NNLO QCD CONTRIBUTIONS TO ε'/ε*

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Recent progress in Lattice QCD and subsequent Next-to-Leading Order analysis of the ratio ε'/ε resulted in a 2.9σ discrepancy between the Standard Model predictions and experimental data. This inconsistency could have several sources, one of which could be the missing contribution of new particles in the theory predictions. However, a reliable Standard Model prediction is essential to disentangle possible new physics effects from the Standard Model background. Indeed, possible higher order corrections could significantly alter the theory prediction. This is particularly true for ε'/ε where the Next-to-Leading Order corrections have been found to be large. To close this gap, we aim to calculate the relevant matching corrections at Next-to-Next-to-Leading Order for this physical quantity and present a more accurate theoretical prediction within the Standard Model.

1. Introduction

The observed matter–antimatter asymmetry is still one of the biggest mysteries of the Universe. The violation of the CP-symmetry is a necessary condition to generate this anomaly. Within the Standard Model (SM), this effect is parametrized by the CP-violating phases generated in the complex Yukawa-type interactions of the fermion fields with the Higgs doublet. Yet, not enough CP violation (CPV) is present in this model to explain the matter–antimatter asymmetry in the Universe. New sources of CPV could modify the Standard Model expectations for direct and indirect CP violation of hadronic decays such as $K \to \pi\pi$. For these decays, the Standard Model

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prediction of CP violation contains an additional flavour suppression due to the smallness of some Cabibbo–Kobayashi–Maskawa (CKM) factors. This mechanism is typically not present in models of new physics (NP). The understanding of CPV could shed some light on this curious puzzle, and it has been of big interest since its discovery more than 50 years ago. One of the most important experimental results in the field was the observation of direct CP violation parametrized by the ratio ε'/ε and measured by a long series of precision counting experiments. The experimental determination of this observable is given by measuring the ratio

$$R = \left|\frac{\eta_{00}}{\eta_{+-}}\right|^2 = \frac{\Gamma\left(K_L \to \pi^0\pi^0\right)/\Gamma\left(K_S \to \pi^0\pi^0\right)}{\Gamma\left(K_L \to \pi^+\pi^-\right)/\Gamma\left(K_S \to \pi^+\pi^-\right)}, \qquad (1)$$

where η_i $(i = 00, +-)$ refer to the CP-violating amplitude ratios of K_L and K_S into two-pion final states. This quantity is directly accessible experimentally and is related to the observable ε'/ε by

$$R \simeq 1 - 6\,\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right). \qquad (2)$$

The world average based on the recent results of NA48 [1] and KTeV [2] collaborations stands at

$$\left(\frac{\varepsilon'}{\varepsilon}\right)_{\mathrm{exp}} = (16.6 \pm 2.3) \times 10^{-4}. \qquad (3)$$

On the other hand, theoretical predictions of ε'/ε rely on a combination of continuum perturbation theory to evaluate the short-distance contributions, that are sensitive to physics beyond the Standard Model, and the evaluation of long-distance contributions to matrix elements using Lattice QCD. This estimation is notoriously difficult due to the presence of the strong interactions and confinement at low-energy scales. Indeed, it is still subject to very large hadronic uncertainties even though in the past years there has been a huge progress in the Lattice community and the matrix elements could be determined with controlled systematics [3, 4]. The latter achievement and the prospects of Lattice improvement open the possibility for a precision theory prediction of direct CP violation. Moreover, the recent theoretical determination of this observable at the Next-to-Leading Order (NLO) within the SM found a 2.9σ tension between theory and experiment [5]

$$\left(\frac{\varepsilon'}{\varepsilon}\right)_{\mathrm{SM}} = (1.9 \pm 4.5) \times 10^{-4}. \qquad (4)$$

This inconsistency could have several sources, one of which could be the missing contribution of new particles in the theory prediction. However, a

reliable SM prediction is essential to disentangle possible NP effects from the SM background. As rapid progress on the lattice is bringing non-perturbative long-distance effects under control, a more precise knowledge of short-distance contributions is needed. In this proceedings, we have focused on the study of the latter and their impact effects on ε'/ε.

2. Effective Hamiltonian for the $\Delta S = 1$ transitions

The transition $K \to \pi\pi$ is described by the following effective Hamiltonian:

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} (z_i(\mu) + \tau y_i(\mu)) Q_i \,, \tag{5}$$

where the Wilson coefficients $z_i(\mu)$ and $y_i(\mu)$ encode information from higher order regimes and can be calculated perturbatively, G_{F} is the Fermi constant, the factor τ is defined in terms of the CKM matrix elements: $\tau = -V_{td} V_{ts}^* / (V_{ud} V_{us}^*)$, the operators Q_i are built up of light particles and classified as: current–current ($i = 1, 2$), QCD penguin operators ($i = 3$–6), and electroweak penguin operators ($i = 7$–10), (see reference [6] for their definition).

In order to study the CP-violating observable ε'/ε , it is important to improve both the estimation of long-distance contributions (hadronic matrix elements) [3, 4] and the theoretical calculations of short-distance contributions [6–14]. The completion of the QCD Next-to-Next-to-Leading Order (NNLO) corrections and their impact on the theoretical estimation of this observable comprise the content of our project.

Several contributions are required to determine the effective Hamiltonian relevant for the study of direct CP violation at the NNLO. First, the initial conditions for the Wilson coefficients at the electroweak scale, $z_i(M_W)$ and $y_i(M_W)$, have to be determined. This is achieved by matching the SM Green's functions to those in the five-flavour theory. Subsequently, the Wilson coefficients are evolved down to the bottom-quark scale using the renormalisation group equations (RGE): $U(\mu_W, \mu_b)$ with $\mu_i \sim \mathcal{O}(m_i)$. At an energy scale below the bottom-quark mass, $\mu < m_b$, the threshold corrections due to the removal of the bottom quark as a dynamical particle has to be included, $M(\mu_b)$. These effects can be obtained perturbatively by matching two effective field theories at scales $\mu_b = \mathcal{O}(m_b)$. The resulting Wilson coefficients are then evolved down to the charm-quark scale using the RGE: $U(\mu_b, \mu_c)$, and at this scale $\mu_c = \mathcal{O}(m_c)$, the matching equation for the charm quark is now evaluated. Here, we match the four-flavour theory onto the three-flavour theory and find the new threshold corrections, $M(\mu_c)$. Finally, we incorporate these results and perform the renormalisation group

evolution down to the scales where the hadronic matrix elements are computed. For technical reasons, another operator basis different from Q_i was employed [7, 9, 15, 16]. In this manner, we avoided the need to calculate traces that involve γ_5. To analyse the impact of these threshold corrections on direct CP violation, we transformed our results to the traditional operator basis [5]. Moreover, we introduced a new formalism in terms of renormalisation group invariant factors (denoted by hats), which are scale- and scheme-independent. More details can be found in references [17] and [18].

3. Formalism for ε'/ε within the Standard Model

The phenomenological analysis of ε'/ε is based on the following formula:

$$\frac{\varepsilon'}{\varepsilon} = -i\frac{\omega_+}{\sqrt{2}|\varepsilon_K|}e^{i(\delta_2-\delta_0-\phi_{\varepsilon_K})}\left[\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\left(1-\hat{\Omega}_{\mathrm{eff}}\right) - \frac{1}{a}\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2}\right], \qquad (6)$$

where the term ω_+ is determined from the charged decay mode, and $A_I \equiv \langle(\pi\pi)_I|\mathcal{H}_{\mathrm{eff}}|K\rangle$ correspond to the amplitudes for the two isospin states with their strong phases $\delta_{0,2}$ removed. The latter, as well as the phase ϕ_{ε_K} and the magnitude $|\varepsilon_K|$ of ε_K, which parametrizes indirect CP violation are all determined from experimental data. Isospin breaking effects are parametrized by the coefficients a and $\hat{\Omega}_{\mathrm{eff}}$ [19]. The numerical value for the latter is extracted from chiral perturbation theory [19–21]. The ratios of the imaginary and the real parts of the isospin limit amplitudes are the main pieces for the theoretical prediction of the observable ε'/ε. The determination of these important terms is based on the work published in reference [6]. Here, the authors assume that the amplitudes $\mathrm{Re}A_0$ and $\mathrm{Re}A_2$ originate already at tree level within the SM. Consequently, these quantities are expected to be only marginally affected by NP contributions. Moreover, some Fierz identities relating current–current and $(V-A)\times(V-A)$ type QCD and electroweak penguin operators allow them to reduce the hadronic uncertainty in the Standard Model prediction. We have extended this formalism to incorporate the non-zero z_i coefficients with $i > 2$ and, consistently, adapt it to incorporate higher order corrections by working with the renormalisation group invariant quantities (denoted by hats).

The operators Q_-, Q_3, Q_5, Q_6 are pure $I = 1/2$ operators, hence in the isospin limit their matrix elements for $I = 2$ vanish: $\langle Q_-\rangle_2 = \langle Q_3\rangle_2 = \langle Q_5\rangle_2 = \langle Q_6\rangle_2 = 0$. As a result, in the isospin limit, they do not contribute to A_2 and we find for the real part of the amplitudes

$$\mathrm{Re}A_2 = \frac{G_{\mathrm{F}}}{\sqrt{2}}\lambda_u\left[\left(\hat{z}_+ + \frac{3}{2}\left[\hat{z}_9 + \hat{z}_{10}\right]\right)\langle\hat{Q}_+\rangle_2 + \hat{z}_7\langle\hat{Q}_7\rangle_2 + \hat{z}_8\langle\hat{Q}_8\rangle_2\right],$$

$$\text{Re}A_0 = \frac{G_F}{\sqrt{2}}\lambda_u \left[\left(\hat{z}_+ + \frac{3}{2}\left[\hat{z}_9 + \hat{z}_{10}\right]\right)\langle \hat{Q}_+\rangle_0 + \left(\hat{z}_- + \frac{1}{2}\left[4\hat{z}_4 - 3\hat{z}_9 + \hat{z}_{10}\right]\right)\langle \hat{Q}_-\rangle_0 \right.$$

$$\left. + \frac{1}{2}(2\hat{z}_3 + 2\hat{z}_4 - \hat{z}_9 - \hat{z}_{10})\langle \hat{Q}_3\rangle_0 + \hat{z}_5\langle \hat{Q}_5\rangle_0 + \hat{z}_6\langle \hat{Q}_6\rangle_0 + \hat{z}_7\langle \hat{Q}_7\rangle_0 + \hat{z}_8\langle \hat{Q}_8\rangle_0 \right], \quad (7)$$

where $\lambda_u = V_{ud}V_{us}^*$ and we explicitly keep the small penguin contribution. The dominant contribution to these terms comes from the current–current operators. Similarly, the imaginary parts of the isospin amplitudes A_0 and A_2 take the following form:

$$\text{Im}A_2 = \frac{G_F}{\sqrt{2}}\lambda_u \text{Im}\tau \left[\frac{3}{2}(\hat{y}_9 + \hat{y}_{10})\langle \hat{Q}_+\rangle_2 + \hat{y}_7\langle \hat{Q}_7\rangle_2 + \hat{y}_8\langle \hat{Q}_8\rangle_2 \right],$$

$$\text{Im}A_0 = \frac{G_F}{\sqrt{2}}\lambda_u \text{Im}\tau \left[\frac{3}{2}b(\hat{y}_9 + \hat{y}_{10})\langle \hat{Q}_+\rangle_0 + \left(2\,\hat{y}_4 - \frac{b}{2}[3\hat{y}_9 - \hat{y}_{10}]\right)\langle \hat{Q}_-\rangle_0 \right.$$

$$+ \left(\hat{y}_3 + \hat{y}_4 - \frac{b}{2}[\hat{y}_9 + \hat{y}_{10}]\right)\langle \hat{Q}_3\rangle_0 + \hat{y}_5\langle \hat{Q}_5\rangle_0 + \hat{y}_6\langle \hat{Q}_6\rangle_0$$

$$\left. + b\hat{y}_7\langle \hat{Q}_7\rangle_0 + b\hat{y}_8\langle \hat{Q}_8\rangle_0 \right], \quad (8)$$

with the important contribution coming from QCD and electroweak penguins, respectively. The factor b, which appears in the previous equation, describes the corrections to the isospin zero amplitude

$$\text{Im}A_0 = (\text{Im}A_0)^{\text{QCDP}} + b\,(\text{Im}_0)^{\text{EWP}}\ , \qquad b = \frac{1}{a(1 - \hat{\Omega}_{\text{eff}})}\ . \quad (9)$$

In the ratio of isospin amplitudes, note that the same $(V-A)\times(V-A)$ operators appear in the numerators and the denominators. This suggests that we split the numerator into $(V-A)\times(V-A)$ and $(V-A)\times(V+A)$ pieces. Whereas the first type are dominated by short distance (Wilson coefficients) due to a cancellation of the matrix elements, the contributions coming from the $(V-A)\times(V+A)$ operators are very sensitive to long-distance effects (hadronic matrix elements). To minimize the non-perturbative uncertainties, one can extract the denominators from CP-averaged $K \to \pi\pi$ decay rates to remove the dependence on the $(V-A)\times(V-A)$ operators. For the isospin 2 amplitude, we obtain

$$\frac{\text{Im}A_2}{\text{Re}A_2} = \text{Im}\tau \left[\frac{3}{2}\frac{\hat{y}_9 + \hat{y}_{10}}{\hat{z}_+}(1 + \delta z_2) + \frac{G_F}{\sqrt{2}}\lambda_u\,(\hat{y}_8 + p_{72}y_7)\frac{\langle \hat{Q}_8\rangle_2}{\text{Re}A_2} \right], \quad (10)$$

where we performed an expansion in the small penguin contribution

$$\delta z_2 = -\frac{3}{2}\frac{(\hat{z}_9 + \hat{z}_{10})}{\hat{z}_+} - \frac{G_F}{\sqrt{2}}\lambda_u\frac{\langle \hat{Q}_8\rangle_2}{\text{Re}A_2}\,(p_{72}\hat{z}_7 + \hat{z}_8) \quad (11)$$

and defined

$$p_{72} = \frac{\langle \hat{Q}_7 \rangle_2}{\langle \hat{Q}_8 \rangle_2} . \tag{12}$$

Note that the first term in (10) is completely free of hadronic matrix elements. This is because data have not been used for the denominator of this part of the ratio. In case of having used $\mathrm{Re}A_0$ and $\mathrm{Re}A_2$ from data also in the $(V-A) \times (V-A)$ part, a dependence on (mainly) the matrix element of the operator Q_4 and its Wilson coefficients would be introduced, and this should be avoided. Indeed, this is the main reason why the prediction of [5] is more accurate than that of RBC and UKQCD [3], and leads to a more pronounced tension with the data, in spite of employing the same non-perturbative matrix elements.

Extending this formalism to the isospin-zero ratio, we have

$$\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} = \mathrm{Im}\tau \left[\frac{(2\,\hat{y}_4 - \frac{b}{2}[3\hat{y}_9 - \hat{y}_{10}])}{\hat{z}_-(1+\hat{q})} + \frac{\frac{3b}{2}[\hat{y}_9 + \hat{y}_{10}]\hat{q}}{\hat{z}_+(1+\hat{q})} \right.$$
$$+ \frac{(\hat{y}_3 + \hat{y}_4 - \frac{b}{2}[\hat{y}_9 + \hat{y}_{10}])}{\hat{z}_-(1+\hat{q})} p_3 + \frac{G_\mathrm{F}}{\sqrt{2}} \frac{\lambda_u}{\mathrm{Re}A_0}$$
$$\times \left. \left([\hat{y}_6 + p_5\hat{y}_5 + p_{8g}\hat{y}_{8g}]\langle \hat{Q}_6 \rangle_0 + b[\hat{y}_8 + p_{70}\hat{y}_7 + p_{70\gamma}\hat{y}_{7\gamma}]\langle \hat{Q}_8 \rangle_0 \right) \right] , \tag{13}$$

where we again expanded in the small penguin contribution

$$\delta z_0 = \frac{(-2\hat{z}_3 - 2\hat{z}_4 + \hat{z}_9 + \hat{z}_{10})}{2(\hat{q}+1)\hat{z}_-} p_3 - \frac{4\hat{z}_4 - 3\hat{z}_9 + \hat{z}_{10}}{2(\hat{q}+1)\hat{z}_-} - \frac{3\hat{q}\,(\hat{z}_9 + \hat{z}_{10})}{2(\hat{q}+1)\hat{z}_+}$$
$$- \frac{G_\mathrm{F}}{\sqrt{2}} \frac{\lambda_u}{\mathrm{Re}A_0} \left[(p_5\hat{z}_5 + \hat{z}_6)\langle \hat{Q}_6 \rangle_0 + (p_7\hat{z}_7 + \hat{z}_8)\langle \hat{Q}_8 \rangle_0 \right] , \tag{14}$$

and defined the following ratios of matrix elements:

$$\hat{p}_3 = \frac{\langle \hat{Q}_3 \rangle_0}{\langle \hat{Q}_- \rangle_0} , \ \hat{p}_5 = \frac{\langle \hat{Q}_5 \rangle_0}{\langle \hat{Q}_6 \rangle_0} , \ \hat{p}_{8g} = \frac{\langle \hat{Q}_{8g} \rangle_0}{\langle \hat{Q}_6 \rangle_0} , \ \hat{p}_{70} = \frac{\langle \hat{Q}_7 \rangle_0}{\langle \hat{Q}_8 \rangle_0} , \ \hat{p}_{70\gamma} = \frac{\langle \hat{Q}_{7\gamma} \rangle_0}{\langle \hat{Q}_8 \rangle_0} , \tag{15}$$

where the first two ratios are colour-suppressed, but still important enough to be included for a proper analysis of direct CP violation. Moreover, the weakly scale-dependent parameter $\hat{q}$, which appears in expressions (13) and (14), is defined in terms of the current–current Wilson coefficients and operators

$$\hat{q} = \hat{z}_+ \langle \hat{Q}_+ \rangle_0 / \hat{z}_- \langle \hat{Q}_- \rangle_0 . \tag{16}$$

The small ratio z_+/z_- implies that only modest accuracy is needed for the hadronic matrix elements entering the isospin-zero ratio through $\hat{q}$. As a

result, the prediction for ε'/ε involves predominantly two hadronic matrix elements (often parametrised in terms of parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$, refer to Section 5.2 of reference [6]), as well as perturbative Wilson coefficients $\hat{z}_{1,2}$ and $\hat{y}_{6,8}$. Our new calculation essentially removes the perturbative uncertainty on $\hat{y}_6$. The uncertainties on $\hat{z}_{1,2}$ are already tiny, leaving $\hat{y}_8$ and an improved treatment of isospin-breaking corrections as the main objectives for the future.

4. Impact of the NNLO QCD corrections

In the absence of QED corrections, the isospin $I = 2$ ratio would vanish, and the isospin-zero part would receive corrections only from QCD and current–current operators. The pure QCD calculation gives the dominant contribution to the observable ε'/ε, even though the QED part is important through its contribution to the isospin $I = 2$ ratio. Incorporating the recent Lattice determination of the isospin zero matrix elements, we can analyse the effect of our results on the $\mathrm{Im}A_0/\mathrm{Re}A_0$ contribution to ε'/ε in the isospin limit. For the numerical analysis, we kept the values of the isospin $I = 2$ amplitudes fixed to the values of the work quoted in reference [5], and we perform the scale variation in the Wilson coefficients that contribute to the isospin $I = 0$ amplitude ratio. The resulting dependence on the matching scale μ_c is shown in Fig. 1. This plot exhibits a significant reduction of the residual scale dependence order-by-order in perturbation theory, where the
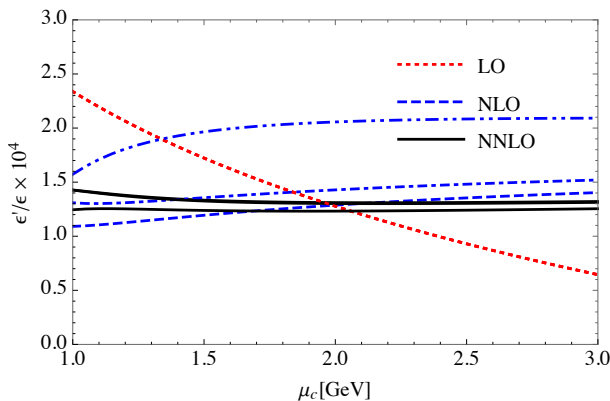

Fig. 1. Residual scale dependence of ϵ'/ϵ using non-perturbative parameter from Lattice QCD at LO, NLO and NNLO. The scale variation is due to the residual scale dependence of the Wilson coefficients $\hat{y}_3$–$\hat{y}_6$, $\hat{z}_+$, $\hat{z}_-$ and $\hat{z}_3$–$\hat{z}_6$ and measures part of the remaining perturbative uncertainty. The different lines at LO (dotted line), NLO (dashed, dash-dotted and dash-double-dotted lines) and NNLO (solid lines) correspond to different solutions of the renormalisation group equations. Their variation provides an additional measure of the remaining perturbative uncertainty.

perturbative corrections are estimated through the respective scale variations. The perturbative uncertainties stemming from current–current and QCD penguin operators are reduced to around 12% at NNLO level. This removes the largest part of the perturbative uncertainty in this quantity and also strengthens the approach to treat the charm quark contribution perturbatively. After incorporating the NNLO corrections to the isospin $I = 0$, we find that the uncertainties in the isospin $I = 2$ decay mode will dominate the perturbative error. Following the procedures of reference [5], we estimate an associated uncertainty of $\pm 0.8 \times 10^{-4}$ to the Standard Model prediction of ε'/ε for the electroweak penguins.

5. Limitations and future improvements

Several subtleties are still present in the phenomenological analysis of ε'/ε. Indeed, Lattice computations of the relevant matrix elements are currently performed only in the isospin limit (α_e, degenerate masses). This limit is not very good since pions are not exact $I = 1$ states and electromagnetic effects cannot be neglected when charged particles are present. In fact, it receives at $\mathcal{O}(10\%)$ corrections, which have been parametrised by the factor $\hat{\Omega}_{\text{eff}}$, together with a particular scheme for ω, which defines ω_+. The parameter a is also an attempt to include a class of higher-order isospin-breaking effects. While the latter factors introduce isospin-breaking effects, the phases $\delta_{0,2}$ are still defined in the isospin limit, even if they are no longer the true strong phases of the amplitudes. Moreover, the definition of the isospin limit is not self-consistent as the electroweak Wilson coefficients have scale dependence that is due to electromagnetism, and this does not cancel against the scale dependence of an isospin-symmetric evaluation of the matrix elements. In practice, one matches a QCD×QED evolution to a pure QCD Lattice calculation at some scale ($\mu = m_c$). This can only be resolved by including electromagnetism into the matrix elements, which introduces an IR problem. Moreover, the practical evaluation of the hadronic matrix elements involves a theory with three light quarks (u, d, s). To control possible non-perturbative effects of virtual charm quarks, it would be desirable to calculate the matrix elements in a theory with four or even five flavours. In the future, Lattice QCD will be able to include the dynamical charm quark and a phenomenological analysis of direct CP violation would be able to be performed at the four-flavour theory. In the meantime, we can use our NNLO results to provide an estimation of the four-flavour matrix elements. More details of this calculation and the extension of the formalism to $n_f > 3$ will be published in a future publication [22] .

6. Summary and outlook

Motivated by the fact that the experimental data for direct CP violation is in tension with the SM prediction, and that further progress is expected in the near future for the non-perturbative sector, we have updated the theoretical prediction of direct CP violation by incorporating the NNLO QCD corrections. The main message of our analysis is that the inclusion of these higher order corrections does not move the central value of ε'/ε and that these new contributions reduce the theoretical uncertainty by a factor 0.12. With this update, the perturbative uncertainty on ε'/ε is now dominated by the NNLO corrections to the $I = 2$ amplitude ratio. All the results presented in this conference are preliminary. For further details and the exact analysis of ε'/ε, we refer to the future publication [22].

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