\rangle = \exp\left\{-\frac{e^2m^2}{\pi}(\lambda\coth\lambda - 1)\right\},$$

where λ is the hyperbolic angle between u and v. Next, we construct the Hilbert subspace $\mathcal{H}_{|m,u\rangle} \subset \mathcal{H}_m$ spanned by

$$U_\alpha|m, u\rangle\,, \qquad \alpha \in \mathrm{SL}(2,\mathbb{C})\,.$$

Note that $\mathcal{H}_{|m,u\rangle} \neq \mathcal{H}_m$. Using the Gelfand–Neumark Fourier analysis on the Lobachevsky space as Staruszkiewicz in [4], we show that

$$U|_{\mathcal{H}_{|m,u\rangle}} = \begin{cases} \mathfrak{D}(\rho_0) \bigoplus \int\limits_{\rho>0} \mathfrak{S}(n=0,\rho)\,\mathrm{d}\rho\,, \quad \rho_0 = 1 - z_0\,, \quad z_0 = \frac{e^2m^2}{\pi}\,, & \text{if } 0 < \frac{e^2m^2}{\pi} < 1\,, \\ \int\limits_{\rho>0} \mathfrak{S}(n=0,\rho)\,\mathrm{d}\rho\,, & \text{if } 1 < \frac{e^2m^2}{\pi}\,, \end{cases} \tag{3}$$

where $\mathrm{d}\rho$ is the Lebesgue measure on $\mathbb{R}_+$.

We need two Lemmas concerning the structure of the representation U of $\mathrm{SL}(2,\mathbb{C})$ in the Hilbert space of the quantum phase field S.

Lemma 2.1.

$$U|_{\mathcal{H}_{m=1}} = U|_{\mathcal{H}_{|u\rangle}} \otimes U|_{\mathcal{H}_{m=0}}\,.$$

☐ First, we show that (all tensor products in this Lemma are the Hilbert-space tensor products)

$$\mathcal{H}_{m=1} = \mathcal{H}_{|u\rangle} \otimes \mathcal{H}_{m=0} = \mathcal{H}_{|u\rangle} \otimes \Gamma\left(\mathcal{H}^1_{m=0}\right)\,, \tag{4}$$

where $\mathcal{H}^1_{m=0}$ is the single-particle subspace of infrared transversal photons spanned by

$$c^+_{lm}|0\rangle\,,$$

and $\Gamma(\mathcal{H}^1_{m=0})$ stands for the boson Fock space over $\mathcal{H}^1_{m=0}$, *i.e.* direct sum of symmetrized tensor products of $\mathcal{H}^1_{m=0}$. The Hilbert subspace $\mathcal{H}_{|u\rangle}$ is spanned by $|u\rangle$, and all its transforms $U_{\Lambda(\alpha)}|u\rangle = |u'\rangle$ with $u' = \Lambda(\alpha)^{-1}u$ ranging over the Lobachevsky space $\mathscr{L}_3 \cong \mathrm{SL}(2,\,\mathbb{C})/\mathrm{SU}(2,\mathbb{C})$ of time-like unit four-vectors u' — the Lorentz images of the fixed u. The Hilbert space structure of $\mathcal{H}_{|u\rangle}$ can be regarded as the one induced by the invariant kernel

$$u \times v \mapsto \langle u|v\rangle = \exp\left\{-\frac{e^2}{\pi}(\lambda\coth\lambda - 1)\right\}$$

on the Lobachevsky space $\mathscr{L}_3$ as the RKHS corresponding to the kernel, compare *e.g.* [10]. Because this kernel is continuous as a map $\mathscr{L}_3 \times \mathscr{L}_3 \mapsto \mathbb{R}$, and the Lobachevsky space is separable, then it is easily seen that there exists a denumerable subset $\{u_1, u_2, \ldots\} \subset \mathscr{L}_3$ such that $|u_1\rangle, u_2\rangle, \ldots$ are linearly independent and such that the denumerable set of finite rational (with $b_i \in \mathbb{Q}$) linear combinations

$$\sum_{i=1}^{k} b_i|u_i\rangle$$

of the elements $|u_1\rangle, |u_2\rangle, \ldots$ is dense in $\mathcal{H}_{|u\rangle}$, *cf. e.g.* [11] Chap. XIII, §3. One can choose (Schmidt orthonormalization, [11], Chap. XIII, §3) out of them a denumerable and orthonormal system

$$e_k(b_{1k}u_1, \ldots, b_{kk}u_k) = \sum_{i=1}^{k} b_{ik}|u_i\rangle = \sum_{i=1}^{k} b_{ik}e^{-iS(u_i)}|0\rangle\,, \qquad k = 1, 2, \ldots\,,$$

which is complete in $\mathcal{H}_{|u\rangle}$. Note that

$$U_{\Lambda(\alpha)}|u\rangle = U_{\Lambda(\alpha)}e^{-iS(u)}|0\rangle = U_{\Lambda(\alpha)}e^{-iS(u)}U^{-1}_{\Lambda(\alpha)}|0\rangle = e^{-iS(u')}|0\rangle\,,$$

where $u' = \Lambda(\alpha)^{-1}u$ is the Lorentz image u' in the Lobachevsky space of u under the Lorentz transformation $\Lambda(\alpha)$, because $|0\rangle$ is Lorentz invariant: $U|0\rangle = |0\rangle$. In particular,

$$\begin{aligned}
&U_{\Lambda(\alpha)}e_k(b_{1k}u_1, \ldots, b_{kk}u_k) = e_k\left(b_{1k}u'_1, \ldots, b_{kk}u'_k\right) \\
&= U_{\Lambda(\alpha)}\left(\sum_{i=1}^{k} b_{ik}e^{-iS(u_i)}|0\rangle\right) = \sum_{i=1}^{k} b_{ik}e^{-iS(u'_i)}|0\rangle\,, \\
&u'_i = \Lambda(\alpha)^{-1}u_i\,, \qquad k = 1, 2, 3, \ldots
\end{aligned}$$

forms another orthonormal and complete system in $\mathcal{H}_{|u\rangle}$. If $y \in \mathcal{H}_{|u\rangle}$, then for some sequence of numbers $b^k \in \mathbb{C}$ such that

$$||y||^2 = \sum_{k} \left|b^k\right|^2 < +\infty\,,$$

we have

$$y=\sum_{k=1,2,\ldots} b^k e_k(b_{1k}u_1,\ldots,b_{kk}u_k)=\sum_{k=1,2,\ldots,\, i=1,\ldots,k} b^k b_{ik} e^{-iS(u_i)}|0\rangle \tag{5}$$

and

$$U_{\Lambda(\alpha)}y=\sum_{k=1,2,\ldots} b^k e_k\left(b_{1k}u'_1,\ldots,b_{kk}u'_k\right)=\sum_{k=1,2,\ldots,\, i=1,\ldots,k} b^k b_{ik} e^{-iS(u'_i)}|0\rangle\,.$$

Similarly, let us write shortly

$$c^+_{lm}=c^+_\alpha \quad \text{and} \quad U_{\Lambda(\alpha)}c^+_{lm}U^{-1}_{\Lambda(\alpha)}=c'^+_{lm}\,.$$

Then if $x\in\Gamma(\mathcal{H}^1_{m=0})=\mathcal{H}_{m=0}$, there exists a multi-sequence of numbers $a^{\alpha_1\ldots\alpha_n}\in\mathbb{C}$ such that

$$||x||^2=\sum_{n=1,2,\ldots,\,\alpha_1,\ldots,\alpha_n}\left(4\pi e^2\right)^n\left|a^{\alpha_1\ldots\alpha_n}\right|^2<+\infty$$

and

$$\begin{aligned} x &= \sum_{n=1,2,\ldots,\,\alpha_1,\ldots,\alpha_n} a^{\alpha_1\ldots\alpha_n}c^+_{\alpha_1}\ldots c^+_{\alpha_n}|0\rangle\,,\\ U_{\Lambda(\alpha)}x &= \sum_{n=1,2,\ldots,\,\alpha_1,\ldots,\alpha_n} a^{\alpha_1\ldots\alpha_n}c'^+_{\alpha_1}\ldots c'^+_{\alpha_n}|0\rangle\,, \end{aligned} \tag{6}$$

where we have shortly written α_i for the pair l_i,m_i with $-l_i\le m_i\le l_i$.

Before giving the definition of $x\otimes y$ for any general elements x,y of the form (6) and respectively (5) giving the algebraic tensor product $\mathcal{H}_{m=0}\widehat{\otimes}\mathcal{H}_{|u\rangle}$ densely included in $\mathcal{H}_{m=1}$, we need some further preliminaries. Namely, note that the operators $c_{lm}=c_\alpha$ depend on the reference frame. For the construction of $\otimes$, we need the operators in several reference frames. If the time-like axis of the reference frame has the unit versor $v\in\mathscr{L}_3$, then for the operator $c_\alpha=c_{lm}$ computed in this reference frame, we will write

$${}^v c_\alpha \quad \text{or} \quad {}^v c_{lm}$$

and

$${}^v c^+_\alpha \quad \text{or} \quad {}^v c^+_{lm}$$

for their adjoints. Only for the fixed vector $u\in\mathscr{L}_3$ we simply write

$${}^u c_\alpha=c^+_\alpha \quad \text{or} \quad {}^u c_{lm}=c_{lm}$$

and

$$ {}^{u}c^{+}_{\alpha} = c^{+}_{\alpha} \quad \text{or} \quad {}^{u}c^{+}_{lm} = c^{+}_{lm} $$

in order to simplify notation.

Now, let

$$ \overset{u\mapsto v}{A}_{\alpha\beta} $$

be the unitary matrix transforming the orthonormal basis vectors $c^{+}_{\alpha}|0\rangle = {}^{u}c^{+}_{\alpha}|0\rangle$ in $\mathcal{H}_{m=0}$

$$ {}^{v}c^{+}_{\alpha}|0\rangle = \sum_{\beta} \overset{u\mapsto v}{A}_{\alpha\beta} \; {}^{u}c^{+}_{\beta}|0\rangle = \sum_{\beta} \overset{u\mapsto v}{A}_{\alpha\beta} c^{+}_{\beta}|0\rangle \,, \tag{7} $$

under the Lorentz transformation $\Lambda_{uv}(\lambda_{uv})$ transforming the reference frame time-like versor $u \in \mathscr{L}_3$ into the reference frame unit time-like versor $v \in \mathscr{L}_3$. In particular, it gives the irreducible representation of the $\mathrm{SL}(2,\mathbb{C})$ group in the single-particle Hilbert subspace $\mathcal{H}^{1}_{m=0}$ of infrared transversal photons spanned by

$$ c^{+}_{\alpha}|0\rangle = \; {}^{u}c^{+}_{\alpha}|0\rangle \,, $$

and equal to the Gelfand–Minlos–Shapiro irreducible unitary representation $(l_0 = 1, l_1 = 0) = \mathfrak{S}(n = 2, \rho = 0)$, computed explicitly in [12]. Then, as shown in [3], it follows that

$$ \begin{aligned} U^{u}_{\Lambda_{uv}(\lambda_{uv})} c_{\alpha} U^{-1}_{\Lambda_{uv}(\lambda_{uv})} &= U_{\Lambda_{uv}(\lambda_{uv})} c_{\alpha} U^{-1}_{\Lambda_{uv}(\lambda_{uv})} =^{v} c_{\alpha} \\ &= \sum_{\beta} \overline{\overset{u\mapsto v}{A}_{\alpha\beta}}^{\,u} c_{\beta} + \overline{\overset{u\mapsto v}{B}_{\alpha}}\, Q \\ &= \sum_{\beta} \overline{\overset{u\mapsto v}{A}_{\alpha\beta}} c_{\beta} + \overline{\overset{u\mapsto v}{B}_{\alpha}}\, Q \,, \end{aligned} \tag{8} $$

and[1]

$$ \begin{aligned} & U_{\Lambda_{uv}(\lambda_{uv})} S(u) U^{-1}_{\Lambda_{uv}(\lambda_{uv})} = S(v) \\ & = S(u) + \frac{1}{4\pi i e} \sum_{\alpha\beta} \left(\overset{u\mapsto v}{B}_{\alpha} \overline{\overset{u\mapsto v}{A}_{\alpha\beta}} \; {}^{u}c_{\beta} - \overline{\overset{u\mapsto v}{B}_{\alpha}} \overset{u\mapsto v}{A}_{\alpha\beta} \; {}^{u}c^{+}_{\beta} \right) \end{aligned} \tag{9} $$

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[1] We are using slightly different convention than [3], with ours $\overset{u\mapsto v}{A}_{\alpha\beta}$ corresponding to the complex conjugation $\overline{A_{\alpha\beta}}$ of the matrix elements $A_{\alpha\beta}$ used in [3] and similarly our numbers $\overset{u\mapsto v}{B}_{\alpha}$ correspond to the complex conjugation $\overline{B_{\alpha}}$ of the numbers B_{α} used in [3].

and thus

$$\begin{aligned} U_{\Lambda_{uv}(\lambda_{uv})}\, {}^{u}c^{+}_{\alpha} U^{-1}_{\Lambda_{uv}(\lambda_{uv})} &= U_{\Lambda_{uv}(\lambda_{uv})} c^{+}_{\alpha} U^{-1}_{\Lambda_{uv}(\lambda_{uv})} = {}^{v}c^{+}_{\alpha} \\ &= \sum_{\beta} \overset{u\mapsto v}{A}_{\alpha\beta}\, {}^{u}c^{+}_{\beta} + \overset{u\mapsto v}{B}_{\alpha} Q \\ &= \sum_{\beta} \overset{u\mapsto v}{A}_{\alpha\beta} c^{+}_{\beta} + \overset{u\mapsto v}{B}_{\alpha} Q \,, \end{aligned} \tag{10}$$

where Q is the charge operator and where $\overset{u\mapsto v}{B}_{\alpha}$ are complex numbers depending on the transformation $\Lambda_{uv}(\lambda_{uv})$ mapping $u \mapsto v = \Lambda_{uv}(\lambda_{uv})^{-1}u$ such that

$$\sum_{\alpha} \left| \overset{u\mapsto v}{B}_{\alpha} \right|^{2} = 8e^{2}(\lambda_{uv}\coth\lambda_{uv} - 1)$$

with λ_{uv} equal to the hyperbolic angle between u and v. Note that the charge operator is invariant (commutes with $U_{\Lambda_{uv}(\lambda_{uv})}$) and is identical in each reference frame so that no superscript u nor v is needed for Q.

The limit on the right-hand side of equality (7) should be understood in the sense of the ordinary Hilbert space norm in the Hilbert space of the quantum phase field S. In general, all limits in the expressions containing linear combinations of operators acting on $|0\rangle$ should be understood in this manner.

Now, let us explain why for each fixed α, we need essentially all ${}^{v}c_{\alpha}$, $v \in \mathscr{L}_3$ for the construction of the bilinear map $x \times y \mapsto x \otimes y$ which serves to define the algebraic tensor product $\mathcal{H}_{m=0} \widehat{\otimes} \mathcal{H}_{|u\rangle}$ of the Hilbert spaces $\mathcal{H}_{m=0}$ and $\mathcal{H}_{|u\rangle}$. In particular, consider two vectors $c^{+}_{\alpha}|0\rangle$ and $e^{-iS(v)}|0\rangle$ with v not equal to the fixed time-like versor u of the reference frame in which the partial waves $f^{(+)}_{lm}$ and the operators $c_{lm} = c_{\alpha} = {}^{u}c_{\alpha}$ are computed. Perhaps it would be tempting to put

$$c^{+}_{\alpha} e^{-iS(v)}|0\rangle$$

for the tensor product of $c^{+}_{\alpha}|0\rangle$ and $e^{-iS(v)}|0\rangle$, but this would be a wrong definition. In particular,

$$\begin{aligned} &\langle 0|e^{iS(v)}\, {}^{u}c_{\beta}\, {}^{u}c^{+}_{\alpha} e^{-iS(v)}|0\rangle = \langle 0|e^{iS(v)} c_{\beta} c^{+}_{\alpha} e^{-iS(v)}|0\rangle \\ &\neq \langle 0|\, {}^{u}c_{\beta}\, {}^{u}c^{+}_{\alpha}|0\rangle\langle 0|e^{iS(v)} e^{-iS(v)}|0\rangle = \langle 0|c_{\beta} c^{+}_{\alpha}|0\rangle\langle 0|e^{iS(v)} e^{-iS(v)}|0\rangle \end{aligned}$$

contrary to what is expected of the inner product for simple tensors. This is mainly because $c_{\alpha} = {}^{u}c_{\alpha}$ do not commute with $e^{-iS(v)}$ for $u \neq v$. However, for any two $u, w \in \mathscr{L}_3$,

$$\langle 0|e^{iS(v)}\, {}^{v}c_{\beta}\, {}^{w}c^{+}_{\alpha} e^{-iS(w)}|0\rangle = \langle 0|\, {}^{v}c_{\beta}\, {}^{w}c^{+}_{\alpha}|0\rangle\langle 0|e^{iS(v)} e^{-iS(w)}|0\rangle \tag{11}$$

which easily follows from (8)–(10) and from the canonical commutation relations. Similarly, for the case when two (or more) creation operators are involved,

$$\begin{aligned}
&\langle 0|e^{iS(v)}\ {}^{v}c_{\beta_1}\ {}^{v}c_{\beta_2}\ {}^{w}c^{+}_{\alpha_1}\ {}^{w}c^{+}_{\alpha_1} e^{-iS(w)}|0\rangle \\
&= \langle 0|\ {}^{v}c_{\beta_1}\ {}^{v}c_{\beta_2}\ {}^{w}c^{+}_{\alpha_1}\ {}^{w}c^{+}_{\alpha_2}|0\rangle\langle 0|e^{iS(v)}e^{-iS(w)}|0\rangle\,, \\
&\langle 0|e^{iS(v)}\ {}^{v}c_{\beta_1}\ldots\ {}^{v}c_{\beta_n}\ {}^{w}c^{+}_{\alpha_1}\ldots\ {}^{w}c^{+}_{\alpha_n} e^{-iS(w)}|0\rangle \\
&= \langle 0|\ {}^{v}c_{\beta_1}\ldots\ {}^{v}c_{\beta_n}\ {}^{w}c^{+}_{\alpha_1}\ldots\ {}^{w}c^{+}_{\alpha_n}|0\rangle\langle 0|e^{iS(v)}e^{-iS(w)}|0\rangle
\end{aligned} \tag{12}$$

as expected of the inner product on simple tensors. This explains the need for using ${}^{v}c_{lm} = {}^{v}c_{\alpha}$ in various reference frames v, as in composing any complete orthomnormal system in $\mathcal{H}_{|u\rangle}$ we need linear combinations of vectors

$$e^{-iS(v)}|0\rangle$$

with various $v \in \mathscr{L}_3$.

Therefore, for any $v \in \mathscr{L}_3$, we put

$$\begin{aligned}
\left(\ {}^{v}c^{+}_{\alpha_1}\ {}^{v}c^{+}_{\alpha_2}|0\rangle\right) \otimes \left(e^{-iS(v)}|0\rangle\right) &= \ {}^{v}c^{+}_{\alpha_1}\ {}^{v}c^{+}_{\alpha_2} e^{-iS(v)}|0\rangle\,, \\
\left(\ {}^{v}c^{+}_{\alpha_1}\ldots\ {}^{v}c^{+}_{\alpha_n}|0\rangle\right) \otimes \left(e^{-iS(v)}|0\rangle\right) &= \ {}^{v}c^{+}_{\alpha_1}\ldots\ {}^{v}c^{+}_{\alpha_n} e^{-iS(v)}|0\rangle\,.
\end{aligned} \tag{13}$$

Let, in particular, U be the unitary representor of a Lorentz transformation which transforms v into v'. Then

$${}^{v}c^{+}_{\alpha} = \sum_{\beta} \overset{w\mapsto v}{A}{}_{\alpha\beta}\ {}^{w}c^{+}_{\alpha} + \overset{w\mapsto v}{B}{}_{\alpha} Q$$

and

$$\begin{aligned}
\left(U\ {}^{v}c^{+}_{\alpha}|0\rangle\right) \otimes \left(Ue^{-iS(w)}|0\rangle\right) &= \left(\ {}^{v'}c^{+}_{\alpha}|0\rangle\right) \otimes \left(e^{-iS(w')}|0\rangle\right) \\
&= \left(\sum_{\beta} \overset{w'\mapsto v'}{A}{}_{\alpha\beta}\ {}^{w'}c^{+}_{\alpha}|0\rangle\right) \otimes \left(e^{-iS(w')}|0\rangle\right) \\
&= \sum_{\beta} \overset{w'\mapsto v'}{A}{}_{\alpha\beta}\ {}^{w'}c^{+}_{\alpha} e^{-iS(w')}|0\rangle \\
&= \sum_{\beta} \overset{w\mapsto v}{A}{}_{\alpha\beta}\ {}^{w'}c^{+}_{\alpha} e^{-iS(w')}|0\rangle \\
&= U\left(\sum_{\beta} \overset{w\mapsto v}{A}{}_{\alpha\beta}\ {}^{w}c^{+}_{\alpha} e^{-iS(w)}|0\rangle\right),
\end{aligned}$$

so that

$$\left(U\ {}^{v}c_{\alpha}^{+}|0\rangle\right)\otimes\left(Ue^{-iS(w)}|0\rangle\right)=U\left(\left(\ {}^{v}c_{\alpha}^{+}|0\rangle\right)\otimes\left(e^{-iS(w)}|0\rangle\right)\right)$$

and, similarly, we show that this is the case for more general simple tensors

$$\left(U\,{}^{v}c_{\alpha_1}^{+}\dots{}^{v}c_{\alpha_n}^{+}|0\rangle\right)\otimes\left(U\,e^{-iS(v)}|0\rangle\right)=U\left(\left({}^{v}c_{\alpha_1}^{+}\dots{}^{v}c_{\alpha_n}^{+}|0\rangle\right)\otimes\left(e^{-iS(v)}|0\rangle\right)\right). \tag{14}$$

Now in order to define $x\otimes y$ for general x,y of the form of (6) and respectively (5), we need to extend formula (13). In fact, $x\otimes y$ is uniquely determined by (13). Now, we prepare the explicit formula for $x\otimes y$ out of (13).

Let $u_1,u_2,\ldots\in\mathscr{L}_3$ be the unit four-vectors which are used in the definition of the complete orthonormal system

$$e_k(b_{1k}u_1,\ldots,b_{kk}u_k)=\sum_{i=1}^{k}b_{ik}|u_i\rangle=\sum_{i=1}^{k}b_{ik}e^{-iS(u_i)}|0\rangle\,,\qquad k=1,2,\ldots$$

in $\mathcal{H}_{|u\rangle}$. Corresponding to them, we define

$${}^{u_i}c_\alpha=\sum_\beta\overline{\overset{u\mapsto u_i}{A}}_{\alpha\beta}\ {}^{u}c_\alpha+\overline{\overset{u\mapsto v}{B}}_{\alpha}Q=\sum_\beta\overline{\overset{u\mapsto u_i}{A}}_{\alpha\beta}\,c_\alpha+\overline{\overset{u\mapsto u_i}{B}}_{\alpha}Q\,,$$

and

$${}^{u_i}c_\alpha^{+}=\sum_\beta\overset{u\mapsto u_i}{A}_{\alpha\beta}\ {}^{u}c_\alpha^{+}+\overset{u\mapsto v}{B}_{\alpha}Q=\sum_\beta\overset{u\mapsto u_i}{A}_{\alpha\beta}c_\alpha^{+}+\overset{u\mapsto u_i}{B}_{\alpha}Q\,.$$

Having defined this, we introduce for each $i=1,2,\ldots$ and the corresponding operator ${}^{u_i}c_\alpha$ the operator

$${}^{i}c_\alpha=\sum_\beta\overline{\overset{u_i\mapsto u}{A}}_{\alpha\beta}\ {}^{u_i}c_\alpha \tag{15}$$

by discarding the part proportional to the total charge Q in the operator

$$c_\alpha=\ {}^{u}c_\alpha=\sum_\beta\overline{\overset{u_i\mapsto u}{A}}_{\alpha\beta}\ {}^{u_i}c_\beta+\overline{\overset{u_i\mapsto u}{B}}_{\alpha}Q$$

as obtained by the transformation $u_i\mapsto u$ transforming the system of operators ${}^{u_i}c_\beta$ into the system of operators ${}^{u}c_\alpha$. Of course, we have

$$c_\alpha^{+}=\ {}^{u}c_\alpha^{+}=\sum_\beta\overset{u_i\mapsto u}{A}_{\alpha\beta}\ {}^{u_i}c_\beta^{+}+\overset{u_i\mapsto u}{B}_{\alpha}Q\,.$$

The crucial facts for the computations which are to follow are the following. For each four-vector $v \in \mathscr{L}_3$,

$$\left[{}^{v}c_{\alpha}, e^{-iS(v)}\right] = 0\,.$$

The commutation rules are preserved and

$$[{}^{v}c_{\alpha}, {}^{v}c_{\beta}] = 0\,, \quad \left[{}^{v}c_{\alpha}, {}^{v}c_{\beta}^{+}\right] = 4\pi e^{2}\,\delta_{\alpha\beta}\,, \quad [Q, {}^{v}c_{\alpha}] = 0\,, \quad {}^{v}c_{\alpha}|0\rangle = \langle 0|{}^{v}c_{\alpha}^{+} = 0\,.$$

Moreover, if we fix arbitrarily $\alpha = (l, m)$, then because the operators ${}^{i}c_{\alpha}$, $i = 1, 2, \ldots$ all differ from the fixed operator $c_{\alpha} = {}^{u}c_{\alpha}$ with fixed $u \in \mathscr{L}_3$ by the operator (depending on i) which is always proportional to the total charge operator Q, as a consequence of the transformation rule (8) and (10), then not only

$$\left[{}^{i}c_{\alpha}, {}^{i}c_{\beta}\right] = 0\,, \quad \left[{}^{i}c_{\alpha}, {}^{i}c_{\beta}^{+}\right] = 4\pi e^{2}\,\delta_{\alpha\beta}\,, \quad \left[Q, {}^{i}c_{\alpha}\right] = 0\,, \quad {}^{i}c_{\alpha}|0\rangle = \langle 0|\,{}^{i}c_{\alpha}^{+} = 0\,,$$
$$i = 1, 2, \ldots$$

for all $i = 1, 2, \ldots$ but likewise

$$\left[{}^{i}c_{\alpha}, {}^{j}c_{\beta}\right] = 0\,, \quad \left[{}^{i}c_{\alpha}, {}^{j}c_{\beta}^{+}\right] = 4\pi e^{2}\,\delta_{\alpha\beta}\,, \quad \left[Q, {}^{i}c_{\alpha}\right] = 0\,, \quad {}^{i}c_{\alpha}|0\rangle = \langle 0|{}^{i}c_{\alpha}^{+} = 0\,,$$
$$i, j = 1, 2, \ldots$$

Note also that

$$c_{\alpha}^{+}|0\rangle = {}^{i}c_{\alpha}^{+}|0\rangle\,, \qquad i = 1, 2, 3, \ldots$$

Furthermore, we have the following orthogonality relations:

$$\langle 0| \left(\sum_{j=1}^{s} b_{js} e^{iS(u_j)}\, {}^{j}c_{\beta_1} \ldots {}^{j}c_{\beta_m} \right) \left(\sum_{i=1}^{k} b_{ik}\, {}^{i}c_{\alpha_1}^{+} \ldots {}^{i}c_{\alpha_n}^{+} e^{-iS(u_i)} \right) |0\rangle$$
$$= \left(4\pi e^{2}\right)^{n} \delta_{sk}\,\delta_{mn}\,\delta_{\{\alpha_1 \ldots \alpha_n\}\ \{\beta_1 \ldots \beta_m\}}\,. \tag{16}$$

Let x, y be general elements, respectively, $x \in \mathcal{H}_{m=0}$ and $y \in \mathcal{H}_{|u\rangle}$ of the general form (6) and respectively (5). We define the following bilinear map $\otimes$ of $\mathcal{H}_{m=0} \times \mathcal{H}_{|u\rangle}$ into $\mathcal{H}_{m=1}$ by the formula:

$$x \times y \mapsto x \otimes y$$
$$= \sum_{n=1,2,\ldots,\, k=1,2,\ldots,\, i=1,\ldots,\, k,\, \alpha_1,\ldots,\alpha_n} a^{\alpha_1 \ldots \alpha_n} b^{k} b_{ik}\, {}^{i}c_{\alpha_1}^{+} \ldots {}^{i}c_{\alpha_n}^{+} e^{-iS(u_i)} |0\rangle\,.$$

We show now that $\mathcal{H}_{m=0}$ and $\mathcal{H}_{|u\rangle}$ are $\otimes$-linearly disjoint [13], compare Part III, Chap. 39, Definition 39.1. Namely, let $y_1, \dots, y_r$ be a finite subset of generic elements

$$y_j = \sum_{k=1,2,\dots} b_j^k e_k(b_{1k}u_1, \dots, b_{kk}u_k) = \sum_{k=1,2,\dots,\, i=1,\dots,\, k} b_j^k b_{ik} e^{-iS(u_i)}|0\rangle$$

in $\mathcal{H}_{|u\rangle}$ for $j = 1, \dots, r$; and similarly let $x_1, \dots, x_r$ be a finite subset of generic elements

$$x_j = \sum_{n=1,2,\dots,\, \alpha_1,\dots\alpha_n} a_j^{\alpha_1\dots\alpha_n} c_{\alpha_1}^+ \dots c_{\alpha_n}^+ |0\rangle$$

in $\mathcal{H}_{m=0}$ for $j = 1, \dots, r$. Let us suppose that

$$\sum_{j=1}^r x_j \otimes y_j$$
$$= \sum_{j=1,\dots,r,\, n=1,2,\dots,\, k=1,2,\dots,\, i=1,\dots,\, k,\, \alpha_1,\dots,\, \alpha_n} a_j^{\alpha_1\dots\alpha_n} b_j^k b_{ik} \; {}^i c_{\alpha_1}^+ \dots {}^i c_{\alpha_n}^+ e^{-iS(u_i)}|0\rangle = 0\,, \tag{17}$$

and that $x_1, \dots, x_r$ are linearly independent. We have to show that $y_1 = \dots = y_r = 0$. The linear independence of x_j means that if for numbers b^j it follows that

$$\sum_{j=1}^r b^j s a_j^{\alpha_1\dots\alpha_n} = 0$$

for all $n = 1, 2, \dots$, $\alpha_i = (1, -1), (1, 0), (1, 1), (2, -2), \dots$, then $b_1 = \dots = b_r = 0$. Now, consider the inner product of the left-hand side of (17) with

$$\sum_{q=1}^k b_{qk} \; {}^q c_{\beta_1}^+ \dots \; {}^q c_{\beta_n}^+ e^{-iS(u_q)}|0\rangle\,.$$

Then from (17) and the orthogonality relations (16), we get

$$\sum_{j=1}^r a_j^{\beta_1\dots\beta_n} b_j^k = 0$$

for each $k = 1, 2, \dots$ Therefore, by the linear independence of x_j, we obtain

$$b_1^k = \dots = b_r^k = 0$$

for each $k = 1, 2, \ldots$, so that

$$y_1 = \ldots = y_r = 0\,.$$

Similarly, from (17) and linear independence of $y_1, \ldots, y_r$, it follows that

$$x_1 = \ldots = x_r = 0\,,$$

so that $\mathcal{H}_{m=0}$ and $\mathcal{H}_{|u\rangle}$ are $\otimes$-linearly disjoint.

By construction, the image of $\otimes : \mathcal{H}_{m=0} \times \mathcal{H}_{|u\rangle} \to \mathcal{H}_{m=1}$ span the Hilbert space $\mathcal{H}_{m=1}$ and is dense in $\mathcal{H}_{m=1}$. Therefore, the image of $\otimes$ defines the algebraic tensor product $\mathcal{H}_{m=0} \otimes_{\mathrm{alg}} \mathcal{H}_{|u\rangle}$ of $\mathcal{H}_{m=0}$ and $\mathcal{H}_{|u\rangle}$ densely included in $\mathcal{H}_{m=1}$.

Now, we show that the inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{H}_{m=1}$, if restricted to the algebraic tensor product subspace $\mathcal{H}_{m=0} \otimes_{\mathrm{alg}} \mathcal{H}_{|u\rangle}$, coincides with the inner product of the algebraic Hilbert space tensor product

$$\langle x \otimes y | x' \otimes y' \rangle = \langle x|x'\rangle \langle y|y'\rangle$$

for any generic elements $x, x' \in \mathcal{H}_{m=0}$ and any generic elements $y, y' \in \mathcal{H}_{|u\rangle}$. Indeed, let x, y be generic elements of the form of (6) and (5), respectively, and similarly for the generic elements x', y', we put

$$x' = \sum_{q=1,2,\ldots,\, \beta_1,\ldots,\beta_q} a'^{\beta_1\ldots\beta_n} c^+_{\beta_1} \ldots c^+_{\beta_q} |0\rangle$$

and

$$y' = \sum_{s=1,2,\ldots} b'^s e_s(b_{1s}u_1, \ldots, b_{ss}u_s) = \sum_{s=1,2,\ldots,\, j=1,\ldots,s} b'^s b_{js} e^{-iS(u_j)} |0\rangle\,.$$

Then

$$\langle x' \otimes y' | x \otimes y \rangle = \sum_{n,k,q,s,\alpha_1,\ldots,\,\alpha_n,\,\beta_1,\ldots,\,\beta_q} \overline{a'^{\beta_1\ldots\beta_q}} a^{\alpha_1\ldots\alpha_n} \overline{b'^s} b^k$$

$$\times \langle 0| \left(\sum_{j=1}^{s} b_{js} e^{iS(u_j)}\ {}^{j}c_{\beta_q} \ldots {}^{j}c_{\beta_1} \right) \left(\sum_{i=1}^{k} {}^{i}c^+_{\alpha_1} \ldots {}^{i}c^+_{\alpha_n} e^{-iS(u_i)} \right) |0\rangle$$

which, on using (12) and the orthogonality relations (16), is equal to

$$\left(\sum_{n,\alpha_1,\ldots\alpha_n} (4\pi e^2)^n\, \overline{a'^{\alpha_1\ldots\alpha_n}} a^{\alpha_1\ldots\alpha_n} \right) \left(\sum_k \overline{b'^k} b^k \right) = \langle x|x'\rangle \langle y|y'\rangle\,.$$

Thus, the proof of equality (4) is now complete.

Now, let x, y be any generic elements of the form of (6) and (5) respectively. Then by repeated application of (14) and the continuity of each representor[2] U, we obtain

$$U(x \otimes y) = Ux \otimes Uy\,.$$

This ends the proof of our Lemma. □

We observe now that the same proof can be repeated in showing validity of the following

Lemma 2.2.

$$U|_{\mathcal{H}_m} = U|_{\mathcal{H}_{|m,u\rangle}} \otimes U|_{\mathcal{H}_{m=0}}\,.$$

□

Now, let[3] 'Integer part x' for any positive real number x be the least natural number among all natural numbers n for which $x \leq n$. Joining the last Lemma with result (3) of Staruszkiewicz [4], we obtain the theorem formulated in Introduction.

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[2] Each representor $U_{\Lambda(\alpha)}$ being unitary is bounded and thus continuous in the topology of the Hilbert space.

[3] Note that the standard definition of the integer part is slightly different.