ANALYSIS OF A VIRTUAL STATE USING THE COMPLEX SCALING METHOD*

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We demonstrate that the complex scaling method (CSM) is a useful tool to study virtual states. We investigate it by applying the CSM to a simple schematic two-body model which simulates the ^{8}Be$+n$ system. The pole position of the virtual state is obtained by using the continuum level density and the phase shift.

1. Introduction

Virtual states correspond to poles of the S-matrix in the second Riemann sheet of the complex energy plane [1] and have a large influence on the scattering cross section at energies just above the threshold. The scattering length of a virtual state is negative, while that of a bound state is positive. The study of virtual states is important for the investigation of scattering observables and interactions.

The complex scaling method (CSM) [2, 3] has been shown to be very useful in studies of weakly-bound states strongly coupled to continuum. However, it was suggested that it is difficult to explicitly describe within this framework broad resonance states and virtual states, as those need scaling

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angles larger than $\pi/4$, which is a limit due to the analyticity of the potential. There is no previous evidence (except for our previous works [4, 5]) that the CSM can be successfully applied to the investigation of a virtual state. Recently, by applying the CSM to the $\alpha+\alpha+n$ three-body model for ^{9}Be, we have shown that a sharp peak of the photo-disintegration cross section experimentally observed just above the ^{8}Be$(0^+)+n$ threshold can be explained as a $1/2^+$ virtual state of the ^{8}Be$(0^+) + n$ two-body configuration [4]. Furthermore, in the framework of the CSM, the structure of a virtual state in an s-wave was discussed by using a simple schematic two-body model [5].

In our previous work [5], we concluded that a virtual state has a strong influence on the scattering observables when it approaches the zero energy near the physical scattering region. In the present report, we discuss the virtual state position on the complex energy plane using the continuum level density and the phase shift corresponding to the virtual state.

2. Complex scaling method

In the CSM, the relative coordinate $\vec{r}$ is rotated as $\vec{r} \rightarrow \vec{r}e^{i\theta}$ in the complex coordinate plane [6]. The complex-scaled Hamiltonian H^θ and wave function $\Psi^\nu_{J^\pi}(\theta)$ are defined as $U(\theta)HU^{-1}(\theta)$ and $U(\theta)\Psi^\nu_{J^\pi}$, respectively (see Refs. [2, 3] for details). Therefore, the Schrödinger equation can be rewritten as

$$H^\theta \Psi^\nu_{J^\pi}(\theta) = E^\theta_\nu \Psi^\nu_{J^\pi}(\theta), \tag{1}$$

where J^π is the spin and parity, ν is the state index, and θ, being a real number, is the scaling angle.

Applying the L^2 basis function method, we expand the wave function as

$$\Psi^\nu_{J^\pi}(\theta) = \sum_{n=1}^{N} c_n^{J^\pi \nu}(\theta)\phi_n(\vec{r}), \tag{2}$$

where $\phi_n(\vec{r})$ is the appropriate set of basis functions. The expansion coefficients $c_n^{J^\pi \nu}$ and the complex energy eigenvalues E^θ_ν are obtained by solving the complex-eigenvalue problem given in Eq. (1). The complex energies of resonant states are obtained as $E_r = E_r^{res} - i\Gamma_r/2$, when $\tan^{-1}(\Gamma_r/2E_r^{res}) < 2\theta$. In the present schematic two-body model, the Hamiltonian is given as

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r), \tag{3}$$

where

$$V(r) = v_0 \exp\left(-ar^2\right), \qquad a = 0.16 \text{ fm}^{-2}. \tag{4}$$

For simplicity, we use $\frac{\hbar^2}{2\mu} = 1$ (MeV fm^2).

3. Results

3.1. Virtual s-state

In our previous works [3, 4], we confirmed that a virtual state is responsible for the enhancement of the photo-disintegration cross section just above the threshold in *s*-waves. When the virtual state approaches the zero energy on the second Riemann sheet, it has a strong influence on the reaction observables. In the present work, we analyse the continuum level density (CLD) and the phase shifts applying the model Hamiltonian simulating the ^{8}Be+n system without spin degrees of freedom.

In Fig. 1, we show the energy levels considered in the present model. The $J^\pi = 0^+$ and 1^- states are obtained by solving Eq. (1) for the Hamiltonian (Eq. (3)). The potential strength v_0 in Eq. (4) is chosen to reproduce one bound $J^\pi = 0^+$ of *s*-waves. But this 0_1^+ solution is assumed to be a Pauli-forbidden state, because in this model we describe the ^{8}Be(0^+) + n system which has the Pauli-forbidden ($0s$) neutron configuration. Therefore, the 1_1^- solution corresponds to the ground state.
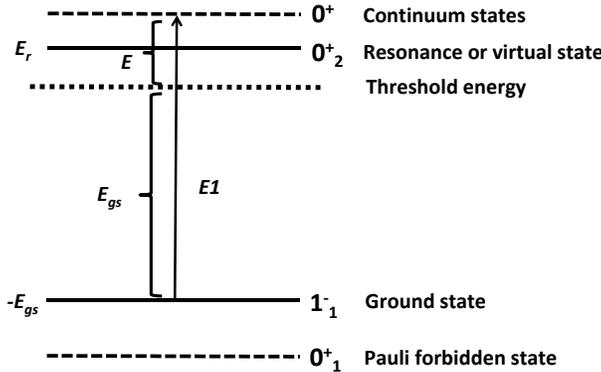

Fig. 1. The energy level diagram of the two-body potential model describing ^{9}Be as a ^{8}Be+n system. The dotted line represents the threshold energy.

3.2. Phase shift of s-waves

In the CSM, a virtual state cannot be obtained as an isolated solution, but the continuum solutions are considered to include components of the virtual state. To confirm this property of the continuum solutions, we calculate the phase shifts using energy eigenvalues (resonance; $E_r^{\mathrm{res}} - i\Gamma_r/2$, $r = 1, \cdots N_r^\theta$, continuum; $\epsilon_c^r - i\epsilon_c^i$, $c = 1, \cdots N_c^\theta$) of Eq. (1) (see Ref. [8])

$$\delta_\theta^N(E) = N_{\mathrm{b}}\pi + \sum_{r=1}^{N_r^\theta}\left\{\tan^{-1}\left(\frac{\Gamma_r/2}{E_r^{\mathrm{res}}-E}\right)\right\} + \sum_{r=c}^{N_c^\theta}\left\{\tan^{-1}\left(\frac{\epsilon_c^i}{\epsilon_c^r-E}\right)\right\}$$

$$- \sum_{k=1}^{N^\theta}\left\{\tan^{-1}\left(\frac{\epsilon_k^{0i}}{\epsilon_k^{0r}-E}\right)\right\} , \tag{5}$$

where N_{b} is the number of bound-state solutions, and $(\epsilon_k^{0r} - i\epsilon_k^{0i}, \quad k = 1\cdots N^\theta)$ are energy eigenvalues of the free Hamiltonian $H_0 = -(\hbar^2/2\mu)\nabla^2$. For the present s-wave solutions, we have no bound states, except for the Pauli-forbidden state, and no resonance solutions, and so the phase shift is described by the third and the forth term.

From the phase shift, we can calculate the scattering length

$$a_s = -\lim_{k\to 0}\frac{\tan\left(\delta_\theta^N(E)\right)}{k} , \tag{6}$$

where $k = \sqrt{2\mu E}/\hbar$. The calculated phase shift shows a sudden change of the scattering length from a positive value for $v_0 \leq 1.43$ MeV to a negative value for $v_0 \geq -1.42$ MeV. This result suggests that a virtual state appears around $v_0 \sim -1.42$ MeV. At $v_0 = -1.43$ MeV, a bound solution is obtained. This bound state is considered to develop from the virtual state at $v_0 = -1.42$ MeV as the attractive potential strength is increased slightly.

3.3. Continuum level density

We try to extract the virtual state obtained at $v_0 = -1.42$ MeV explicitly. For this purpose, we calculate the continuum level density. The continuum level density $\Delta_\theta^N(E)$ is related to the phase shift via $\Delta_\theta^N(E) = (1/\pi)\mathrm{d}\delta_\theta^N(E)/\mathrm{d}E$ [8]. The continuum level density of continuum solutions at $v_0 = -1.42(\equiv v^-)$ MeV, which include the virtual state, is calculated as

$$\Delta_\theta^N(E;v^-) = -\frac{1}{\pi}\left[\sum_{c=1}^{N_c^\theta}\frac{1}{E-\epsilon_c^r+i\epsilon_c^i} - \sum_{k=1}^{N}\frac{1}{E-\epsilon_k^{0r}+i\epsilon_k^{0i}}\right] . \tag{7}$$

The number of continuum solutions for v^- is $N_c^\theta = N$ because of no bound-state and resonance solutions. However, in the case of $v_0 = -1.43(\equiv v^+)$ one bound state appears and the number of continuum solutions is $N_c^\theta = N - 1$. We put the continuum level density for v^+ as $\Delta_\theta^N(E;v^+)$, which has no virtual state.

Assuming that the continuum level density $\Delta_\theta^N(E)$ is a smooth function for the coupling constant v_0, we can extract the contribution from the virtual state to the continuum level density by taking a difference between $\Delta_\theta^N(E; v^-)$ and $\Delta_\theta^N(E; v^+)$

$$\Delta_\theta^{\mathrm{vir}}(E) = \Delta_\theta^N\left(E; v^-\right) - \Delta_\theta^N\left(E; v^+\right) . \tag{8}$$

In a similar way, we can extract the virtual state component $\delta^{\mathrm{vir}}(E)$ of the phase shifts.

3.4. Position of the virtual state

The results for $\Delta_\theta^{\mathrm{vir}}(E)$ and $\delta^{\mathrm{vir}}(E)$ are shown in Fig. 2. The behaviour of the phase shift $\delta^{\mathrm{vir}}(E)$ seems to be a logarithmic function of E, and the continuum level density $\Delta_\theta^{\mathrm{vir}}(E)$ behaves like a function of $1/(E - E^{\mathrm{vir}})$. Then, assuming $\Delta_\theta^{\mathrm{vir}}(E) \propto 1/(E - E^{\mathrm{vir}})$, we try to extract the energy E^{vir} corresponding to the virtual pole position on the second Riemann sheet. The obtained result is $E^{\mathrm{vir}} \approx -0.001$ MeV.
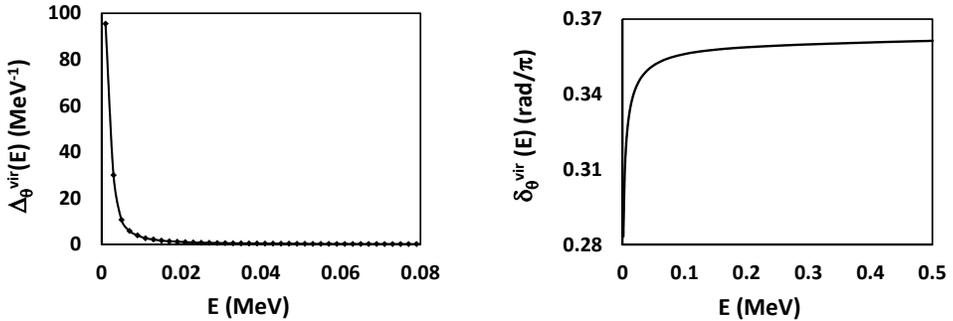

Fig. 2. Left panel: The continuum level density $\Delta_\theta^{\mathrm{vir}}(E)$ for the virtual state solution. Right panel: The phase shift of the virtual state, calculated from the virtual state continuum level density $\Delta_\theta^{\mathrm{vir}}(E)$.

To verify the reliability of this result, we compare it with that obtained using the Jost function method [7]. The Jost function method can be easily applied to the present two-body model, and we obtain a solution for the virtual state at $E^{\mathrm{vir}} = -4.97 \times 10^{-6}$ MeV. Comparing this with the present result calculated by the CSM, we can consider the latter reasonable because the complex eigenvalues solved with the basis-function method in the CSM have only three digits of precision. Therefore, it is difficult for the CSM to keep a high numerical accuracy.

4. Summary

We employed a simple schematic two-body model and the CSM to investigate a virtual state. The calculated continuum level density and the phase shifts indicate a virtual state solution near the threshold.

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