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CONFORMAL YANO–KILLING TENSORS FOR SPACE-TIMES WITH COSMOLOGICAL CONSTANT

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We present a new method for constructing conformal Yano–Killing tensors in the five-dimensional Anti-de Sitter space-time. The found tensors are represented in two different coordinate systems. We also discuss, in terms of CYK tensors, global charges which are well-defined for asymptotically (five-dimensional) Anti-de Sitter space-time. Additionally, in Appendix A, we present our own derivation of conformal Killing one-forms in four-dimensional Anti-de Sitter space-time as an application of the Theorem 4.

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1. Introduction

We generalize the construction of the conformal Yano–Killing tensors presented in [1] to the case of the five-dimensional Anti-de Sitter space-time which is intensively explored in the context of the AdS/CFT correspondence. In particular, one can try to generalize formulae (3.11)–(3.17) from Section 3.2 in [10] to the case of conformal tensors. More precisely, tensor product of two conformal Killing vector fields K^μ can be replaced by a symmetric conformal Killing tensor $K^{\mu\nu}$: $K^\mu K^\nu \langle \mathcal{O}_{\mu\nu} \rangle \longrightarrow K^{\mu\nu} \langle \mathcal{O}_{\mu\nu} \rangle$, but for the skew-symmetric tensor $F_{\mu\nu}$ (primary operator) one can consider the expression $Q_{\mu\nu} \langle F^{\mu\nu} \rangle$, where Q is the conformal Yano–Killing two-form[1]. We also generalize constructions from [1] to show how the conformal Yano–Killing tensors can be used to define global gravitational charges in the case of the five-dimensional Anti-de Sitter space-time.

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1 Obviously, higher rank tensors with more indices are also possible, both symmetric and skew-symmetric.

1.1. Construction of the five-dimensional Anti-de Sitter space-time

Anti-de Sitter space-time can be constructed in the following way. We consider six-dimensional affine space $\tilde{V}$, which is modeled on the vector space V. The vector space V is equipped with a pseudo-scalar product

$$(v, w) = -v^0 w^0 + \sum_{k=1}^{4} v^k w^k - v^5 w^5 \,.$$

The affine space $\tilde{V}$ is naturally a flat manifold. If we choose one point in $\tilde{V}$, then it effectively turns our affine space $\tilde{V}$ into vector space V which is isomorphic to $\mathbb{R}^6$ but not canonically. Next, we consider in the affine space $\tilde{V}$ identified with the vector space V the locus of the equation

$$(x, x) = -l^2 \,. \tag{1.1}$$

Bound $(x, x) = -l^2$ defines five-dimensional submanifold of $\tilde{V}$, which as a manifold is the five-dimensional Anti-de Sitter space-time. In the case of affine space $\tilde{V}$ we have canonical isomorphism $\forall_{a \in \tilde{V}} \, T_a \tilde{V} \simeq V$. This means that $\tilde{V}$ is a (pseudo)-Riemannian manifold. Therefore, we can pull back the metric to the locus $(x, x) = -l^2$ from the ambient space $\tilde{V}$. That way our five-dimensional Anti-de Sitter, later denoted AdS5, gains the structure of the (pseudo)-Riemannian manifold.

Each linear transformation of V that respects quadratic form $(\cdot, \cdot)$, that is a transformation from $\mathrm{SO}\,(2, 4)$ group, preserves AdS5 as a subset, because if $f \in \mathrm{SO}\,(2, 4)$ then it follows that $(x, x) = (fx, fx)$. The metric of AdS5 is also preserved, because transformation f preserves the (pseudo)-scalar product of the space V, so it also preserves (pseudo)-Riemannian metric of the affine space $\tilde{V}$. That means that it also preserves metric induced on AdS5, which is submanifold of $\tilde{V}$. This shows that $\mathrm{SO}\,(2, 4)$ is a subgroup of the isometry group of AdS5 (it is, in fact, the whole isometry group).

2. Description of the metric submanifold

Inspired by this example, let us consider general situation. We have a pair (N, g), where g is a metric of manifold N. We also have submanifold M of codimension 1 with metric $\tilde{g}$ induced from N. Let $\overset{N}{\nabla}_X Y$ be the Levi-Civita derivative of the vector field Y tangent to N with respect to the field X, which is also tangent to N. If fields X, Y are tangent to M, that is $X, Y \in \Gamma(TM)$, we will also write $\overset{N}{\nabla}_X Y$ understanding that in this notation, fields X, Y are substituted by their arbitrary local extensions. The result on M

does not depend on the choice of those extensions. Next, we denote as $\overset{M}{\nabla}_X Y$ the derivative of the field Y tangent to M with respect to the field X also tangent to M with respect to the metric connection on M. In this notation, we have $X, Y \in \Gamma(TM) \Rightarrow \overset{N}{\nabla}_X Y = \overset{M}{\nabla}_X Y + \tilde{K}(X,Y)$, where $\tilde{K}(X,Y)$ is the form of external curvature. It is also called the second fundamental form. It is known that $\tilde{K}(X,Y) \perp TM$ and $\tilde{K}(X,Y) = \tilde{K}(Y,X)$.

If we choose a local coordinate system $x_1, \ldots, x_{n+1}$ on some open subset of N which satisfies $\emptyset \neq \{p \in N \mid x_{n+1} = 0\} \subset M$, then the collection of functions $x_1, \ldots, x_n$ is a local coordinate system on M. Let Latin indices go from 1 to n, whereas Greek letters go from 1 to $n+1$. We also choose normal field n defined on M such that $n \in \Gamma(TN)$, $n \perp TM$, and $(n,n) = \pm 1$. Here, we choose the sign depending on the type of the surface (null surfaces are not considered here). We can now write $\tilde{K}(X,Y) = K(X,Y)n$ which defines K as a symmetric tensor of rank 2. We also use convenient notation, in which $v^\mu{}_{|\nu} = \overset{N}{\nabla}_\nu v^\mu$ and $v^a{}_{;b} = \overset{M}{\nabla}_b v^a$.

Theorem 1. *Let ω be a one-form on the manifold N. Then*

$$\omega_{b|a} = \omega_{b;a} - K_{ab}\omega_\mu n^\mu \,.$$

The proof of this theorem can be found in Appendix B.

Theorem 2. *The external curvature form K_{ab} satisfies equation $K = -\frac{1}{2}\mathcal{L}_n g$.*

The proof of this theorem can be found in Appendix B.

Theorem 3. *In the case of $N = \tilde{V}$ and $M = AdS5$, we have $K = C\tilde{g}$, where C is some real function on M.*

Proof. We have the identity $K = -\frac{1}{2}\mathcal{L}_n g$. On the vector space with pseudo-scalar product we can always choose coordinates (r, ϕ_i), where $r(p) = \sqrt{|\, g(p-0, p-0)\,|}$, $p \in N$ (0 here is an arbitrarily chosen point in N) is the distance from the zero vector, whereas ϕ_i are some angles that are coordinate system of the pseudo-sphere of constant r. Additionally, we can choose coordinates ϕ_i in such a way that the metric has the form of

$$g = \pm \mathrm{d}r^2 \pm r^2 \hat{g} \,, \tag{2.1}$$

where $\hat{g}$ is a metric of the unit pseudo-sphere parametrized by ϕ_i and does not depend on the coordinate r. The Lie derivative along the field $n = \partial_r$ of the metric g is, of course, proportional to $\hat{g}$ which is proportional to the induced metric on pseudo-sphere. □

So far the results are repeated to fix the notation — the subject of Theorems 1–3 is well-established.

3. Pulling back conformal tensors to submanifolds

Theorem 3 suggests restricting our considerations to the case when $K \sim \tilde{g}$. From now on we assume that this condition holds. Now, we can prove the following theorem.

Theorem 4. *If k is a conformal Killing one-form on N, then its pullback to M is a conformal Killing one-form on M.*

Proof. Let us compute

$$k_{(a|b)} + K_{ab} n^\mu k_\mu = k_{(a;b)} \quad (= A g_{ab}) \,,$$

where A is a function. We see that $k_{(a;b)} \sim \tilde{g}_{ab}$ because both terms above are proportional to the metric tensor g_{ab} on M. Let us notice that in this case, the restriction of a Killing one-form, that is one-form such that $k_{(a|b)} = 0$, in some cases will not be a Killing one-form on M but only a conformal one. □

Theorem 5. *We have the following identity for computing the covariant derivative of the two-form Q on the manifold N:*

$$Q_{ac|b} = Q_{ac;b} - K_{ab} Q_{\nu c} n^\nu - K_{bc} Q_{a\mu} n^\mu \,.$$

Proof. We contract the two-form Q with arbitrary vector field v tangent to M. We can compute the derivative of the resulting one-form using formula (B.7).

$$(Q_{a\mu} v^\mu)_{|b} = (Q_{ac} v^c)_{;b} - K_{ab} Q_{\nu\mu} v^\mu n^\nu = Q_{ac;b} v^c + Q_{ac} v^c{}_{;b} - K_{ab} Q_{\nu\mu} v^\mu n^\nu \,. \tag{3.1}$$

On the other hand,

$$(Q_{a\mu} v^\mu)_{|b} = Q_{ac|b} v^c + Q_{ac} v^c_{;b} + Q_{a\mu} K_{bc} v^c n^\mu \,. \tag{3.2}$$

Comparison of the two sides of equations leads to the conclusion that

$$Q_{ac|b} v^c = Q_{ac;b} v^c - K_{ab} Q_{\nu c} n^\nu v^c - Q_{a\mu} K_{bc} v^c n^\mu \,, \tag{3.3}$$

so

$$Q_{ac|b} = Q_{ac;b} - K_{ab} Q_{\nu c} n^\nu - K_{bc} Q_{a\mu} n^\mu \,. \tag{3.4}$$

It is easy to generalize this identity to arbitrary n-forms. For a three-form, the identity is given by Theorem 9. The identity for a two-form can be written as

$$Q_{ab|c} = Q_{ab;c} - q_a K_{cb} + q_b K_{ac} \,, \tag{3.5}$$

where $q_a = Q_{a\mu} n^\mu$. □

Definition 6. *The two-form Q satisfying equation $Q_{\alpha(\beta;\gamma)} = 0$ is called the Yano–Killing tensor.*

Theorem 7. *If $\tilde{Q}$ is a Yano–Killing tensor on the manifold N, then its pullback to sub-manifold M denoted by Q satisfies*

$$Q_{ab;c} + Q_{ac;b} = 2\tilde{q}_a g_{cb} - \tilde{q}_b g_{ac} - \tilde{q}_c g_{ab} \,,$$

where $\tilde{q}$ is a certain one-form[2].

Proof. Let us check what equation is satisfied by the pullback of the form Q to M. We have

$$0 = Q_{ab|c} + Q_{ac|b} = Q_{ab;c} + Q_{ac;b} - 2q_a K_{cb} + q_b K_{ac} + q_c K_{ab} \,. \tag{3.6}$$

So it turns out that pullback of the Yano–Killing tensor is satisfying a bit different equation then Yano–Killing tensors. This equation looks like this

$$Q_{ab;c} + Q_{ac;b} = 2\tilde{q}_a g_{cb} - \tilde{q}_b g_{ac} - \tilde{q}_c g_{ab} \,, \tag{3.7}$$

where $\tilde{q}_c$ is a certain one-form. □

Last theorem suggests the following definition:

Definition 8. *If M is a Riemannian manifold and Q is the two-form satisfying equation*

$$Q_{ab;c} + Q_{ac;b} = 2\tilde{q}_a g_{cb} - \tilde{q}_b g_{ac} - \tilde{q}_c g_{ab}$$

for some one-form $\tilde{q}$, then Q is called the conformal Yano–Killing tensor. We will often use abbreviation CYK tensor for the conformal Yano–Killing tensor.

We decided to find CYK tensors on the five-dimensional Anti-de Sitter space-time. In the ambient vector space V that surrounds five-dimensional Anti-de Sitter space-time, it is easy to find some CYK tensors. We can just choose two-forms which have constant coefficients in the Cartesian coordinates. This way, we can obtain 15 CYK tensors on the Anti-de Sitter space-time. However, it is known that there are 35 linearly-independent CYK tensors on this space-time. We will now present a way to find the remaining 20 CYK tensors.

[2] $\tilde{q}$ is obviously related to the divergence of Q by contraction of the indices in the above equation.

Theorem 9. *Analogously to equation (3.3), it can be proved that the covariant derivative of the three-form $T_{\alpha\beta\gamma}$ looks like this*

$$T_{abc|d} = T_{abc;d} - Q_{bc}K_{ad} + Q_{ac}K_{bd} - Q_{ab}K_{cd}\,, \tag{3.8}$$

where $Q_{ab} = T_{ab\mu}n^\mu$.

Proof. Analogous to the proof of Theorem 5. □

Theorem 10. *If a three-form T on the manifold N satisfies the equation $T_{\alpha\beta(\gamma|\delta)} = 0$, then its pullback to the manifold M satisfies the equation*

$$2T_{ab(c;d)} = 2Q_{ab}g_{cd} - Q_{ac}g_{bd} - Q_{ad}g_{bc} + Q_{bc}g_{ad} + Q_{bd}g_{ac}$$

with a certain two-form Q.

Proof.

$$2T_{ab(c|d)} = 2T_{ab(c;d)} - 2Q_{ab}K_{cd} + Q_{ac}K_{bd} + Q_{ad}K_{bc} - Q_{bc}K_{ad} - Q_{bd}K_{ac}\,. \tag{3.9}$$

From this equation it follows that if T satisfies $T_{\alpha\beta(\gamma|\delta)} = 0$, then pullback T to M satisfies

$$2T_{ab(c;d)} = 2Q_{ab}g_{cd} - Q_{ac}g_{bd} - Q_{ad}g_{bc} + Q_{bc}g_{ad} + Q_{bd}g_{ac}\,. \tag{3.10}$$

□

Definition 11. *The three-form T_{abc} satisfying equation (3.10) is called the CYK three-form.*

4. Five-dimensional case

In this section, we do not take into considerations the surrounding manifold N. Additionally, all tensor fields are defined on M and $\dim M = 5$. In this case, we have the following well-known theorems.

Theorem 12. *Hodge dual of the CYK three-form is a CYK tensor.*

The proof can be found in Appendix B but a general case is also given in Proposition 3.2 in [2].

Theorem 13. *If k is conformal Killing one-form for the metric g, and Ω^2 is a positive smooth function, then $\Omega^2 k$ is a conformal Killing one-form for the metric $\Omega^2 g$.*

Proof. Let us denote as X the vector field associated with one-form k as follows $X^i = g^{ij}k_j$ (in this proof, indices i, j go through all functions from our coordinate system). It is known that conformal Killing equation $\nabla_{(a}k_{b)} = \lambda' g_{ab}$, where λ' is some function is equivalent to the equation $\mathcal{L}_X g = \lambda g$. Now let us compute $\mathcal{L}_X(\Omega^2 g) = \mathcal{L}_X(\Omega^2)g + \Omega^2\mathcal{L}_X g = (\frac{\mathcal{L}_X\Omega^2}{\Omega^2} + \lambda)(\Omega^2 g)$. This shows that vector field X is related to conformal Killing one-form b for the metric $\Omega^2 g$. This one-form is equal to $b_i = \Omega^2 g_{ij}X^j = \Omega^2 k_i$. □

Theorem 14. *If Q is a CYK tensor for the metric g, then $\Omega^3 Q$ is a CYK tensor for the metric $\Omega^2 g$.*

The proof can be found in [3].

4.1. Construction of conserved charges in asymptotically Anti-de Sitter space-times

Definition 15. *Tensor field W is called spin-2 field if it satisfies*

$$\begin{aligned} W_{\alpha\beta\gamma\delta} &= W_{\gamma\delta\alpha\beta} = W_{[\alpha\beta][\gamma\delta]}\,, \qquad W_{\alpha[\beta\gamma\delta]} = 0\,, \\ W^{\alpha}_{\ \beta\alpha\delta} &= 0\,, \qquad \nabla_{[\lambda}W_{\mu\nu]\alpha\beta} = 0\,. \end{aligned}$$

An example of the spin-2 field is the Weyl tensor. In the case of this tensor, we also know that conformal transformations do not change Weyl components $W^{\alpha}{}_{\beta\gamma\delta}$.

Next theorem enables one to define conserved charges on space-times that are asymptotically similar to the Anti-de Sitter space-time.

Theorem 16. *If Q is a CYK tensor and W is a spin-2 field, then the three-form $T_{\alpha\beta\gamma} = \frac{1}{2}\epsilon_{\alpha\beta\gamma}{}^{\delta\sigma}W_{\delta\sigma\mu\nu}Q^{\mu\nu}$ is closed.*

Proof. Let us define

$$F_{\mu\nu} = W_{\mu\nu\lambda\kappa}Q^{\lambda\kappa}\,, \tag{4.1}$$

where Q is a certain CYK tensor. We will show that

$$F^{\mu\nu}{}_{;\nu} = \tfrac{2}{3}W^{\mu\nu\alpha\beta}\mathcal{Q}_{\alpha\beta\nu}\,, \tag{4.2}$$

where

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \tfrac{1}{2}\left(g_{\lambda\sigma}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right)\,, \tag{4.3}$$

so $\mathcal{Q} = 0$ if Q is a CYK tensor (it follows from contraction of a pair of indices in equation (3.7)). We can prove equation (4.2) in the following way:

$$\begin{aligned} W^{\mu\nu\alpha\beta}\mathcal{Q}_{\alpha\beta\nu} &= W^{\mu\nu\alpha\beta}\left(Q_{\alpha\beta;\nu} + Q_{\nu\beta;\alpha}\right) = \left(W^{\mu\nu\alpha\beta} + W^{\mu\alpha\nu\beta}\right) Q_{\alpha\beta;\nu} \\ &= \left(W^{\mu\nu\alpha\beta} + \tfrac{1}{2}W^{\mu\alpha\nu\beta} - \tfrac{1}{2}W^{\mu\beta\nu\alpha}\right) Q_{\alpha\beta;\nu} \\ &= \left[\tfrac{3}{2}W^{\mu\nu\alpha\beta} - \tfrac{1}{2}\left(W^{\mu\nu\alpha\beta} + W^{\mu\alpha\beta\nu} + W^{\mu\beta\nu\alpha}\right)\right] Q_{\alpha\beta;\nu} \\ &= \tfrac{3}{2}W^{\mu\nu\alpha\beta}Q_{\alpha\beta;\nu} \,. \end{aligned}$$

For this reason,

$$\nabla_\nu F^{\mu\nu} = \nabla_\nu\left(W^{\mu\nu\alpha\beta}Q_{\alpha\beta}\right) = \left(\nabla_\nu W^{\mu\nu\alpha\beta}\right) Q_{\alpha\beta} + W^{\mu\nu\alpha\beta}\nabla_\nu Q_{\alpha\beta} \,. \tag{4.4}$$

Let us notice that if we contract indices μ and α in the equation

$$\nabla_\lambda W_{\mu\nu\alpha\beta} + \nabla_\mu W_{\nu\lambda\alpha\beta} + \nabla_\nu W_{\lambda\mu\alpha\beta} = 0 \,, \tag{4.5}$$

then we will end up with

$$\nabla_\alpha W_{\lambda\mu}{}^{\alpha}{}_{\beta} = 0 \,, \tag{4.6}$$

and finally,

$$\nabla_\nu F^{\mu\nu} = W^{\mu\nu\alpha\beta}\mathcal{Q}_{\alpha\beta;\nu} = \tfrac{2}{3}W^{\mu\nu\alpha\beta}\mathcal{Q}_{\alpha\beta\nu} \,. \tag{4.7}$$

In our case, Q is a CYK tensor, so

$$\nabla_\nu F^{\mu\nu} = 0 \,. \tag{4.8}$$

We can always express F as $F = *T$. We can use identity $**F = (-1)^s$, $s = \mathrm{sgn}\,\det(g)$ (valid for two-forms F in the five-dimensional pseudo-Riemannian space) to obtain

$$T = (-1)^s * F \,. \tag{4.9}$$

Next, we have

$$\nabla_\nu F^{\nu\mu} = \tfrac{1}{6}\nabla_\nu\left(\epsilon^{\nu\mu\alpha\beta\gamma}T_{\alpha\beta\gamma}\right) = \tfrac{1}{6}\epsilon^{\nu\mu\alpha\beta\gamma}\nabla_\nu T_{\alpha\beta\gamma} = \tfrac{1}{6}\epsilon^{\nu\mu\alpha\beta\gamma}\partial_{[\nu}T_{\alpha\beta\gamma]} \,. \tag{4.10}$$

We can change covariant derivatives to partial derivatives because the Christoffel symbols are symmetric in their lower indices. This shows that $\partial_{[\nu}T_{\alpha\beta\gamma]} = 0$, and, therefore, $\mathrm{d}T = 0$. □

4.2. A way to construct a quasi-local charge

In Theorem 16, we have found a way to obtain a closed three-form. For space-time which is asymptotically similar to the Anti-de Sitter space-time (it means that there exists a coordinate system in which metric is similar to the Anti-de Sitter metric close to infinity, see [4]), we can construct an asymptotic CYK tensor which asymptotically satisfy CYK equation from Definition 8. In this way, if we also have spin-2 field on our asymptotically Anti-de Sitter space-time (for instance, the Weyl tensor), then we can construct asymptotically closed three-form. We can now consider the slice of constant time. We integrate our three-form on a large three-dimensional sphere belonging to this slice and located in the asymptotic region. We will end up with a quantity that asymptotically does not depend on the size of this sphere or rather approaches (possibly finite) limit at infinity. This way, we obtain some quasi-local charge. It turns out that if we change the metric g (by conformal rescaling) to the $\Omega^2 g$, and if our spin-2 field W is chosen to be the Weyl tensor, then the corresponding form T transforms to $\Omega^2 T$.

5. CYK tensors in coordinate systems

It follows from our past considerations that on AdS5 one can find CYK tensors as pullbacks of constant two-forms on $\tilde{V}$ and as the Hodge duals of pullbacks of constant three-forms on the surrounding space $\tilde{V}$.

We will use the convention that indices a, b, c go from 1 to 3, indices i, j, k go from 1 to 4, indices μ, ν, λ go from 0 to 4, and indices A, B, C go from 0 to 5.

5.1. Poincaré coordinate system

Let us consider a parametrization of the Anti-de Sitter space-time with coordinates t, x^1, x^2, x^3, y. This means that t has index 0, x^1 has index 1, and so on. Quantity l is a parameter that is related to the size of Anti-de Sitter. This parameter is also a part of equation $(X, X) = -l^2$ (this is equation (1.1)) defining AdS5. In these coordinates, we have

$$\begin{aligned} X^0 &= \frac{1}{2y}\left(y^2 + l^2 + \|\bar{x}\|^2 - t^2\right), \\ X^a &= \frac{x^a}{y} l \qquad a \in \{1, 2, 3\}, \\ X^4 &= \frac{1}{2y}\left(y^2 - l^2 + \bar{x}^2 - t^2\right), \\ X^5 &= \frac{t}{y} l, \end{aligned} \tag{5.1}$$

where metric on the space $\tilde{V}$ is equal to

$$\mathrm{d}s^2 = -\left(\mathrm{d}X^0\right)^2 + \sum_{k=1}^{4}\left(\mathrm{d}X^k\right)^2 - \left(\mathrm{d}X^5\right)^2 . \tag{5.2}$$

AdS5 is a locus

$$-l^2 = -\left(X^0\right)^2 + \sum_{k=1}^{4}\left(X^k\right)^2 - \left(X^5\right)^2 . \tag{5.3}$$

It turns out that in these coordinates, the induced metric is conformally flat and equal to

$$\mathrm{d}s^2 = \frac{l^2}{y^2}\left(-\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}\bar{x}^2\right) , \tag{5.4}$$

where $\|\bar{x}\|^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$, whereas $\mathrm{d}\bar{x}^2 = \sum_{a=1}^{3}\left(\mathrm{d}x^a\right)^2$. For this reason, if we choose conformal factor $\Omega = \frac{l}{y}$ in Theorem 14, then we see that CYK tensors on AdS5 divided by Ω^3 are CYK tensors on the five-dimensional Minkowski space-time. Let us denote the pullbacks of constant two-forms as

$$C_{AB} := \mathfrak{i}^*\left(\mathrm{d}X^A \wedge \mathrm{d}X^B\right) , \tag{5.5}$$

and the Hodge duals of pullbacks of constant three-forms as

$$H_{ABC} := *\mathfrak{i}^*\left(\mathrm{d}X^A \wedge \mathrm{d}X^B \wedge \mathrm{d}X^C\right) , \tag{5.6}$$

where $\mathfrak{i}$ is an immersion of the Anti-de Sitter space-time into 6 dimensional ambient vector space V.

Let us adopt the following notation: $D = x^a\mathrm{d}x^a + y\mathrm{d}y - t\mathrm{d}t$, $\mathcal{D} = x^a\mathrm{d}x^a + y\mathrm{d}y$, $\tau_a = \mathrm{d}x^a$, $\tau_4 = \mathrm{d}y$, $\mathcal{K}_a = x^a\mathcal{D} - \frac{1}{2}\left(\bar{x}^2 + y^2\right)\tau_a$, $\mathcal{K}_4 = y\mathcal{D} - \frac{1}{2}\left(\bar{x}^2 + y^2\right)\tau_4$, $\mathcal{L}_{ab} = x^a\mathrm{d}x^b - x^b\mathrm{d}x^a$, $\mathcal{L}_{a4} = x^a\mathrm{d}y - y\mathrm{d}x^a$.

We calculated the tensors C_{AB} and H_{ABC} in Mathematica. We can express them in the above notation. Let us consider an array of numbers ϵ^{ijkl} which gives to the collection of indices $i, j, k, l \in \{1, 2, 3, 4\}$ the sign of the permutation associated with them or zero (if this collection of indices is not a permutation, then the result is 0). We also use here the old summation convention. This means that we contract the same indices even if they are on the same level:

$$
\begin{aligned}
C_{0,4} &= \Omega^3 \frac{1}{l}\left[\tau_4 \wedge (-D)\right] , \\
C_{0,5} &= \Omega^3 \frac{1}{l^2}\left[-\mathrm{d}t \wedge \mathcal{K}_4 + \frac{1}{2}l^2 \mathrm{d}t \wedge \tau_4 + t\tau_4 \wedge \mathcal{D} - \frac{1}{2}t^2 \tau_4 \wedge \mathrm{d}t\right] , \\
C_{4,5} &= \Omega^3 \frac{1}{l^2}\left[\mathrm{d}t \wedge (-\mathcal{K}_4) - \frac{1}{2}l^2 \mathrm{d}t \wedge \tau_4 + t\tau_4 \wedge \mathcal{D} - \frac{1}{2}t^2 \tau_4 \wedge \mathrm{d}t\right] , \\
C_{0,a} &= \Omega^3 \frac{1}{l^2}\left[\mathcal{L}_{a,4} \wedge D + \frac{1}{2}(D,D)\tau_a \wedge \tau_4 - \frac{1}{2}l^2 \tau_4 \wedge \tau_a\right] , \\
C_{a,4} &= \Omega^3 \frac{1}{l^2}\left[-\frac{1}{2}l^2 \tau_4 \wedge \tau_a + D \wedge \mathcal{L}_{a,4} + \frac{1}{2}D^2 \tau_4 \wedge \tau_a\right] , \\
C_{a,5} &= \Omega^3 \frac{1}{l}\left[\mathrm{d}y \wedge (t\mathrm{d}x^a - x^a \mathrm{d}t) + y\mathrm{d}x^a \wedge \mathrm{d}t)\right] , \\
C_{ab} &= \Omega^3 \frac{1}{l}\left[\mathrm{d}y \wedge (\mathcal{L}_{ba}) + y\mathrm{d}x^a \wedge \mathrm{d}x^b\right] ,
\end{aligned}
$$

$$
\begin{aligned}
H_{0,4,5} &= \Omega^3 \frac{\operatorname{sgn} y}{l}\left[\frac{1}{2}\epsilon^{ijk4} x_i \tau_j \wedge \tau_k\right] , \\
H_{0,d,4} &= \Omega^3 \frac{\operatorname{sgn} y}{l}\left[-\frac{1}{2}t\epsilon^{dij4}\tau_i \wedge \tau_j - \frac{1}{2}\epsilon^{dab4}\mathcal{L}_{a,b} \wedge \mathrm{d}t\right] , \\
H_{0,d,5} &= \frac{1}{2}\Omega^3 \frac{\operatorname{sgn} y}{l^2}\left[\epsilon^{dab4}\left(-\mathcal{L}_{a,b} \wedge D - \frac{1}{2}D^2 \tau_a \wedge \tau_b + \frac{1}{2}l^2 \tau_a \wedge \tau_b\right)\right] , \\
H_{d,4,5} &= \frac{1}{2}\Omega^3 \frac{\operatorname{sgn} y}{l^2}\left[\epsilon^{dab4}\left(\mathcal{L}_{a,b} \wedge D + \frac{1}{2}D^2 \tau_a \wedge \tau_b + \frac{1}{2}l^2 \tau_a \wedge \tau_b\right)\right] , \\
H_{0,d} &= \Omega^3 \frac{\operatorname{sgn} y}{l^2}\left[t\mathcal{D} \wedge \mathrm{d}x^d + \left(-\mathcal{K}_d + \frac{1}{2}\left(-l^2 + t^2\right)\mathrm{d}x^d\right) \wedge \mathrm{d}t\right] , \\
H_{d,4} &= \Omega^3 \frac{\operatorname{sgn} y}{l^2}\left[t\mathcal{D} \wedge \mathrm{d}x^d + \left(-\mathcal{K}_d + \frac{1}{2}\left(l^2 + t^2\right)\mathrm{d}x^d\right) \wedge \mathrm{d}t\right] , \\
H_{d,5} &= \Omega^3 \frac{\operatorname{sgn} y}{l}\left[\mathrm{d}x_d \wedge D\right] , \\
H_{1,2,3} &= \Omega^3 \frac{\operatorname{sgn} y}{l}\left[\mathrm{d}t \wedge \mathcal{D}\right] .
\end{aligned}
$$

All those CYK tensors are written in the form of $\alpha[\beta]$, where α consists of conformal coefficient multiplied by locally constant terms like $\operatorname{sgn} y$ and l. Theorem 14 ensures us that β is a CYK tensor for the metric $\Omega^{-2}(\frac{l^2}{z^2}(-\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}\bar{x}^2)) = -\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}\bar{x}^2$ which is equal to the five-dimensional Minkowski metric.

5.2. Spherical coordinate system

Coordinates y, x^1, x^2, x^3, t are not convenient because we are interested in the form of the CYK tensors on the conformal verge — scri. That means that we want to set y equal to 0. The scri of AdS5 has the topology of $\mathbb{R}\times S^3$, however, in those coordinates the sphere is parameterized inconveniently. For this reason, we consider the following parametrization:

$$\begin{aligned} X_0 &= \sqrt{l^2+r^2}\cos\left(\frac{t}{l}\right) , \\ X_k &= rn^k , \\ X_5 &= \sqrt{l^2+r^2}\sin\left(\frac{t}{l}\right) , \end{aligned}$$

where $\sum_{i=1}^4 \left(n^i\right)^2 = 1$. That means that n^k can be parameterized with 3 angles. We also introduce the coordinate z which replaces the coordinate r

$$r = l\frac{1-z^2}{2z} , \qquad z \in [0,1] . \tag{5.7}$$

The choice of this coordinate is justified by the observation that it solves the equation

$$\left\|\frac{l\mathrm{d}z}{z}\right\|^2 = 1 . \tag{5.8}$$

This means that in the conformally equivalent metric $\frac{z^2}{l^2}g$, the coordinate z is easily related to the distance from the center of AdS5. In those coordinates, the Anti-de Sitter metric equals

$$g = \frac{l^2}{z^2}\left(\mathrm{d}z^2 - \left(\frac{1+z^2}{2}\right)^2\frac{\mathrm{d}t^2}{l^2} + \left(\frac{1-z^2}{2}\right)^2\mathrm{d}\Omega_3\right) . \tag{5.9}$$

We can divide it by the conformal factor $\frac{l^2}{z^2} = \Omega^2$, and then go to the conformal scri $z=0$. Scri is a manifold that has metric defined up to the conformal rescaling (this ambiguity arises because we could divide the Anti-de Sitter metric by an arbitrary conformal factor). In those coordinates, scri $\mathbb{R}\times S^3$ is conveniently parameterized because t parameterizes $\mathbb{R}$, whereas $(n_k)_{k\in\{1,2,3,4\}}$ parameterize S^3.

During calculations involving the Hodge dual, we used the following coordinates:

$$\begin{aligned} X_0 &= l\sqrt{1+\frac{\|p\|^2}{l^2}}\cos\left(\frac{t}{l}\right) , \\ X_k &= p^k \qquad k\in\{1,2,3,4\} , \end{aligned}$$

$$X_5 \;=\; l\sqrt{1+\frac{\|p\|^2}{l^2}}\sin\left(\frac{t}{l}\right) ,$$

where we denoted $\|p\|^2 := \sum_{i=1}^4 \left(p^i\right)^2$, and then we expressed the resulting CYK tensors through functions t, z, n^1, n^2, n^3, n^4 and their exterior derivatives.

This way we obtained CYK tensors on the Anti-de Sitter space-time. We adhered to our convention that Latin indices go from 1 to 4. We carried out calculations in Mathematica.

$$\begin{aligned}
C_{0,k} \;&=\; \mathrm{d}t\wedge\left(\frac{l\left(z^2+1\right)^2\sin\left(\frac{t}{l}\right)n^k}{4z^3}\mathrm{d}z+\frac{l\left(z^4-1\right)\sin\left(\frac{t}{l}\right)}{4z^2}\mathrm{d}n^k\right)\\
&\quad-\frac{l^2\left(z^2-1\right)^2\cos\left(\frac{t}{l}\right)}{4z^3}\mathrm{d}z\wedge\mathrm{d}n^k ,\\
C_{0,5} \;&=\; -\frac{l\left(z^4-1\right)}{4z^3}\mathrm{d}t\wedge\mathrm{d}z ,\\
C_{i,j} \;&=\; \frac{l^2\left(z^4-1\right)}{4z^3}\mathrm{d}z\wedge\left(n^i\mathrm{d}n^j-n^j\mathrm{d}n^i\right)+\frac{l^2\left(z^2-1\right)^2}{4z^2}\mathrm{d}n^i\wedge\mathrm{d}n^j ,\\
C_{k,5} \;&=\; \mathrm{d}t\wedge\left(\frac{l\left(z^2+1\right)^2\cos\left(\frac{t}{l}\right)n_k}{4z^3}\mathrm{d}z+\frac{l\left(z^4-1\right)\cos\left(\frac{t}{l}\right)}{4z^2}\mathrm{d}n^k\right)\\
&\quad+\frac{l^2\left(z^2-1\right)^2\sin\left(\frac{t}{l}\right)}{4z^3}\mathrm{d}z\wedge\mathrm{d}n^k ,
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
H_{0,i,j} \;&=\; \frac{1}{2}\epsilon^{ijkl}\left[\left(\frac{l\left(z^2-1\right)^2\left(z^2+1\right)\cos\left(\frac{t}{l}\right)}{8z^3}\mathrm{d}t-\frac{l^2\left(z^2-1\right)\sin\left(\frac{t}{l}\right)}{2z^2}\mathrm{d}z\right)\right.\\
&\quad\left.\wedge\left(n^l\mathrm{d}n^k-n^k\mathrm{d}n^l\right)-\frac{l^2\left(z^2-1\right)^2\sin\left(\frac{t}{l}\right)\left(z^2+1\right)}{8z^3}\mathrm{d}n^l\wedge\mathrm{d}n^k\right] ,\\
H_{5,i,j} \;&=\; \frac{1}{2}\epsilon^{ijkl}\left[\left(\frac{l\left(z^2-1\right)^2\left(z^2+1\right)\sin\left(\frac{t}{l}\right)}{8z^3}\mathrm{d}t+\frac{l^2\left(z^2-1\right)\cos\left(\frac{t}{l}\right)}{2z^2}\mathrm{d}z\right)\right.\\
&\quad\left.\wedge\left(n^l\mathrm{d}n^k-n^k\mathrm{d}n^l\right)+\frac{l^2\left(z^2-1\right)^2\left(z^2+1\right)\cos\left(\frac{t}{l}\right)}{8z^3}\mathrm{d}n^l\wedge\mathrm{d}n^k\right] ,\\
H_{ijk} \;&=\; \epsilon^{ijkl}\mathrm{d}t\wedge\left(\frac{l\left(z^2+1\right)n^l}{2z^2}\mathrm{d}z+\frac{l\left(z^6+z^4-z^2-1\right)}{8z^3}\mathrm{d}n^l\right) ,\\
H_{0,m,5} \;&=\; -\frac{l^2\left(z^2-1\right)^3}{16z^3}\left(\epsilon^{mijk}n^k\mathrm{d}n^i\wedge\mathrm{d}n^j\right) .
\end{aligned}$$

6. Analysis of the five-dimensional black hole with negative cosmological constant

6.1. Energy as the mass charge

Let us consider the solution of Einstein equations with the negative cosmological constant of the spherically symmetric black hole. The metric is equal to

$$\mathrm{d}s^2 = -\left(\frac{r^2}{l^2} + 1 - \frac{2m}{r^2}\right)\mathrm{d}t^2 + \left(\frac{r^2}{l^2} + 1 - \frac{2m}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3\,, \tag{6.1}$$

see *e.g.* equation (2.1) in [7]. Here, $\mathrm{d}\Omega_3$ denotes the metric of the unit three-dimensional sphere. It turns out that if we use the CYK tensor C_{05} from equation (5.11) and the Weyl tensor of metric (6.1), we will end up with the three-form T from Theorem 16 equal to

$$T = \frac{12m}{l}\omega\,, \tag{6.2}$$

where ω is the volume three-form of the three-dimensional unit sphere. This result was calculated in Mathematica. Therefore, the quasi-local charge equals $\frac{24m\pi^2}{l}$. It means that the mass of the Anti-de Sitter is related with (asymptotic) the CYK tensor $\frac{l}{24\pi^2}C_{05} = -\frac{l^2(z^4-1)}{96\pi^2 z^3}\mathrm{d}t\wedge\mathrm{d}z$. In this case, the three-form (4.9) does not depend on z and is closed, so the energy charge in this case is not only asymptotic — it is exact. In the asymptotically flat case, we have the so-called ADM mass defined as

$$m_{\mathrm{ADM}} := \frac{1}{6\pi^2}\int\limits_{S^3}\left(g_{ij,i} - g_{ii,j}\right)\mathrm{d}S^j\,.$$

The coefficient $\frac{1}{6\pi^2}$ arises from the volume of three-dimensional sphere and from the coefficient in the Einstein equation in this dimension (see Appendix D in [6]). More precisely,

$$2\gamma = \frac{2(n-1)\omega_{n-1}}{n-2} = \left\{\begin{array}{ll} 16\pi & \text{for } n=3 \\ 6\pi^2 & \text{for } n=4 \end{array}\right.\,.$$

We think that in our case, which is not asymptotically flat, we should also multiply the result of integral on sphere by such factor. This means that it is sufficient to take $\frac{l}{4}C_{05}$ in the definition of the CYK tensor responsible for energy. In that case, the asymptotic three-form will be equal to the ADM form.

6.2. Canonical coordinates on five-dimensional black hole with negative cosmological constant

Let us try to find the solution of equation (5.8) for metric (6.1). We have

$$1=\left\|\frac{l}{z}\mathrm{d}z\right\|^2=\left(\frac{l}{z}\right)^2\left(\frac{\partial z}{\partial r}\right)^2\|\mathrm{d}r\|^2=\frac{l^2}{z^2}\left(\frac{\partial z}{\partial r}\right)^2\left(\frac{r^2}{l^2}+1-\frac{2m}{r^2}\right). \quad (6.3)$$

Since we expect that $z\sim\frac{1}{r}$, we demand $z>0$ and $\frac{\partial z}{\partial r}<0$. For this reason, we have

$$\frac{\partial z}{\partial r} = -\frac{z}{l}\frac{1}{\sqrt{\frac{r^2}{l^2}+1-\frac{2m}{r^2}}}, \quad (6.4)$$

$$\log z+C = -\int\frac{\mathrm{d}r}{l\sqrt{\frac{r^2}{l^2}+1-\frac{2m}{r^2}}}. \quad (6.5)$$

We substitute $w=\frac{l}{r}$ to obtain

$$-\int\frac{\mathrm{d}r}{l\sqrt{\frac{r^2}{l^2}+1-\frac{2m}{r^2}}}=\int\frac{\mathrm{d}w}{w^2\sqrt{w^{-2}+1-\frac{2mw^2}{l^2}}}=\int\frac{\mathrm{d}w}{w\sqrt{1+w^2-w^4\frac{2m}{l^2}}}. \quad (6.6)$$

Denoting $b:=\frac{2m}{l^2}$, we get

$$z=\exp\left(\int\frac{\mathrm{d}w}{w\sqrt{1+w^2-bw^4}}\right). \quad (6.7)$$

It is easy to notice that if z satisfies equation (5.8), then αz with α being arbitrary constant also satisfies that equation. For this reason, the constant arising from the integral in equation (6.7) is irrelevant. For small w, we have $\int\frac{\mathrm{d}w}{w\sqrt{1+w^2-bw^4}}\simeq\log w$ so $z\simeq w$. We can also calculate the asymptotic

$$\begin{aligned}\int\frac{\mathrm{d}w}{w\sqrt{1+w^2-bw^4}} &\simeq \int\frac{\mathrm{d}w}{w}\left(1-\frac{1}{2}\left(w^2-bw^4\right)+\frac{3}{8}\left(w^2-bw^4\right)^2\right)\\ &\simeq \int\frac{\mathrm{d}w}{w}\left(1-\frac{1}{2}w^2+\left(\frac{1}{2}b+\frac{3}{8}\right)w^4\right)\\ &\simeq \log w-\frac{1}{4}w^2+\left(\frac{1}{8}b+\frac{3}{32}\right)w^4, \end{aligned} \quad (6.8)$$

$$\begin{aligned} z &= w\exp\left(-\frac{1}{4}w^2+\left(\frac{1}{8}b+\frac{3}{32}\right)w^4\right) \\ &= w\left(1-\frac{1}{4}w^2+\left(\frac{1}{8}b+\frac{3}{32}+\frac{1}{32}\right)w^4\right) \\ &= w-\frac{1}{4}w^3+\frac{1}{8}\left(b+1\right)w^5+\cdots . \end{aligned} \tag{6.9}$$

It is easy to check that if $z=w+\alpha w^3+\beta w^5+\cdots$, then $w=z-\alpha z^3+\left(3\alpha^2-\beta\right)z^5+\cdots$. Therefore, we have

$$w=z+\frac{1}{4}z^3+\frac{1-2b}{16}z^5 . \tag{6.10}$$

We can now express the Schwarzschild metric in the coordinates w, t and angles. We get the following metric:

$$\mathrm{d}s^2=\left(-1-w^{-2}+bw^2\right)\mathrm{d}t^2+\left(\frac{l^2}{-bw^6+w^4+w^2}\right)\mathrm{d}w^2+\left(\frac{l}{w}\right)^2\mathrm{d}\Omega . \tag{6.11}$$

From the construction, we know that

$$\left(\frac{l^2}{-bw^6+w^4+w^2}\right)\mathrm{d}w^2=\left(\frac{l}{z}\right)^2\mathrm{d}z^2 .$$

For this reason, we can now write everything in terms of z. We will obtain the approximation of the real metric. Let us calculate the coefficient that multiplies $\mathrm{d}t^2$. Substituting $w=z+\omega z^3+\tau z^5$, we obtain

$$\begin{aligned} -1-w^{-2}+bw^2 &= -1-\left(z+\omega z^3+\tau z^5\right)^{-2}+b\left(z+\omega z^3+\tau z^5\right)^2 \\ &= \frac{1}{z^2}\left[-z^2-\left(1+\omega z^2+\tau z^4\right)^{-2}+bz^2\left(z+\omega z^3+\tau z^5\right)^2\right] \\ &\simeq \frac{1}{z^2}\left[-z^2-1-1\left(-2\right)\left(\omega z^2+\tau z^4\right)-2\,\frac{3}{2}\left(\omega z^2\right)^2+bz^4\right] \\ &\simeq \frac{1}{z^2}\left[-1+z^2\left(-1+2\omega\right)+z^4\left(2\tau-3\omega^2+b\right)\right] \\ &= \frac{1}{z^2}\left[-1-\frac{1}{2}z^2+\left(2\,\frac{1-2b}{16}-\frac{3}{16}+b\right)\right] \\ &= \frac{1}{z^2}\left[-1-\frac{1}{2}z^2+\left(-\frac{1}{16}+\frac{3}{4}b\right)z^4\right] . \end{aligned} \tag{6.12}$$

Comparing this coefficient with analogous coefficient in equation (5.9) we see that we should get $-\left(\frac{1+z^2}{2}\right)^2$. However, looking at equation (5.9) we see that in equation (6.12) coordinate z is 2 times bigger, because it is behaving like $z \sim \frac{1}{r}$. For this reason, we are introducing $z = 2\tilde{z}$. Now, we have

$$-1 - w^{-2} + bw^2 = \frac{1}{\tilde{z}^2}\left[-\frac{1}{4} - \frac{1}{2}\tilde{z}^2 + \left(-\frac{1}{4} + 3b\right)\tilde{z}^4\right]. \tag{6.13}$$

This result is in accordance with equation (5.9) when $b = 0$ (that is when $m = 0$).

Now we only need to calculate the coefficient that multiplies $\mathrm{d}\Omega$. We have

$$\begin{aligned}\left(\frac{l}{w}\right)^2 &= \left(\frac{l}{z}\right)^2\left(1 + \omega z^2 + \tau z^4\right)^{-2} \\ &\simeq \left(\frac{l}{z}\right)^2\left(1 - 2\left(\omega z^2 + \tau z^4\right) + 3\left(\omega z^2\right)^2\right) \\ &= \left(\frac{l}{z}\right)^2\left(1 - 2\omega z^2 + z^4\left(-2\tau + 3\omega^2\right)\right) \\ &= \left(\frac{l}{z}\right)^2\left(1 - \frac{1}{2}z^2 + z^4\left(-2\frac{1-2b}{16} + \frac{3}{16}\right)\right) \\ &= \left(\frac{l}{z}\right)^2\left(1 - \frac{1}{2}z^2 + \left(\frac{1}{16} + \frac{1}{4}b\right)z^4\right). \end{aligned} \tag{6.14}$$

Again, let us express it with function $\tilde{z}$, so we get

$$\left(\frac{l}{w}\right)^2 = \left(\frac{l}{\tilde{z}}\right)^2\left[\frac{1}{4} - \frac{1}{2}\tilde{z}^2 + \left(\frac{1}{4} + b\right)\tilde{z}^4\right]. \tag{6.15}$$

This result is in accordance with equation (5.9). We finally obtain

$$\begin{aligned}\mathrm{d}s^2 \simeq \left(\frac{l}{\tilde{z}}\right)^2 &\left[\left(3b\tilde{z}^4 - \left(\frac{1+\tilde{z}^2}{2}\right)^2\right)\left(\frac{\mathrm{d}t}{l}\right)^2\right. \\ &\left. + \mathrm{d}\tilde{z}^2 + \left(\left(\frac{1-\tilde{z}^2}{2}\right)^2 + b\tilde{z}^4\right)\mathrm{d}\Omega\right]. \end{aligned} \tag{6.16}$$

This metric is in accordance with equation (5.9). It turns out that our metric differs from the Anti-de Sitter metric by terms that have rank 4 in $\tilde{z}$. We now use the fact that $b = \frac{2m}{l^2}$ to express the metric in terms of m. We get

$$\mathrm{d}s^2 \simeq \left(\frac{l}{\tilde{z}}\right)^2 \left[\left(\frac{6m}{l^2}\tilde{z}^4 - \left(\frac{1+\tilde{z}^2}{2}\right)^2\right)\left(\frac{\mathrm{d}t}{l}\right)^2 \right.$$
$$\left. +\mathrm{d}\tilde{z}^2 + \left(\left(\frac{1-\tilde{z}^2}{2}\right)^2 + \frac{2m}{l^2}\tilde{z}^4\right)\mathrm{d}\Omega\right] . \tag{6.17}$$

The above asymptotic form is in accordance with the general form of the asymptotically Anti-de Sitter metrics given by (1.2) in [4] which simply means that Schwarzschild–AdS is an asymptotically Anti-de Sitter spacetime.

7. Conclusion

We propose a new construction of CYK tensors in AdS5 using the observation that *constant tensors* in the ambient space restricted to the pseudosphere AdS5 generate all solutions of CYK equation. We would like to stress that theorems in Section 3 are nice tools and we show in Appendix A how to use them to construct in explicit form standard 4D conformal covector fields. One can argue that AdS5 is conformally equivalent to 5D Minkowski, hence, using conformal transformation, we can translate the solution in flat space to the solution in constant curvature space. However, the construction of solutions in flat space and corresponding conformal rescaling is not so simple. We think that our construction is simple and natural, one can say that the CYK tensors in AdS are simpler than in flat Minkowski because they are naturally obtained from constant tensors. Obviously, CYK tensors in Minkowski can be reconstructed from AdS via conformal transformation or by limiting procedure (in the tangent space).

It turns out that in the case of five-dimensional Anti-de Sitter spacetime, one can carry out constructions very similar to those in the four-dimensional case. Specifically in the five-dimensional case, one can find all conformal Yano–Killing tensors in a way that is analogous to the reasoning in [1] which solves the same problem in the four-dimensional case.

CYK tensors obtained in AdS5 enable us to define quasi-local charges that have good asymptotic properties. We have chosen the CYK tensor which defines the energy for the example of five-dimensional Schwarzschild blackhole. Probably in the case of the five-dimensional Kerr black hole with negative cosmological constant, it is possible to find CYK tensor which is responsible for the angular momentum, but it seems to be quite heavy calculation, so it will not be analyzed in this publication.

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Appendix A

Conformal Killing one-forms in four dimensions

We will now find all conformal Killing one-forms on four-dimensional Anti-de Sitter space-time. According to equation (1.8) in [9], the Anti-de Sitter metric has the following form:

$$\tilde{g} = \frac{l^2}{\cos^2 x}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \sin^2(x)\,\sigma\right), \tag{A.1}$$

where σ is a metric on a two-dimensional sphere, $l \in \mathbb{R}$ is a size of our Anti-de Sitter spacetime, $t \in \mathbb{R}$, $x \in [0, \pi]$. t has index 0, coordinates on the sphere have indices 1 and 2, and x has index 3. According to Theorem 13, we can find conformal Killing one-forms on the conformally equivalent metric

$$g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \sin^2(x)\,\sigma\,, \tag{A.2}$$

and later multiply the found one-forms by the conformal factor $\frac{l^2}{\cos^2 x}$. This is our strategy.

We are now abandoning previous index conventions. Let us denote that indices A, B go from 1 to 2. They are used to parameterize the sphere. We will also use ; to denote covariant derivatives on the spheres of constant x and t (using metric connection of this sphere). Greek indices go from 0 to 3. Additionally, η is a metric induced on the slice of constant time. We have therefore $\eta_{AB} = \sin^2 x\,\sigma_{AB}$, $\eta_{3A} = 0$ and $\eta_{33} = 1$. Latin indices denote spacial coordinates. In the following (unless stated otherwise), we will use $|$ to denote covariant derivative on the spacial slice (slice with constant t). We see that according to Theorem 4 if we pull back a conformal Killing one-form to the slice of constant time, we will obtain a conformal Killing one-form on the slice. This is so because the slice has external curvature equal to 0. Analogously, if we pull back a conformal Killing one-form to the slice of constant time and constant x, we will obtain a conformal Killing one-form on this slice.

We need to calculate the Christoffel symbols $\Gamma^{\alpha}{}_{\beta\gamma}$ for the metric g. Coefficients $\Gamma^{A}{}_{BC}$ are the same as Christoffel symbols on the unit sphere. $\Gamma^{A}{}_{3B} = \frac{1}{2}\eta^{AC}\eta_{CA|3} = \cot x\,\delta^{A}_{B}$, $\Gamma^{3}{}_{AB} = -\frac{1}{2}g^{33}\eta_{AB|3} = -\cot x\,\eta_{AB}$. Those are all non-vanishing Christoffel symbols $\Gamma^{\alpha}{}_{\beta\gamma}$.

A.1 Two dimensional problem on the sphere

ξ is our conformal Killing one-form on the whole space-time (with metric g conformally equivalent to the metric of Anti-de Sitter). Let us now consider its pullback to the slice of constant t and x (to the sphere). As we previously stated, this restriction is a conformal Killing one-form on the sphere, so it satisfies the equation

$$\xi_{A;B} + \xi_{B;A} - \eta_{AB}\xi^{C}{}_{;C} = 0\,, \tag{A.3}$$

where ; denotes covariant derivative on the sphere (for a given point we pull back one-form to the sphere and then we covariantly differentiate it using metric connection on the sphere). Equality (A.3) means that ξ_A is a dipole one-form. This can be shown in the following way. For every (pseudo)-Riemannian manifold K with Riemann tensor R and one-form ω, we have

$$\omega_{c|ba} - \omega_{c|ab} = R_{abc}{}^{d}\omega_d\,. \tag{A.4}$$

We are using here | to denote Levi-Civita derivative on the manifold K.

This can be generalized to the following identity:

$$2T^{\alpha_1\alpha_2\cdots\alpha_n}{}_{\beta_1\cdots\beta_m|[ba]} = \sum_{k=1}^{m} R_{ab\beta_k}{}^{d}T^{\alpha_1\cdots\alpha_n}{}_{\beta_1\cdots d\cdots\beta_m} - \sum_{k=1}^{n} R_{abd}{}^{\alpha_k}T^{\alpha_1\cdots d\cdots\alpha_n}{}_{\beta_1\cdots\beta_m}\,. \tag{A.5}$$

In the future calculations, we will use the character | to denote the covariant derivative on the spacial part of Anti-de Sitter space-time with respect to the metric induced from the metric g which is conformally equivalent to the metric of Anti-de Sitter.

We are choosing a sphere of constant x and t. This sphere has a natural structure of the pseudo-Riemannian manifold which is the same as the structure of sphere, which has radius $r = \sin x$ (that means that there exists isomorphism from the category of pseudo-Riemannian manifolds between a sphere of constant x and t and a sphere of the radius $r = \sin x$). We know that Ricci tensor on the sphere of the radius r equals to $R_{AB} = \frac{1}{r^2}\eta_{AB}$. For this reason, the Riemann tensor equals $R_{ABCD} = \frac{1}{r^2}\left(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}\right)$. In the following calculations for the sphere, we will lower and rise the indices using metric η induced from the metric g from equation (A.2). ϵ_{AB} will denote the volume form of our sphere with respect to the metric η. According to the Hodge–Kodaira theorem, each one-form can be expressed as the sum of external derivative, coderivative and harmonic form. It is also well-known that there are no harmonic one-forms on the sphere. For this reason, we have

$$\xi_A = \overset{1}{v}_{;A} + \epsilon_A{}^{B}\overset{2}{v}_{;B}\,, \tag{A.6}$$

where $\overset{1}{v}$ are $\overset{2}{v}$ are some functions.

We will show that ξ_A is a dipole function. We can find functions $\overset{1}{v}$ and $\overset{2}{v}$ in the following way:

$$\xi_A{}^{;A} = \overset{1}{v}_{;A}{}^{A}\,, \tag{A.7}$$

(we are using convention that all indices to the right of the ; are differentiating the tensor field) and

$$\xi_{A;C}\epsilon^{AC} = \epsilon_A{}^B \overset{2}{v}_{;BC}\epsilon^{AC} = \overset{2}{v}_{;A}{}^{A}\,. \tag{A.8}$$

Therefore, if we will show that $\xi_{A;C}\epsilon^{AC}$ and $\xi_A{}^{;A}$ are dipole functions, then also ξ_A will be a dipole one-form.

Let us differentiate equation (A.3) covariantly with index A at the top. We will end up with

$$\begin{aligned} 0 &= \xi_{A;B}{}^{A} + \xi_{B;A}{}^{A} - \xi^C{}_{;CB} \\ &= \xi_A{}^{;A}{}_B + R^A{}_{BA}{}^{D}\xi_D + \xi_{B;A}{}^{A} - \xi^C{}_{;CB} \\ &= \frac{1}{r^2}\xi_B + \xi_{B;A}{}^{A} = 0\,. \end{aligned} \tag{A.9}$$

Equality (A.9) can be written in the form of

$$\left(\overset{0}{\Delta} + 1\right)\xi = 0\,, \tag{A.10}$$

where $\overset{0}{\Delta}$ denotes Laplacian created from the structure of the metric of the unit sphere, and 1 denotes identity operator. This means that we are pulling back ξ to the sphere, and then we are using the metric of the unit sphere to calculate Laplacian of resulting ξ_A. Equality (A.10) means that ξ_A is a dipole one-form. Now let us covariantly differentiate equation (A.9) and then contract resulting index with B

$$\begin{aligned} 0 &= \frac{1}{r^2}\xi_B{}^{;B} + \xi_{B;A}{}^{BA} + R^{BA}{}_B{}^D\xi_{D;A} + R^{BA}{}_A{}^D\xi_{B;D} \\ &= \frac{1}{r^2}\xi_B{}^{;B} + \xi_{B;A}{}^{BA} \\ &= \frac{1}{r^2}\xi_B{}^{;B} + \xi_B{}^{;B}{}_A{}^A + \left(R^B{}_{AB}{}^D\xi_D\right)^{;A} = \xi_B{}^{;B}{}_A{}^A + \frac{2}{r^2}\xi_A{}^{;A}\,. \end{aligned} \tag{A.11}$$

This proves that $\xi_A{}^{;A}$ is a dipole function.

To prove that $\xi_{B;C}\epsilon^{BC}$ is a dipole function, we start with the following identity:

$$R_{CAB}{}^D\epsilon^{BC} = -\frac{1}{r^2}\delta_C{}^D g_{AB}\epsilon^{BC} = -\frac{1}{r^2}\epsilon_A{}^D\,. \tag{A.12}$$

We will now differentiate covariantly equality (A.9) and then contract the result with ϵ^{BC}. We will end up with

$$
\begin{aligned}
0 &= \frac{1}{r^2}\xi_{B;C}\epsilon^{BC} + \xi_{B;A}{}^{A}{}_{C}\epsilon^{BC} \\
&= \frac{1}{r^2}\xi_{B;C}\epsilon^{BC} + \xi_{B;AC}{}^{A}\epsilon^{BC} + R_{C}{}^{A}{}_{B}{}^{D}\xi_{D;A}\epsilon^{BC} + R_{C}{}^{A}{}_{A}{}^{D}\xi_{B;D}\epsilon^{BC} \\
&= \frac{1}{r^2}\xi_{B;C}\epsilon^{BC} + \xi_{B;AC}{}^{A}\epsilon^{BC} - \frac{1}{r^2}\epsilon^{AD}\xi_{D;A} - \frac{1}{r^2}\xi_{B;D}\epsilon^{BD} \\
&= \frac{1}{r^2}\xi_{B;C}\epsilon^{BC} + \xi_{B;CA}{}^{A}\epsilon^{BC} + R_{CAB}{}^{D}\xi_{D}{}^{;A}\epsilon^{BC} \\
&= \frac{1}{r^2}\xi_{B;C}\epsilon^{BC} + \left(\xi_{B;C}\epsilon^{BC}\right)_{;A}{}^{A} - \frac{1}{r^2}\epsilon_{A}{}^{D}\xi_{D}{}^{;A} \\
&= \frac{2}{r^2}\xi_{B;C}\epsilon^{BC} + \left(\xi_{B;C}\epsilon^{BC}\right)_{;A}{}^{A}.
\end{aligned}
\tag{A.13}
$$

The last equality means that $\xi_{B;C}\epsilon^{BC}$ is a dipole function.

Equations (A.13), (A.11), (A.7) and (A.8) prove that $\Delta\Delta\overset{1}{v}$ and $\Delta\Delta\overset{2}{v}$ are dipole functions. Dipole functions belong to the eigenspace of Laplacian with non-zero eigenvalue. That is why Laplacian Δ acts on them as an isomorphism. For this reason, functions $\overset{1}{v}$ and $\overset{2}{v}$ are sums of dipole functions and elements of the kernel of Δ, which are monopole functions. According to equation (A.6), monopole parts of these functions do not matter because in equation (A.6) functions $\overset{1}{v}$ and $\overset{2}{v}$ are differentiated.

A.2 Expanding to the spacial slice

In this section, we will denote covariant derivative with respect to the slice of constant time with the character $|$. We have to remember here that we are using metric (A.2). The conformal Killing one-forms on the sphere enable one to find conformal Killing one-forms on the whole slice of the constant t. We are calculating the covariant derivatives

$$
\begin{aligned}
\xi_{k|l} &= \xi_{k,l} - \Gamma^{i}{}_{kl}\xi_i\,, \\
\xi_{3|3} &= \xi_{3,3}\,, \\
\xi_{3|A} &= \xi_{3,A} - \Gamma^{i}{}_{3A}\xi_i = \xi_{3,A} - \cot x\,\xi_A\,, \\
\xi_{A|3} &= \xi_{A,3} - \cot x\,\xi_A\,, \\
\xi_{A|B} &= \xi_{A;B} - \Gamma^{3}{}_{AB}\xi_3 = \xi_{A;B} + \cot x\,\eta_{AB}\xi_3\,.
\end{aligned}
$$

On the spacial slice, we have

$$\xi_{k|l} + \xi_{l|k} = \alpha\,\eta_{kl}\,. \tag{A.14}$$

Let us calculate spacial derivative

$$\xi^{k}{}_{|k} = \eta^{33}\xi_{3|3} + \eta^{AB}\xi_{A|B} = \xi_{3,3} + \xi^{A}{}_{;A} + 2\cot x\,\xi_3\,. \tag{A.15}$$

We have $2\xi^k{}_{|k} = 3\alpha$ so

$$\alpha = \tfrac{2}{3}\xi^k{}_{|k}\,. \tag{A.16}$$

It follows from the conformal Killing equation that

$$\begin{aligned} \xi_{3|A} + \xi_{A|3} &= 0\,, && \text{(A.17)}\\ 2\xi_{3|3} &= \alpha\,, && \text{(A.18)}\\ \xi_{A|B} + \xi_{B|A} &= \alpha\eta_{AB}\,, \end{aligned}$$

so contracting the last equation with η^{AB}, we get

$$\eta^{AB}\xi_{A|B} = \alpha\,. \tag{A.19}$$

We derive equation (A.20) from equation (A.17), whereas combined equations (A.18) and (A.19) lead to $\xi^A{}_{;A} + 2\cot x\,\xi_3 = \eta^{AB}\xi_{A|B} = \alpha = 2\xi_{3,3}$. This last equation is equivalent to equation (A.21).

$$\xi_{3,A} + \xi_{A,3} - 2\cot x\,\xi_A = 0\,, \tag{A.20}$$

$$2\xi_{3,3} = \xi^A{}_{;A} + 2\cot x\,\xi_3\,. \tag{A.21}$$

We apply covariant derivative $;A$ to equation (A.20) and we obtain

$$\xi_{3;A}{}^{A} + \xi_{A;B,3}\eta^{AB} - 2\cot x\,\xi_A{}^{;A} = 0\,. \tag{A.22}$$

Here, we used the fact that partial derivative ∂_3 and covariant derivative $;A$ commute. This is the consequence of the fact that Christoffel symbols do not depend on x. Equation (A.22) proves that $\Delta\xi_3$ is a dipole function, because the rest of this equality is a dipole function. For this reason, ξ_3 has only monopole and dipole parts.

We can also rewrite equation (A.21) in the form of

$$2\left(\frac{\xi_3}{\sin x}\right)_{,3}\sin x = \xi^A{}_{;A}\,. \tag{A.23}$$

We see, therefore, that monopole and dipole parts of ξ_3 evolve independently. Let us rewrite equality (A.23) in the form of

$$2\left(\frac{{}^m\xi_3}{\sin x}\right)_{,3}\sin x = 0\,, \tag{A.24}$$

$$2\left(\frac{{}^d\xi_3}{\sin x}\right)_{,3}\sin x = \xi^A{}_{;A}\,, \tag{A.25}$$

where ${}^{m}\xi_3$ is the monopole part of ξ_3, whereas ${}^{d}\xi_3$ is the dipole part of ξ_3. From the first of those equations, we get

$$ {}^{m}\xi_3 = a \sin x \,, \tag{A.26} $$

where a is some constant. It is denoted with small letter because it is a monopole function. Functions that are dipole will be denoted with capital letters.

We can also rewrite equation (A.20) in the form of

$$ {\xi_3}^{;A} + {\xi^A}_{,3} = 0 \,. \tag{A.27} $$

Equation (A.27) was obtained by noticing that equation (A.27) can be expressed as ${\xi^A}_{,3} = \left(\sin^{-2} x \, \sigma^{AB} \xi_B\right)_{,3} = \sin^{-2} x \, \sigma^{AB} \xi_{B,3} - 2\sin^{-3} x \cos x \sigma^{AB} \xi_B$ (here σ^{AB} is a metric inverse to σ_{AB}) so it looks like $\xi_{B,3} - 2\cot x \xi_B$ with raised index. It is worth to remember that here ξ_3 is treated as a function, so in the expression ${\xi_3}^{;A}$ covariant derivative acts as partial derivative. We remember that in the covariant derivative $;A$ Christoffel symbols ${\Gamma^A}_{BC}$ are independent of x. This means that covariant derivative $;A$ commutes with ∂_3. For this reason, we can calculate covariant derivative $;A$ of equation (A.27) and contract the indices. We end up with

$$ {{\xi_3}^{;A}}_{A} + \left({\xi^A}_{;A}\right)_{,3} = 0 \,. \tag{A.28} $$

This equation can be rewritten as follows:

$$ \frac{1}{\sin^2 x} \overset{0}{\Delta} \xi_3 + \left({\xi^A}_{;A}\right)_{,3} = 0 \,, \tag{A.29} $$

where $\overset{0}{\Delta}$ denotes Laplacian on the unit sphere. We, therefore, see that

$$ \left({\xi^A}_{;A}\right)_{,3} = 2 \frac{{}^{d}\xi_3}{\sin^2 x} \,. \tag{A.30} $$

By combining equations (A.30) and (A.25), we get

$$ \left(\left(\frac{{}^{d}\xi_3}{\sin x} \right)_{,3} \sin x \right)_{,3} = \frac{{}^{d}\xi_3}{\sin^2 x} \,. \tag{A.31} $$

This equation can be solved. We introduce $u = \frac{{}^{d}\xi_3}{\sin x}$ and $z = \int \frac{\mathrm{d}x}{\sin x}$. Now equation (A.31) has the following form: $\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = u$. It has a solution of

$u = Be^{z} + Ce^{-z}$, where B, C are independent of z. This can be written as $u = \log\left(\sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right)\right)$.

$$u = B\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} + C\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}, \tag{A.32}$$

$$^{d}\xi_3 = \sin(x)\left(B\tan\frac{x}{2} + C\cot\frac{x}{2}\right). \tag{A.33}$$

In the last equation, the left-hand side is a dipole function, so A, B are also dipole functions

$$\begin{aligned}\xi_3 &= {}^{m}\xi_3 +^{d}\xi_3 \\ &= a\sin x + \sin(x)\left(B\tan\frac{x}{2} + C\cot\frac{x}{2}\right) \\ &= a\sin x + 2\left(B\sin^2\frac{x}{2} + C\cos^2\frac{x}{2}\right) \\ &= a\sin x + 2\left(B\sin^2\frac{x}{2} + C\cos^2\frac{x}{2}\right) \\ &= a\sin x + B(1-\cos x) + C(1+\cos x) \\ &= a\sin x + K - J\cos x. \end{aligned}\tag{A.34}$$

Constants K and J are replacing the constants B and C in the following way: $K = B + C$ and $J = B - C$.

We now use equation (A.21) to get

$$\xi^{A}{}_{;A} = 2\xi_{3,3} - 2\cot x\,\xi_3,$$

$$2a\cos x + 2J\sin x - 2\cot x\,(a\sin x + K - J\cos x) = -2K\cot x + 2J\frac{1}{\sin x}.$$

Now, we only need to find the rotational part of ξ_A. In the previous subsection, we defined ϵ as the volume form of the sphere of constant x and t. Now, we want to think about the ϵ as a tensor on the whole Anti-de Sitter space-time with conformally equivalent metric from equation (A.2). We define ϵ on the whole Anti-de Sitter space-time by imposing relations $\epsilon_{\mu 3} = \epsilon_{\mu 0} = 0$, $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, and ϵ_{AB} is the metric volume form of the sphere of constant x and t with respect to the metric induced g from equation (A.2). It is easy to calculate that $\epsilon^{AB}_{,3} = -2\cot x\epsilon^{AB}$. We are differentiating covariantly equation (A.20) with respect to the $;B$ index, and then we are contracting resulting equation with ϵ^{AB}. By taking into account the commutation of the ∂_3 and $;A$, we get

$$\xi_{A;B,3}\epsilon^{AB} - 2\cot x\,\xi_{A;B}\epsilon^{AB} = 0. \tag{A.35}$$

We have, therefore,

$$\xi_{A;B,3}\epsilon^{AB} = \left(\xi_{A;B}\epsilon^{AB}\right)_{,3} - \xi_{A;B}\epsilon^{AB}_{,3} \tag{A.36}$$

and finally,

$$\xi_{A;B}\epsilon^{AB} = D\,, \tag{A.37}$$

where D is a dipole function.

In that way we obtained the following:

$$\begin{aligned} \xi_{A;B}\epsilon^{AB} &= D\,, &\text{(A.38)}\\ \xi_A{}^{;A} &= -2K\cot x + 2J\frac{1}{\sin x}\,, &\text{(A.39)}\\ \xi_3 &= a\sin x + K - J\cos x\,. &\text{(A.40)} \end{aligned}$$

The space of solutions is ten-dimensional, which is the maximal possible number in three dimensions.

We now use equation (A.15) to get

$$\begin{aligned} \xi_k{}^{|k} &= \xi_{3,3} + \xi^A{}_{;A} + 2\cot x\,\xi_3 \\ &= 3a\cos x + 3J\sin x\,. \end{aligned} \tag{A.41}$$

This means that α from the equation $\xi_{\alpha|\beta} + \xi_{\beta|\alpha} = \alpha g_{\alpha\beta}$ (Greek indices may be both spacial and temporal) is equal to

$$\alpha = \tfrac{2}{3}\xi_k{}^{|k} = 2a\cos x + 2J\sin x\,. \tag{A.42}$$

A.3 One-forms in space-time

We should now consider the dependence of a, J, K, D, which are functions that characterize ξ, on time

$$\begin{aligned} 2\xi_{0,0} &= -\alpha\,, &\text{(A.43)}\\ \xi_{0,3} + \xi_{3,0} &= 0\,, &\text{(A.44)}\\ \xi_{0,A} + \xi_{A,0} &= 0\,. &\text{(A.45)} \end{aligned}$$

Let us apply the covariant derivative $;A$ to equation (A.45). We obtain the following equation:

$$\xi_{0;A}{}^A = -\xi^A{}_{;A,0}\,. \tag{A.46}$$

This equation proves that $\overset{0}{\Delta}\xi_0$ is a dipole function, because $\xi^A{}_{;A}$ is a dipole function.

For this reason,

$$\xi_0 =^d \xi_0 +^m \xi_0\,. \tag{A.47}$$

We will now differentiate equation (A.46) with respect to time. We obtain

$$\xi_{0,0;A}{}^A = -\xi_{A;}{}^A{}_{,00}\,. \tag{A.48}$$

We can use the last equation together with equations (A.43) and (A.42) to obtain

$$\xi_{A;}{}^{A}{}_{,00} = \frac{1}{2}\alpha_{;A}{}^{A} = \frac{1}{2}\frac{1}{\sin^2 x}(-2)\,2J\sin x\,. \tag{A.49}$$

We now use equation (A.40) to obtain

$$-2K_{,00}\cot x + 2J_{,00}\frac{1}{\sin x} = -2J\frac{1}{\sin x}\,. \tag{A.50}$$

To obtain the second equation for the coefficients B, C, we have to differentiate equation (A.44) with respect to time

$$\xi_{3,00} = -\xi_{0,03} = \frac{1}{2}\alpha_{,3} = \frac{1}{2}\left(-2a\sin x + 2J\cos x\right)$$

so

$$a_{,00}\sin x + K_{,00} - J_{,00}\cos x = -a\sin x + J\cos x\,.$$

This equation splits into the dipole and monopole parts and we get

$$a_{,00} = -a\,, \tag{A.51}$$

$$K_{,00} - J_{,00}\cos x = J\cos x\,, \tag{A.52}$$

$$J_{,00} - K_{,00}\cos x = -J\,. \tag{A.53}$$

The last equation is equivalent to equation (A.50). Equation (A.51) has a solution

$$a = a_0\sin t + a_1\cos t\,. \tag{A.54}$$

Let us multiply equation (A.52) by $\cos x$ and add the result to equation (A.53). We get

$$J_{,00}\sin^2 x = -J\sin^2 x\,. \tag{A.55}$$

We have, therefore,

$$J_{,00} = -J \tag{A.56}$$

and

$$J = J_0\sin t + J_1\cos t\,. \tag{A.57}$$

Let us add equation (A.52) to equation (A.53) multiplied by $\cos x$. We obtain

$$K_{,00} = 0\,, \tag{A.58}$$

$$K = G + Ht\,. \tag{A.59}$$

We can also covariantly differentiate equation (A.45) with respect to the index $;B$ and contract the result with ϵ^{AB}. We obtain

$$\left(\xi_{A;B}\epsilon^{AB}\right)_{,0} = 0\,. \tag{A.60}$$

Equation (A.60) has a solution

$$\xi_{A;B}\epsilon^{AB} = D\,, \tag{A.61}$$

where D is a dipole function which is time-independent. Additionally,

$$\begin{aligned}\xi_3 &= a\sin x + K - J\cos x \\ &= (a_0\sin t + a_1\cos t)\sin x + G + Ht - (J_0\sin t + J_1\cos t)\cos x \end{aligned} \tag{A.62}$$

and

$$\xi_A{}^{;A} = -2K\cot x + 2J\frac{1}{\sin x}\,. \tag{A.63}$$

Now, we only have to calculate the coefficient ξ_0. Let us calculate

$$\alpha = 2a\cos x + 2J\sin x\,. \tag{A.64}$$

We can use equation (A.43) to get

$$\xi_{0,0} = -\tfrac{1}{2}\alpha = -\,(a_0\sin t + a_1\cos t)\cos x + (-J_0\sin t - J_1\cos t)\sin x\,. \tag{A.65}$$

We have, therefore,

$$\xi_0 = (a_0\cos t - a_1\sin t)\cos x + (J_0\cos t - J_1\sin t)\sin x + F\,, \tag{A.66}$$

where F is a certain function with both monopole and dipole parts which are time-independent. We do not know yet how they depend on x. If we now substitute our results to (A.44), we will find that $H = 0$. More precisely,

$$\xi_{3,0} = (a_0\cos t - a_1\sin t)\sin x + H - (J_0\cos t - J_1\sin t)\cos x\,, \tag{A.67}$$

$$\xi_{0,3} = -\,(a_0\cos t - a_1\sin t)\sin x + (J_0\cos t - J_1\sin t)\cos x + F_{,3}\,. \tag{A.68}$$

We have, therefore, from equation (A.44)

$$H + F_{,3} = 0\,. \tag{A.69}$$

Equation (A.46) proves that

$$\xi_{0;A}{}^{;A} = \frac{-2}{\sin^2 x}(J_0\cos t - J_1\sin t)\sin x + F_{;A}{}^{A}\,, \tag{A.70}$$

$$\xi_A{}^{;A}{}_{,0} = -2H\cot x + \frac{2}{\sin x}(J_0\cos t - J_1\sin t)\;, \tag{A.71}$$

so

$$0 = F_{;A}{}^{A} - 2H\cot x = \frac{-2}{\sin^2 x}{}^{d}F - 2H\cot x\,, \tag{A.72}$$

$${}^{d}F = -H\sin x\cos x\,. \tag{A.73}$$

This combined with equation (A.69) leads to $H=0$. F is independent of x, t and angles. Let us denote this constant quantity as $F=c$.

To sum up, we have the following solutions:

$$\xi_A{}^{;A} = -2G\cot x + \frac{2}{\sin x}\left(J_0 \sin t + J_1 \cos t\right) , \tag{A.74}$$

$$\xi_{A;B}\epsilon^{AB} = D , \tag{A.75}$$

$$\xi_3 = (a_0 \sin t + a_1 \cos t)\sin x + G - (J_0 \sin t + J_1 \cos t)\cos x , \tag{A.76}$$

$$\xi_0 = (a_0 \cos t - a_1 \sin t)\cos x + (J_0 \cos t - J_1 \sin t)\sin x + c . \tag{A.77}$$

Here, a_0, a_1, c are constants, whereas G, D, J_0, J_1 are dipole functions independent of x and t.

The space of solutions has dimension 15, which is exactly the number that was expected.

We can now write the basis of the space of all the conformal Killing one-forms. According to equation (A.6), we have

$$\xi_A = \overset{1}{v}_{;A} + \epsilon_A{}^B \overset{2}{v}_{;B} . \tag{A.78}$$

Functions $\overset{1}{v}$ and $\overset{2}{v}$ may be calculated using previously derived formulas

$$\xi_A{}^{;A} = \overset{1}{v}_{;A}{}^{A} = -\frac{2}{\sin^2 x}\overset{1}{v}$$

and

$$\xi_{A;C}\epsilon^{AC} = \epsilon_A{}^B \overset{2}{v}_{;BC}\epsilon^{AC} = \overset{2}{v}_{;A}{}^{A} = -\frac{2}{\sin^2 x}\overset{2}{v} .$$

We, therefore, have the following (linearly independent) conformal Killing one-forms for the metric g from equation (A.2)

$$\begin{aligned} R &= -\frac{1}{2}\sin^2 x\,\epsilon_A{}^B D_{,B}\,\mathrm{d}x^A , \\ P &= G\mathrm{d}x + \sin x \cos x\,\mathrm{d}G , \\ T &= c\mathrm{d}t , \end{aligned}$$

$$\begin{aligned} B_{J_0} &= -\sin x \sin t \mathrm{d}J_0 - J_0 \sin t \cos x \mathrm{d}x + J_0 \cos t \sin x \mathrm{d}t , \\ D &= a_1 \cos t \sin x \mathrm{d}x - a_1 \sin t \cos x \mathrm{d}t , \\ K_t &= a_0 \sin t \sin x \mathrm{d}x + a_0 (\cos t \cos x - 1)\,\mathrm{d}t , \\ K_{J_1} &= (\sin x \cos x - \sin x \cos t)\,\mathrm{d}J_1 - J_1 (\cos t \cos x - 1)\,\mathrm{d}x \\ &\quad - J_1 \sin t \sin x \mathrm{d}t . \end{aligned}$$

From those conformal Killing one-forms for the metric g from equation (A.2), we can easily obtain conformal Killing one-forms for the 4-dimensional Anti-de Sitter metric from equation (A.1) by multiplying them by the conformal factor $\frac{1}{\cos^2 x}$.

Close to $x = 0$, our metric g from equation (A.2) is similar to the Minkowski metric. We are, therefore, expecting that for $x \to 0$, our Killing forms will look similarly to the known conformal one-forms in the Minkowski space-time. This turns out to be true. We see that R corresponds to the generators of rotations in the Minkowski space-time, P corresponds to spacial translations, T corresponds to time translation, B_{J_0} to boosts, D to dilation, K_t to time acceleration, whereas K_{J_1} to space accelerations.

Appendix B

Additional proofs

In this appendix, we will present proofs for some of the theorems used in this paper.

Let us prove Theorem 1. It states that if ω is a one-form on the manifold N, then

$$\omega_{b|a} = \omega_{b;a} - K_{ab}\omega_\mu n^\mu \,.$$

Proof. We can assume that

$$\tilde{K}\left(X,Y\right) = K\left(X,Y\right)n = K_{ac}X^aY^cn \,. \tag{B.1}$$

We have then

$$\overset{N}{\nabla}_a v^b = \overset{M}{\nabla}_a v^b + K_{ac}v^cn^b \,. \tag{B.2}$$

Using | and ; (in the convention of Section 2), we obtain

$$v^b{}_{|a} = v^b{}_{;a} + K_{ac}v^cn^b \,, \tag{B.3}$$

$$v^{n+1}{}_{|a} = K_{ac}v^cn^{n+1} \,. \tag{B.4}$$

Here, v is tangent to M. We later have

$$\left(v^b\omega_b\right)_{|a} = \left(v^\mu\omega_\mu\right)_{|a} = v^\mu{}_{|a}\omega_\mu + v^\mu\omega_{\mu|a} = v^b{}_{;a}\omega_b + K_{ac}v^cn^\mu\omega_\mu + v^b\omega_{b|a} \,. \tag{B.5}$$

However, on the other hand,

$$\left(v^b\omega_b\right)_{|a} = \left(v^b\omega_b\right)_{;a} = v^b{}_{;a}\omega_b + v^b\omega_{b;a} \,, \tag{B.6}$$

hence, we get the result

$$\omega_{b|a} = \omega_{b;a} - K_{ab}\omega_\mu n^\mu \,. \tag{B.7}$$

□

Now we will prove Theorem 2. It states that the external curvature form K_{ab} satisfies equation $K = -\frac{1}{2}\mathcal{L}_n g$, where n is a normal normalized field.

Proof. Let X and Y be vector fields tangent to M. Now

$$\begin{aligned}
(\mathcal{L}_n g)(X,Y) &= \mathcal{L}_n(g(X,Y)) - g(\mathcal{L}_n X, Y) - g(X, \mathcal{L}_n Y) \\
&= \overset{N}{\nabla}_n (g(X,Y)) - g([n,X], Y) - g([n,Y], X) \\
&= g\left(\overset{N}{\nabla}_n X, Y\right) + g\left(\overset{N}{\nabla}_n Y, X\right) \\
&\quad - g\left(\overset{N}{\nabla}_n X - \overset{N}{\nabla}_X n, Y\right) - g\left(\overset{N}{\nabla}_n Y - \overset{N}{\nabla}_Y n, X\right) \\
&= -2K(X,Y)\,,
\end{aligned}$$

where in the last step, we used the following equation:

$$0 = \overset{N}{\nabla}_X (g(n,Y)) = g\left(\overset{N}{\nabla}_X n, Y\right) + g\left(n, \overset{N}{\nabla}_n Y\right) . \tag{B.8}$$

□

Now, we will present the proof of Theorem 12. It states that the Hodge dual of the CYK three-form is a CYK tensor.

Proof. Let us define $s = \mathrm{sgn}\,(\det g)$. In the following calculations, we will be using standard notations for symmetrization and skew-symmetrization *i.e.* $\alpha_{(ab)} := \frac{1}{2}(\alpha_{ab} + \alpha_{ba})$ and $\alpha_{[ab]} := \frac{1}{2}(\alpha_{ab} - \alpha_{ba})$. We define the Hodge dual as $*T_{ef} = \frac{1}{3!}\epsilon_{ef}{}^{abc}T_{abc}$. That is why we contracted (3.10) with tensor $\frac{1}{6}\epsilon_{ef}{}^{abc}$, where ϵ is a metric volume form of M. Additionally, we denote $\chi_f = *T_f{}^c{}_{;c}$. We end up with some identities

$$2T_{ab(c;d)}\tfrac{1}{6}\epsilon_{ef}{}^{abc} = *T_{ef;d} + \tfrac{1}{6}\epsilon_{ef}{}^{abc}T_{abd;c}\,.$$

Let us evaluate $\frac{1}{6}\epsilon_{ef}{}^{abc}T_{abd;c}$. To this end, we remind ourselves that for the k-form on n dimensional manifold we have $**\alpha = (-1)^{k(n-k)+s}\,\alpha$, where

$s := \operatorname{sgn}(\det g)$. We have

$$\begin{aligned}
2\epsilon_{ef}{}^{abc}T_{abd;c} &= 2s\epsilon_{ef}{}^{abc} * *T_{abd;c} = 2s\epsilon_{ef}{}^{abc}\tfrac{1}{2}\epsilon_{abd}{}^{kh} *T_{kh;c} \\
&= s\epsilon_{ef}{}^{abc}\epsilon_{abd}{}^{kh} *T_{kh;c} = s *T_{kh;c}\epsilon_{abef\tilde{c}}\epsilon^{ab\tilde{d}kh}g^{\tilde{c}c}g_{\tilde{d}d} \\
&= *T_{kh;c}g^{c\tilde{c}}g_{\tilde{d}d}2\delta^{\tilde{d}kh}{}_{ef\tilde{c}} \\
&= 2\Big(*T_{fh;c}g^{hc}g_{ed} + *T_{ke;c}g^{kc}g_{fd} + *T_{ef;d} - *T_{eh;c}g^{hc}g_{fd} \\
&\quad - * T_{fe;d} - *T_{kf;c}g^{kc}g_{ed}\Big) \\
&= 2\left(\chi_f g_{ed} - \chi_e g_{fd} + 2 *T_{ef;d} - \chi_e g_{fd} + \chi_f g_{ed}\right) \\
&= 8\chi_{[f}g_{e]d} + 4 *T_{ef;d}\,,
\end{aligned}$$

where

$$\delta^{a_1a_2a_3}{}_{b_1b_2b_3} = \sum_{\pi\in S(3)} \operatorname{sgn}(\pi) \prod_{i\in\{1,2,3\}} \delta^{a_{\pi(i)}}{}_{b_i}\,. \tag{B.9}$$

So

$$2T_{ab(c;d)}\tfrac{1}{6}\epsilon_{ef}{}^{abc} = \tfrac{2}{3}\chi_{[f}g_{e]d} + \tfrac{4}{3} *T_{ef;d}\,. \tag{B.10}$$

However, from equality (3.10), it follows that

$$\begin{aligned}
2T_{ab(c;d)}\tfrac{1}{6}\epsilon_{ef}{}^{abc} &= \tfrac{1}{6}\epsilon_{ef}{}^{abc}\left(-2Q_{[ab}g_{c]d} + Q_{[ac}g_{b]d} - Q_{[bc}g_{a]d}\right) \\
&= \tfrac{1}{6}\epsilon_{ef}{}^{abc}(-4)\,Q_{ab}g_{cd} = -4 *Q_{efd}\,,
\end{aligned} \tag{B.11}$$

so

$$*T_{ef;d} = -3 *Q_{efd} + \tfrac{1}{2}\chi_{[e}g_{f]d}\,. \tag{B.12}$$

We check that

$$\begin{aligned}
2 *T_{e(f;d)} &= \tfrac{1}{4}\left(\chi_e g_{fd} - \chi_f g_{ed} + \chi_e g_{df} - \chi_d g_{ef}\right) \\
&= \tfrac{1}{4}\left(2\chi_e g_{fd} - \chi_f g_{ed} - \chi_d g_{ef}\right)\,,
\end{aligned} \tag{B.13}$$

which is an equation satisfied by the CYK tensor. □

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