SUBLEADING SHAPE FUNCTIONS IN $\bar{B} \to X_{s,d}\ell\ell$*

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We analyse the resolved power corrections to the inclusive decays $\bar{B} \to X_s\ell^+\ell^-$ and also $\bar{B} \to X_d\ell^+\ell^-$. As a distinctive feature, the resolved contributions remain non-local when the hadronic mass cut is released. Therefore, they reflect an irreducible uncertainty not dependent on the hadronic mass cut. They factorize in hard functions describing physics at the high scale m_b, in so-called jet functions characterizing the physics at the hadronic final state X_s which corresponds to an invariant mass of the order of $\sqrt{m_b \Lambda_{\rm QCD}}$, and in soft functions, so-called shape functions, parametrizing the hadronic physics at the scale $\Lambda_{\rm QCD}$. Knowing the explicit form of the latter, one can derive general properties of such shape functions which allow for precise estimates of the corresponding uncertainties.

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1. Introduction

We discuss so-called resolved power corrections to the inclusive decays $\bar{B} \to X_{s,d}\ell^+\ell^-$. These decays are flavour changing neutral current processes which are theoretically clean and highly sensitive to possible new degrees of freedom (for reviews, see Refs. [1–3]), thus, they belong to the golden modes of the forthcoming Belle II experiment at KEK [4]. Their theoretical precision has already reached a highly sophisticated level [5–7]. The large data sets collected at the Belle II experiment call for even higher precision of the theoretical predictions and, in particular, for the calculation of subleading power corrections.

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The resolved photon contributions to the inclusive decay $\bar{B} \to X_{s,d}\ell^+\ell^-$, considered here, contain subprocesses in which the virtual photon couples to light partons instead of connecting directly to the effective weak-interaction vertex [8, 9]. They can be estimated in a systematic way [10, 11].

Within the inclusive decay $\bar{B} \to X_{s,d}\ell^+\ell^-$, the hadronic ($M_X$) and dilepton invariant (q^2) masses are independent kinematical quantities. An invariant mass cut on the hadronic final-state system ($M_X \lesssim 2$ GeV) is required in order to suppress potential huge backgrounds. This cut poses no further constraints in the high-dilepton mass region, in the low-dilepton mass region, however, the cut on the hadronic mass leads to a specific kinematics in which the standard OPE breaks down and one has to introduce non-perturbative b-quark distributions, so-called shape functions. Given the specific kinematics of low-dilepton masses q^2 and of small hadronic masses M_X, one has to deal with a multi-scale problem, $M_X \lesssim \sqrt{M_B \Lambda_{\rm QCD}} \sim M_B\sqrt{\lambda}$ with $\lambda := \Lambda_{\rm QCD}/M_B$, which can be solved by soft-collinear effective theory (SCET).

A previous SCET analysis made use of the universality of the leading shape function to demonstrate that the reduction resulting from the M_X-cut can be precisely calculated in all angular observables of the inclusive decay $\bar{B} \to X_s\ell^+\ell^-$. The effects of subleading shape functions imply an additional uncertainty of 5% [12, 13]. All these former analyses, however, rely on a problematic assumption, namely that q^2 represents a hard scale in the kinematical region of low q^2 and of small M_X. By contrast, our present SCET analysis [10, 11] has explicitly demonstrated that the hadronic cut implies the scaling of q^2 being not hard but (anti-) hard-collinear in the low-q^2 region and that the resolved contributions would not exist if q^2 was hard: The hard momentum of the leptons would also imply a hard momentum of the intermediate parton. The latter would be integrated out at the hard scale and the lepton pair would be directly connected to the effective electroweak interaction vertex. Moreover, we have shown that the resolved contributions — as a special feature — stay non-local when the hadronic mass cut is released. In this sense, they represent an irreducible uncertainty independent of the hadronic mass cut.

In the following, we discuss the estimate of the effect of the subleading shape functions. In Refs. [10, 11], we have calculated all resolved power corrections to the inclusive decay $\bar{B} \to X_s\ell^+\ell^-$ to first order in $1/m_b$. Here, we also derive the corresponding contributions to the inclusive decay $\bar{B} \to X_d\ell^+\ell^-$.

2. Resolved contributions to the decays $\bar{B} \to X_{s,d}\ell^+\ell^-$

In Ref. [10], we have found three resolved operator combinations to the order of $1/m_b$ for the decay $\bar{B} \to X_s\ell^+\ell$, namely from the interference terms $\mathcal{O}_{7\gamma}$–$\mathcal{O}_{8g}$, $\mathcal{O}_{8g}$–$\mathcal{O}_{8g}$, and $\mathcal{O}_1^c$–$\mathcal{O}_{7\gamma}$. We have shown that the smooth limit $q^2 \to 0$ reproduces the known results for the decay $\bar{B} \to X_s\gamma$, first derived in Ref. [14]. For more details, we refer the reader to Ref. [10].

For the $\mathcal{O}_{7\gamma}$–$\mathcal{O}_{8g}$ part, we have found two contributions denoted by (b) and (c):

$$\begin{aligned} \mathrm{d}\Gamma_{78}^{(\mathrm{b})} &= -\frac{1}{m_b}\,\mathrm{Re}\left[\hat{\Gamma}_{78}\right]\mathrm{d}\Lambda_{\alpha\beta}\,16\pi\alpha_s\,e_q\,m_b\,n\cdot q\,\left(g_\perp^{\alpha\beta}+i\epsilon_\perp^{\alpha\beta}\right) \\ &\quad\times\mathrm{Re}\int \mathrm{d}\omega\delta(\omega+m_b-n\cdot q)\int\frac{\mathrm{d}\omega_1}{\omega_1+\bar{n}\cdot q+i\epsilon}\,\frac{\mathrm{d}\omega_2}{\omega_2-i\epsilon} \\ &\quad\times\left[\bar{g}_{78}(\omega,\omega_1,\omega_2,\mu)-\bar{g}_{78}^{\mathrm{cut}}(\omega,\omega_1,\omega_2,\mu)\right]\,. \end{aligned} \tag{1}$$

Here, we have used the short-hand notation

$$\hat{\Gamma}_{ij} = \frac{G_{\mathrm{F}}^2\alpha m_b^2}{4\pi^2}\,C_iC_j^*\,|\lambda_t^s|^2 \tag{2}$$

with $\lambda_i^j = V_{ib}^*V_{ij}$. For the case of $\bar{B} \to X_d\ell^+\ell^-$, the $|\lambda_t^s|^2$ has to be replaced by $|\lambda_t^d|^2$. The shape functions g_{78} are defined as follows:

$$\begin{aligned} &\bar{g}_{78}(\omega,\omega_1,\omega_2,\mu) = \int\frac{\mathrm{d}r}{2\pi}\,e^{-i\omega_1 r}\int\frac{\mathrm{d}u}{2\pi}\,e^{i\omega_2 u}\int\frac{\mathrm{d}t}{2\pi}\,e^{-i\omega t} \\ &\times\frac{\langle\bar{B}|\,(\bar{h}S_n)\,(tn)\,T^A\,\bar{\Gamma}_n\,\left(S_n^\dagger s\right)(un)\,(\bar{s}S_{\bar{n}})\,(r\bar{n})\,\Gamma_{\bar{n}}\,\left(S_{\bar{n}}^\dagger S_n\right)(0)\,T^A\,\left(S_n^\dagger h\right)(0)\,|\bar{B}\rangle}{2M_B} \\ &\times\bar{g}_{78}^{\mathrm{cut}}(\omega,\omega_1,\omega_2,\mu) = \int\frac{\mathrm{d}r}{2\pi}\,e^{-i\omega_1 r}\int\frac{\mathrm{d}u}{2\pi}\,e^{i\omega_2 u}\int\frac{\mathrm{d}t}{2\pi}\,e^{-i\omega t} \\ &\times\frac{\langle\bar{B}|\,(\bar{h}S_n)\,(tn)\,T^A\,\bar{\Gamma}_n\,\left(S_n^\dagger s\right)((t+u)n)\,(\bar{s}S_{\bar{n}})\,(r\bar{n})\,\Gamma_{\bar{n}}\,\left(S_{\bar{n}}^\dagger S_n\right)(0)\,T^A\,\left(S_n^\dagger h\right)(0)\,|\bar{B}\rangle}{2M_B}\,. \end{aligned} \tag{3}$$

S_n and $S_{\bar{n}}$ are soft Wilson lines connecting the soft fields in the matrix element and thereby ensuring gauge invariance. For the case of $\bar{B} \to X_d\ell^+\ell^-$, the s-quark fields have to be replaced by d-quark fields within in the shape functions.

The integral measure is given in the first subleading order by

$$\mathrm{d}\Lambda_{\alpha\beta} = \mathrm{d}n\cdot q\,\mathrm{d}\bar{n}\cdot q\,\mathrm{d}z\frac{\alpha}{128\pi^3}\left(1+z^2\right)\,\frac{n\cdot q}{\bar{n}\cdot q}\,g_{\perp,\alpha\beta} \tag{4}$$

with $z = \cos\theta$. θ is the angle between the ℓ^+ and B meson three momenta in the di-lepton rest frame. There is no odd term in the variable z. Thus,

there is no resolved contribution to the forward–backward asymmetry in this order [1]. The second contribution is given by

$$\begin{aligned}
\mathrm{d}\Gamma_{78}^{(\mathrm{c})} &= \frac{1}{m_b}\mathrm{Re}\left[\hat{\Gamma}_{78}\right]\mathrm{d}\Lambda_{\alpha\beta}\,4\pi\alpha_s\,m_b\,n\cdot q\,\left(g_\perp^{\alpha\beta}-i\epsilon_\perp^{\alpha\beta}\right)\mathrm{Re}\int\mathrm{d}\omega\delta(\omega+m_b-n\cdot q)\\
&\times\int\frac{\mathrm{d}\omega_1}{\omega_1-\omega_2+\bar{n}\cdot q+i\epsilon}\,\mathrm{d}\omega_2\Bigg[\left(\frac{1}{\omega_1+\bar{n}\cdot q+i\epsilon}+\frac{1}{\omega_2-\bar{n}\cdot q-i\epsilon}\right)\\
&\times g_{78}^{(1)}(\omega,\omega_1,\omega_2,\mu)-\left(\frac{1}{\omega_1+\bar{n}\cdot q+i\epsilon}-\frac{1}{\omega_2-\bar{n}\cdot q-i\epsilon}\right)g_{78}^{(5)}(\omega,\omega_1,\omega_2,\mu)\Bigg]\,. \qquad (6)
\end{aligned}$$

The shape functions are defined as follows:

$$\begin{aligned}
g_{78}^{(1)}(\omega,\omega_1,\omega_2,\mu) &= \int\frac{\mathrm{d}r}{2\pi}\,e^{-i\omega_1 r}\int\frac{\mathrm{d}u}{2\pi}\,e^{i\omega_2 u}\int\frac{\mathrm{d}t}{2\pi}\,e^{-i\omega t}\\
&\times\frac{\langle\bar{B}|\,(\bar{h}S_n)\,(tn)\left(S_n^\dagger S_{\bar{n}}\right)(0)\,T^A\not{\bar{n}}(1+\gamma_5)\left(S_{\bar{n}}^\dagger h\right)(0)\,\boldsymbol{T}\sum_q e_q\,(\bar{q}S_{\bar{n}})\,(r\bar{n})\not{\bar{n}}\,T^A\left(S_{\bar{n}}^\dagger q\right)(u\bar{n})\,|\bar{B}\rangle}{2M_B}\,,\\
g_{78}^{(5)}(\omega,\omega_1,\omega_2,\mu) &= \int\frac{\mathrm{d}r}{2\pi}\,e^{-i\omega_1 r}\int\frac{\mathrm{d}u}{2\pi}\,e^{i\omega_2 u}\int\frac{\mathrm{d}t}{2\pi}\,e^{-i\omega t}\\
&\times\frac{\langle\bar{B}|\,(\bar{h}S_n)\,(tn)\left(S_n^\dagger S_{\bar{n}}\right)(0)\,T^A\not{\bar{n}}(1+\gamma_5)\left(S_{\bar{n}}^\dagger h\right)(0)\,\boldsymbol{T}\sum_q e_q\,(\bar{q}S_{\bar{n}})\,(r\bar{n})\not{\bar{n}}\gamma_5 T^A\left(S_{\bar{n}}^\dagger q\right)(u\bar{n})\,|\bar{B}\rangle}{2M_B}\,.
\end{aligned} \qquad (7)$$

For the case of $\bar{B}\to X_d\ell^+\ell^-$, no modification is needed in these shape functions. The difference to the radiative decay $\bar{B}\to X_{s,d}\gamma$ is introduced through the non-vanishing $\bar{n}\cdot q$ that shifts the small component of the anti-hard-collinear propagator, which corresponds to the anti-hard-collinear jet function. With the same argument, we can already see that the direct contributions will not be affected in such a way, since $\bar{n}\cdot q$ is suppressed relative to the large component of any hard-collinear propagator.

For the double resolved $\mathcal{O}_{8g}$–$\mathcal{O}_{8g}$ contribution involving twice the QCD dipole operator, we have found in Ref. [10]

$$\begin{aligned}
\mathrm{d}\Gamma_{88} &= \frac{1}{m_b}\mathrm{Re}\left[\hat{\Gamma}_{88}\right]\mathrm{d}\Lambda_{\alpha\beta}\,8\pi\alpha_s\,e_s^2\,m_b^2\,\left(g_\perp^{\alpha\beta}+i\epsilon_\perp^{\alpha\beta}\right)\mathrm{Re}\int\mathrm{d}\omega\delta(\omega+m_b-n\cdot q)\\
&\times\int\frac{\mathrm{d}\omega_1}{\omega_1+\bar{n}\cdot q+i\epsilon}\,\frac{\mathrm{d}\omega_2}{\omega_2+\bar{n}\cdot q-i\epsilon}\bar{g}_{88}(\omega,\omega_1,\omega_2,\mu)\,. \qquad (8)
\end{aligned}$$

[1] The triple differential rate in the form of

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}q^2\mathrm{d}z}=\frac{3}{8}\left[\left(1+z^2\right)H_\mathrm{T}\left(q^2\right)+2\left(1-z^2\right)H_\mathrm{L}\left(q^2\right)+2zH_\mathrm{A}\left(q^2\right)\right] \qquad (5)$$

shows that there are three independent angular observables which have different dependences on the Wilson coefficients [19]. The sum $H_\mathrm{T}+H_\mathrm{L}$ corresponds to the q^2 spectrum, while H_A to the forward–backward asymmetry. The special form of integral measure given in Eq. (4) implies that we only have resolved contributions to H_T in the first order in $1/m_b$.

Note that $e_s = e_d$, so there is no modification necessary for the $b \to d$ case. The shape function $\bar{g}_{88}$ is defined as follows (for the $b \to d$ case, only the s-quark fields have to be replaced by d-quark fields again):

$$\bar{g}_{88}(\omega,\omega_1,\omega_2,\mu) = \int \frac{\mathrm{d}r}{2\pi}\, e^{-i\omega_1 r} \int \frac{\mathrm{d}u}{2\pi}\, e^{i\omega_2 u} \int \frac{\mathrm{d}t}{2\pi}\, e^{-i\omega t} \times \frac{\langle \bar{B}|\, (\bar{h}S_n)\,(tn)\, T^A \left(S_n^\dagger S_{\bar{n}}\right)(tn)\, \bar{\Gamma}_{\bar{n}} \left(S_{\bar{n}}^\dagger s\right)(tn+u\bar{n})\, (\bar{s}S_{\bar{n}})\,(r\bar{n})\Gamma_{\bar{n}} \left(S_{\bar{n}}^\dagger S_n\right)(0)\, T^A \left(S_n^\dagger h\right)(0)\, |\bar{B}\rangle}{2M_B}\,. \tag{9}$$

There is a subtlety regarding the convolution integral in Eq. (8). The calculation of the asymptotic behaviour of the soft function for $\omega_{1,2} \gg \Lambda_{\mathrm{QCD}}$ leads to the finding that the convolution integrals are UV divergent. However, this divergence is mirrored by an IR divergence in the direct contribution to $\mathcal{O}_{8g}$–$\mathcal{O}_{8g}$, and scale and scheme independence of the sum of the two contributions can be shown (for further details, see Ref. [10]).

For the $\mathcal{O}_1^c$–$\mathcal{O}_{7\gamma}$ contribution, we have explicitly derived in Ref. [10]

$$\begin{aligned}
\mathrm{d}\Gamma_{17} &= \frac{1}{m_b}\,\mathrm{Re}\left[\hat{\Gamma}_{17}\frac{-(\lambda_t^s)^*\lambda_c^s}{|\lambda_t^s|^2}\right] \mathrm{d}\Lambda_{\alpha\beta}\, e_c\,(n\cdot q)^2\,\mathrm{Re}\int \mathrm{d}\omega\delta(\omega+m_b-n\cdot q)\\
&\times\int \mathrm{d}\omega_1 \frac{1}{\omega_1+i\epsilon}\frac{1}{\omega_1}\left[(\bar{n}\cdot q+\omega_1)\left(1-F\left(\frac{m_c^2}{n\cdot q(\bar{n}\cdot q+\omega_1)}\right)\right)\right.\\
&-\bar{n}\cdot q\left(1-F\left(\frac{m_c^2}{n\cdot q\bar{n}\cdot q}\right)\right)-\bar{n}\cdot q\left(G\left(\frac{m_c^2}{n\cdot q(\bar{n}\cdot q+\omega_1)}\right)\right.\\
&\left.\left.-G\left(\frac{m_c^2}{n\cdot q\bar{n}\cdot q}\right)\right)\right]\int\frac{\mathrm{d}t}{2\pi}e^{-i\omega t}\\
&\times\int\frac{\mathrm{d}r}{2\pi}e^{-i\omega_1 r}\frac{\langle B|\,\bar{h}(nt)\not{\bar{n}}\left[1+\gamma^5\right]\frac{i}{2}\left[\gamma_\perp^\mu,\gamma_\perp^\beta\right]\gamma_\perp^\alpha\bar{n}^\kappa gG_{\mu\kappa}(\bar{n}r)h(0)|B\rangle}{2M_B}\,.
\end{aligned}\tag{10}$$

The decomposition of the Lorentz structure leads to

$$\begin{aligned}
\mathrm{d}\Gamma_{17} &= \frac{1}{m_b}\,\mathrm{Re}\left[\hat{\Gamma}_{17}\frac{-(\lambda_t^s)^*\lambda_c^s}{|\lambda_t^s|^2}\right]\frac{\alpha}{24\pi^3}\,\mathrm{d}n\cdot q\mathrm{d}\bar{n}\cdot q\,\frac{(n\cdot q)^3}{\bar{n}\cdot q}\\
&\times\mathrm{Re}\int\mathrm{d}\omega\delta(\omega+m_b-n\cdot q)\int\mathrm{d}\omega_1\frac{1}{\omega_1+i\epsilon}\\
&\times\frac{1}{\omega_1}\left[(\bar{n}\cdot q+\omega_1)\left(1-F\left(\frac{m_c^2}{n\cdot q(\bar{n}\cdot q+\omega_1)}\right)\right)-\bar{n}\cdot q\left(1-F\left(\frac{m_c^2}{n\cdot q\bar{n}\cdot q}\right)\right)\right.\\
&\left.-\bar{n}\cdot q\left(G\left(\frac{m_c^2}{n\cdot q(\bar{n}\cdot q+\omega_1)}\right)-G\left(\frac{m_c^2}{n\cdot q\bar{n}\cdot q}\right)\right)\right]\,g_{17}(\omega,\omega_1,\mu)\,,
\end{aligned}\tag{11}$$

with

$$g_{17}(\omega,\omega_1,\mu) = \int \frac{\mathrm{d}r}{2\pi}\, e^{-i\omega_1 r} \int \frac{\mathrm{d}t}{2\pi}\, e^{-i\omega t} \times \frac{\langle \bar{B}\big|\big(\bar{h}S_n\big)(tn)\,\bar{n}\!\!\!/(1+\gamma_5)\Big(S_n^\dagger S_{\bar{n}}\Big)(0)\, i\gamma_\alpha^\perp \bar{n}_\beta \Big(S_{\bar{n}}^\dagger\, g G_s^{\alpha\beta} S_{\bar{n}}\Big)(r\bar{n})\Big(S_{\bar{n}}^\dagger h\Big)(0)\big|\bar{B}\rangle}{2M_B}\,.$$

The penguin functions F and G are defined as follows:

$$F(x) = 4x \arctan^2 \frac{1}{\sqrt{4x-1}}\,, \qquad G(x) = 2\sqrt{4x-1}\arctan\frac{1}{\sqrt{4x-1}} - 2\,. \quad (12)$$

In this contribution, the modifications for the $b \to d$ case are more involved. Within the $\mathcal{O}_1^c$–$\mathcal{O}_{7\gamma}$ contribution, all CKM parameter combinations λ_i^s have to be replaced by λ_i^d only, but there is an additional contribution from the interference $\mathcal{O}_1^u$–$\mathcal{O}_{7\gamma}$ which can be neglected due to the CKM suppression in the $b \to s$ case. However, in both cases, in $b \to d$ and also in the $b \to s$, this contribution from the u-quark loop vanishes within the integrated rate at the order of $1/m_b$ anyway: Using the explicit formulae of the penguin functions in Eqs. (12), the last two lines of Eq. (11) can be reduced in the limit $m_c \to m_u = 0$ to

$$\times \frac{1}{\omega_1}\,[\ \omega_1\ ]\ g_{17}(\omega,\omega_1,\mu)\,. \quad (13)$$

Using the trace formalism of HQET, one shows (see Ref. [14] for details) that

$$\int_{-\infty}^{\bar{\Lambda}} \mathrm{d}\omega\, g_{17}(\omega,\omega_1,\mu) = \int_{-\infty}^{\bar{\Lambda}} \mathrm{d}\omega\, \big(g_{17}(\omega,-\omega_1,\mu)\big)^*\,. \quad (14)$$

PT invariance implies that the shape function g_{17} is real, so that the integration of ω_1 in Eq. (11) leads to the final result that the interference term $\mathcal{O}_1^u$–$\mathcal{O}_{7\gamma}$ vanishes within the integrated rate.

Finally, there is an important remark in order: As the different results make clear, the operators defining the shape functions are non-local in both light-cone directions. Therefore, the resolved contributions stay non-local even when the hadronic mass cut is relaxed. In this case, $n\cdot P_X = M_B - n\cdot q$ is not necessarily small any more. We can, thus, expand the shape function in powers of $\Lambda_{\mathrm{QCD}}/(m_b - n\cdot q)$ by which we arrive at a series of matrix elements that are local in the n-direction. However, the non-locality in the $\bar{n}$-direction is not removed. In this sense, the resolved contributions constitute an irreducible uncertainty within the inclusive decay $\bar{B} \to X_{s,d}\ell^+\ell^-$.

3. Numerical estimate of the subleading shape functions

For the input parameters used in the present analysis, we refer to Ref. [10]. We are interested in the relative magnitude of the resolved contributions compared to the total decay rate, *i.e.* the leading direct contributions to the decay rate which one also would take into account when the decay rate was calculated within the OPE

$$\mathcal{F}\left(q^2_{\rm min}, q^2_{\rm max}, M^2_{X,\rm max}\right) = \frac{\Gamma^q_{\rm resolved}\left(q^2_{\rm min}, q^2_{\rm max}, M^2_{X,\rm max}\right)}{\Gamma^q_{\rm OPE}\left(q^2_{\rm min}, q^2_{\rm max}, M^2_{X,\rm max}\right)}\,, \qquad q = d, s\,, \tag{15}$$

where the rate $\Gamma^q_{\rm OPE}$ is given by

$$\begin{aligned}\Gamma^q_{\rm OPE} &= \frac{G_{\rm F}^2 \alpha m_b^5}{32\pi^4}\,|V_{tb}^* V_{tq}|^2 \frac{1}{3}\frac{\alpha}{\pi}\int \frac{{\rm d}\bar{n}\cdot q}{\bar{n}\cdot q}\left(1-\frac{\bar{n}\cdot q}{m_b}\right)^2 \\ &\times\left[C_{7\gamma}^2\left(1+\frac{1}{2}\frac{\bar{n}\cdot q}{m_b}\right)+\left(C_9^2+C_{10}^2\right)\left(\frac{1}{8}\frac{\bar{n}\cdot q}{m_b}+\frac{1}{4}\left(\frac{\bar{n}\cdot q}{m_b}\right)^2\right)+C_{7\gamma}C_9\frac{3}{2}\frac{\bar{n}\cdot q}{m_b}\right] \\ &\equiv \frac{G_{\rm F}^2 \alpha m_b^5}{32\pi^4}\,|V_{tb}^* V_{tq}|^2 \frac{1}{3}\frac{\alpha}{\pi}\,C_{\rm OPE}\,.\end{aligned} \tag{16}$$

The last line defines the quantity $C_{\rm OPE}$. The first term in the square brackets is the leading power in the $1/m_b$ expansion and corresponds to the direct contribution due to the interference of $\mathcal{O}_{7\gamma}$ with itself. The other terms are formally suppressed in the shape function region in which we evaluate these direct contributions. But the large magnitude of the Wilson coefficients $|C_{9/10}| \sim 13|C_{7\gamma}|$ demands their inclusion into our uncertainty.

For the resolved contribution from the interference of $\boldsymbol{\mathcal{O}_1^c}$ with $\boldsymbol{\mathcal{O}_{7\gamma}}$, we have found in Ref. [10]

$$\begin{aligned}\mathcal{F}^q_{17} &= \frac{1}{m_b^4}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}}e_c\,{\rm Re}\left[\frac{-(\lambda_t^q)^*\lambda_c^q}{|\lambda_t^q|^2}\right]{\rm Re}\int\limits_{\frac{q^2_{\rm min}}{M_B}}^{\frac{q^2_{\rm max}}{M_B}}{\rm d}\bar{n}\cdot q\,\frac{m_b^3}{\bar{n}\cdot q}\int\limits_{-\infty}^{+\infty}\frac{{\rm d}\omega_1}{\omega_1+i\epsilon} \\ &\times\frac{1}{\omega_1}\left[(\bar{n}\cdot q+\omega_1)\left(1-F\left(\frac{m_c^2}{m_b(\bar{n}\cdot q+\omega_1)}\right)\right)-\bar{n}\cdot q\left(1-F\left(\frac{m_c^2}{m_b\bar{n}\cdot q}\right)\right)\right. \\ &\left.-\bar{n}\cdot q\left(G\left(\frac{m_c^2}{m_b(\bar{n}\cdot q+\omega_1)}\right)-G\left(\frac{m_c^2}{m_b\bar{n}\cdot q}\right)\right)\right]\int\limits_{-\infty}^{\bar{\Lambda}}{\rm d}\omega\,g_{17}(\omega,\omega_1,\mu)\end{aligned} \tag{17}$$

with $\bar{\Lambda} = M_B - m_b$. Here, we have already used the fact that the soft function only has support for $\omega \sim \Lambda_{\rm QCD}$.

In the $\bar{B} \to X_s \ell^+ \ell^-$ case ($q = s$), we find for the CKM ratio

$$\mathrm{Re}\left[-(\lambda_t^s)^* \lambda_c^s \Big/ \left(|\lambda_t^s|^2\right)\right] = 0.99\,,$$

while in the $\bar{B} \to X_d \ell^+ \ell^-$ ($q = d$), we get

$$\mathrm{Re}\left[-\left(\lambda_t^d\right)^* \lambda_c^d \Big/ \left(\left|\lambda_t^d\right|^2\right)\right] = 1.19\,.$$

Thus, the effect of this resolved contribution in the $b \to d$ case is just 20% larger.

The integration of the soft function over ω eliminates the t integral in the shape function and sets $t = 0$. Defining $h(\omega_1, \mu) := \int_{-\infty}^{\bar{\Lambda}} \mathrm{d}\omega\, g_{17}(\omega, \omega_1, \mu)$, we find

$$h_{17}(\omega_1, \mu) = \int \frac{\mathrm{d}r}{2\pi}\, e^{-i\omega_1 r} \frac{\langle B|\bar{h}(0) \not{\bar{n}} i\gamma_\alpha^\perp \bar{n}_\beta g G^{\alpha\beta}(r\bar{n}) h(0)|B\rangle}{2M_B}\,. \tag{18}$$

Knowing the explicit form of the HQET matrix element, we can derive general properties of the integrated shape function h_{17}. As mentioned above, one can derive from PT invariance that the function is real and even in ω_1. One can also explicitly derive the general normalization of this soft function [14]

$$\int_{-\infty}^{\infty} \mathrm{d}\omega_1 h_{17}(\omega_1, \mu) = 2\,\lambda_2\,. \tag{19}$$

Finally, the soft function h_{17} should not have any significant structure (maxima or zeros) outside the hadronic range, and the values of h_{17} should be within the hadronic range. In summary, we can write the relative contribution due to the interference of $\mathcal{O}_1^c$ with $\mathcal{O}_{7\gamma}$ as

$$\mathcal{F}_{17}^q = \frac{1}{m_b} \frac{C_1(\mu) C_{7\gamma}(\mu)}{C_{\mathrm{OPE}}} e_c\, \mathrm{Re}\left[\frac{-(\lambda_t^q)^* \lambda_c^q}{|\lambda_t^q|^2}\right] \int_{-\infty}^{+\infty} \mathrm{d}\omega_1\, J_{17}\big(q^2_{\mathrm{min}}, q^2_{\mathrm{max}}, \omega_1\big)\, h_{17}(\omega_1, \mu)$$

$$\text{with} \quad J_{17}\left(q^2_{\mathrm{min}}, q^2_{\mathrm{max}}, \omega_1\right) = \mathrm{Re} \frac{1}{\omega_1 + i\epsilon} \int_{\frac{q^2_{\mathrm{min}}}{M_B}}^{\frac{q^2_{\mathrm{max}}}{M_B}} \frac{\mathrm{d}\bar{n}\cdot q}{\bar{n}\cdot q} \frac{1}{\omega_1}$$

$$\times \left[(\bar{n}\cdot q + \omega_1)\left(1 - F\left(\frac{m_c^2}{m_b(\bar{n}\cdot q + \omega_1)}\right)\right) - \bar{n}\cdot q \left(1 - F\left(\frac{m_c^2}{m_b \bar{n}\cdot q}\right)\right)\right.$$

$$\left. -\bar{n}\cdot q \left(G\left(\frac{m_c^2}{m_b(\bar{n}\cdot q + \omega_1)}\right) - G\left(\frac{m_c^2}{m_b \bar{n}\cdot q}\right)\right)\right]\,. \tag{20}$$

For the standard value of $q^2_{\rm min}$ and $q^2_{\rm max}$, the function J_{17} is plotted in Fig. 1. It is largest around $\omega_1 = 0$. As a first trial for a model function for h_{17}, we use a Gaussian

$$h_{17}(\omega_1) = \frac{2\lambda_2}{\sqrt{2\pi}\sigma} e^{-\frac{\omega_1^2}{2\sigma^2}}\,, \tag{21}$$

with $\sigma = 0.5$ GeV as a typical hadronic scale. This model function has all properties one derives from the explicit HQET matrix element. Calculating the convolution integral, we find

$$\mathcal{F}^s_{17\rm Gaussian} = \frac{1}{m_b}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}} e_c\,(-0.252\ {\rm GeV})\,. \tag{22}$$

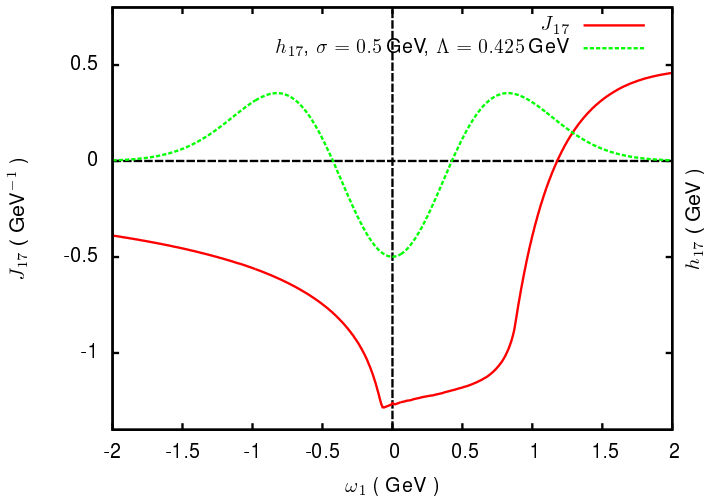


Fig. 1. J_{17} for $q^2_{\rm min} = 1$ GeV^2 and $q^2_{\rm max} = 6$ GeV^2 together with the model function of Eq. (24).

Using a smaller $\sigma = 0.1$ GeV leads to -0.304 GeV. We can express our numbers in percentages

$$\mathcal{F}^s_{17\rm exp} \approx +1.9\%\,. \tag{23}$$

Using a Gaussian for the soft function only yields negative numbers (positive percentages) for the expression in the square brackets. Thus, this model function does not yield to a conservative bound on the size of $\mathcal{F}^q_{17}$. The usage of the same function as in Ref. [14]

$$h_{17}(\omega_1) = \frac{2\lambda_2}{\sqrt{2\pi}\sigma}\frac{\omega_1^2 - \Lambda^2}{\sigma^2 - \Lambda^2} e^{-\frac{\omega_1^2}{2\sigma^2}}\,, \tag{24}$$

also leads to positive numbers for this expression. If Λ and σ are chosen of the order of $\Lambda_{\rm QCD}$, again, all general properties derived for h_{17} are fulfilled. For a parameter choice of $\sigma = 0.5$ GeV and $\Lambda = 0.425$ GeV, one finds

$$\mathcal{F}_{17}^s = \frac{1}{m_b}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}} e_c\,(+0.075\ \text{GeV})\,. \tag{25}$$

For a different parameter choice, $\Lambda = 0.575$ GeV, on the other hand,

$$\mathcal{F}_{17}^s = \frac{1}{m_b}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}} e_c\,(-0.532\ \text{GeV}) \tag{26}$$

which leads us to the conservative estimates for both inclusive modes

$$\mathcal{F}_{17}^s \in [-0.5, +3.4]\%\,, \qquad \mathcal{F}_{17}^d \in [-0.6, +4.1]\%\,. \tag{27}$$

A reduction of the separation between Λ and σ could lead to larger values, but the reduction would also lead to an increase of the values of the soft function to outside the hadronic range.

The relation of our result to the Voloshin term can be easily established: For the decay $\bar{B} \to X_s\gamma$, one can expand our non-local contribution to local operators *if* the charm quark is treated as heavy. In this expansion, the first term is the dominating one [8, 15–17] which corresponds to the so-called Voloshin term. This non-perturbative correction is suppressed by λ_2/m_c^2. But if one assumes the charm to scale as $m_c^2 \sim \Lambda_{\rm QCD} m_b$, which appears to be a more reasonable assumption, one has to describe the charm penguin contribution by the matrix element of a non-local operator [14].

This also holds true for the decay $\bar{B} \to X_s\ell^+\ell^-$. In Ref. [17], the local Voloshin term was derived from a local expansion assuming $\Lambda_{\rm QCD} m_b/m_c^2$ to be small. We rederive the leading term (according to our power counting) of their result from our general result above if we observe the following assumptions.

If one uses a Gaussian as a shape function and assumes this function to be narrow enough, one can expand the part of the integrand in square brackets in Eq. (17) around $\omega_1 = 0$ [2]

$$\begin{aligned}[\dots] &= \omega_1^2 \bar{n}\cdot q \left[\frac{1}{2\bar{n}\cdot q^2} - \frac{2m_c^2}{\bar{n}\cdot q^2}\frac{1}{4m_c^2 - m_b\bar{n}\cdot q}\sqrt{\frac{4m_c^2 - m_b\bar{n}\cdot q}{m_b\bar{n}\cdot q}}\arctan\frac{1}{\sqrt{\frac{4m_c^2 - m_b\bar{n}\cdot q}{m_b\bar{n}\cdot q}}}\right] \\ &= -\frac{m_b\omega_1^2}{12m_c^2}F_{\rm V}(r)\,,\end{aligned} \tag{28}$$

[2] The variable $(m_b\omega_1)/m_c^2$ corresponds to the parameter $t = k\cdot q/m_c^2$ in Ref. [17] which is used there as an expansion parameter. Note that we have already expanded in $\bar{n}\cdot q/m_b$ within the non-local contribution in order to single out the $1/m_b$ term.

where $F_{\rm V}(r)$ is defined in Eq. (4) of [17] with $r = q^2/(4m_c^2)$ (which is different from the function F defined in Eq. (12)). This corresponds exactly to the leading power in $1/m_b$ of the Voloshin term for $\bar{B} \to X_s\ell^+\ell^-$ given in Ref. [17]. For $F_{\rm V}(0) = 1$, this results in the Voloshin term for $\bar{B} \to X_s\gamma$. Numerically, this approach is not advisable. Evaluating the leading $1/m_b$ Voloshin term yields

$$\begin{aligned}\mathcal{F}^s_{\text{Voloshin},m_b^{-1}} &= \frac{1}{m_b}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}} e_c \int\limits_{\frac{q^2_{\rm min}}{M_B}}^{\frac{q^2_{\rm max}}{M_B}} \frac{{\rm d}\bar{n}\cdot q}{\bar{n}\cdot q}\left(-\frac{m_b 2\lambda_2}{12m_c^2}\right) F_{\rm V}\left(\frac{m_b \bar{n}\cdot q}{4m_c^2}\right) \\ &= \frac{1}{m_b}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}} e_c\,(-0.306\ \text{GeV})\,. \end{aligned} \tag{29}$$

Compared to our final estimate, we find that the Voloshin term significantly underestimates the possible charm contributions.

For comparison, we finally consider the higher orders in $1/m_b$ of the Voloshin term derived in Ref. [17]. They are given by

$$\begin{aligned}\mathcal{F}^s_{\text{Voloshin}} &= \frac{1}{m_b}\frac{C_1(\mu)}{C_{\rm OPE}} e_c \int\limits_{\frac{q^2_{\rm min}}{M_B}}^{\frac{q^2_{\rm max}}{M_B}} \frac{{\rm d}\bar{n}\cdot q}{\bar{n}\cdot q}\left(-\frac{m_b 2\lambda_2}{12m_c^2}\right) F_{\rm BI}\left(\frac{m_b \bar{n}\cdot q}{4m_c^2}\right) \\ &\times \left[C_{7\gamma}(\mu)\left(1 + 6\frac{\bar{n}\cdot q}{m_b} - \left(\frac{\bar{n}\cdot q}{m_b}\right)^2\right) + C_9(\mu)\left(2\frac{\bar{n}\cdot q}{m_b} + \left(\frac{\bar{n}\cdot q}{m_b}\right)^2\right)\right] \\ &= \frac{1}{m_b}\frac{C_1(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}}\, e_c\,(+0.481\ \text{GeV})\,. \end{aligned} \tag{30}$$

We note that the higher order in $\bar{n}\cdot q$ are numerically small but the first subleading C_9 is numerically significant taking into account $|C_{9/10}| \sim 13|C_{7\gamma}|$. We also find that these subleading contributions change the sign

$$\mathcal{F}^s_{\text{Voloshin},m_b^{-1}} \approx +1.9\%\,, \qquad \mathcal{F}^s_{\text{Voloshin}} \approx -3.0\%\,. \tag{31}$$

Obviously, within the Voloshin term, there is a cancellation between the $C_{7\gamma}$ and the subleading C_9 contribution. However, in our analysis which uses $m_c^2 \sim m_b\Lambda_{\rm QCD}$, both terms are smeared out by different shape functions and, thus, the corresponding uncertainties have to be added up. These findings call for a calculation of the resolved contributions to the order of $1/m_b^2$ to collect all numerically relevant contributions [18].

The relative uncertainty due to the interference of $\mathcal{O}_{7\gamma}$ and $\mathcal{O}_{8g}$ consists of two contributions $\mathcal{F}_{78}^{(\mathrm{b})}$ and $\mathcal{F}_{78}^{(\mathrm{c})}$. From the explicit form of the shape functions given in Eqs. (3) and (7), one can deduce (see Ref. [14]) that the soft functions $\bar{g}_{78}$ and $g_{78}^{(1,5)}$ have support for $-\infty < \omega \leq \bar{\Lambda}$ and $-\infty < \omega_{1,2} < \infty$, and

$$\int_{-\infty}^{\bar{\Lambda}} \mathrm{d}\omega \left[g_{78}^{(1,5)}(\omega, \omega_1, \omega_2, \mu) \right]^* = \int_{-\infty}^{\bar{\Lambda}} \mathrm{d}\omega \, g_{78}^{(1,5)}(\omega, \omega_2, \omega_1, \mu) \,. \tag{32}$$

From PT invariance of the matrix element, one can draw the consequence that all the shape functions are real implying that the functions

$$h_{78}^{(1,5)} := \int_{-\infty}^{\bar{\Lambda}} \mathrm{d}\omega \, g_{78}^{(1,5)}(\omega, \omega_1, \omega_2) \tag{33}$$

are symmetric under the exchange of ω_1 and ω_2. Moreover, one also derives from the explicit form of the shape functions that

$$\int \mathrm{d}\omega \, \bar{g}_{78}(\omega, \omega_1, \omega_2) = \int \mathrm{d}\omega \, \bar{g}_{78}^{\mathrm{cut}}(\omega, \omega_1, \omega_2) \,. \tag{34}$$

Thus, the contribution $\mathcal{F}_{78}^{(\mathrm{b})}$ vanishes. The other contribution is given by

$$\begin{aligned} \mathcal{F}_{78}^{(\mathrm{c})} = \frac{1}{m_b} \frac{C_{8g}(\mu) C_{7\gamma}(\mu)}{C_{\mathrm{OPE}}} 4\pi\alpha_{\mathrm{s}}(\mu) \, \mathrm{Re} \int_{\frac{q^2_{\mathrm{min}}}{M_B}}^{\frac{q^2_{\mathrm{max}}}{M_B}} \frac{\mathrm{d}\bar{n}\cdot q}{\bar{n}\cdot q} \int s \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \, \frac{1}{\omega_1 - \omega_2 + \bar{n}\cdot q + i\epsilon} \\ \times \left[\left(\frac{1}{\omega_1 + \bar{n}\cdot q + i\epsilon} + \frac{1}{\omega_2 - \bar{n}\cdot q - i\epsilon} \right) h_{78}^{(1)}(\omega_1, \omega_2, \mu) \right. \\ \left. - \left(\frac{1}{\omega_1 + \bar{n}\cdot q + i\epsilon} - \frac{1}{\omega_2 - \bar{n}\cdot q - i\epsilon} \right) h_{78}^{(5)}(\omega_1, \omega_2, \mu) \right] . \end{aligned} \tag{35}$$

In the vacuum insertion approximation, we find for both shape functions [14]

$$h_{78}^{(1,5)}(\omega_1, \omega_2, \mu) = -e_{\mathrm{spec}} \frac{F^2(\mu)}{8} \left(1 - \frac{1}{N_{\mathrm{c}}^2} \right) \phi_+^B(-\omega_1, \mu) \, \phi_+^B(-\omega_2, \mu) \,, \tag{36}$$

where $F = f_B \sqrt{M_B}$, e_{spec} is the charge of the B-meson spectator quark, and ϕ_+^B is the light-cone distribution amplitude (LCDA). Since the LCDAs

vanish for $\omega_i \to 0$, the ω_i integrals yield

$$-e_{\rm spec}\,\frac{F^2(\mu)}{8}\left(1-\frac{1}{N_c^2}\right)(-2)\,P\int\frac{{\rm d}\omega_1}{\omega_1-\bar{n}\cdot q}\,\phi_+^B(-\omega_1)$$
$$\times\,P\int\frac{{\rm d}\omega_2}{\omega_1-\omega_2-\bar{n}\cdot q}\,\phi_+^B(-\omega_2)\,.$$

In order to estimate the magnitude of this contribution, we use the model for the LCDAs given in Ref. [20]

$$\phi_+^B(\omega)=\frac{\omega}{\omega_0}e^{-\omega/\omega_0}\,, \tag{37}$$

where $\omega_0=\frac{2}{3}\bar{\Lambda}$. Then the principal value integrals of (37) can be computed analytically and we find for the uncertainty

$$\mathcal{F}_{78}^{(c)}=\frac{1}{m_b}\,\frac{C_{8g}(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}}\,4\pi\alpha_s(\mu)\,e_{\rm spec}\int\limits_{\frac{q^2_{\rm min}}{M_B}}^{\frac{q^2_{\rm max}}{M_B}}\frac{{\rm d}\bar{n}\cdot q}{\bar{n}\cdot q}\,\frac{F^2(\mu)}{4}\left(1-\frac{1}{N_c^2}\right)\frac{1}{4\omega_0^3}$$
$$\times\left[-2\omega_0-(2\bar{n}\cdot q+\omega_0)e^{\frac{\bar{n}\cdot q}{\omega_0}}\,{\rm Ei}\left(-\frac{\bar{n}\cdot q}{\omega_0}\right)+\omega_0 e^{-\frac{\bar{n}\cdot q}{\omega_0}}\,{\rm Ei}\left(\frac{\bar{n}\cdot q}{\omega_0}\right)\right]\,, \tag{38}$$

where the exponential integral is defined as

$${\rm Ei}(z)=-P\int\limits_{-z}^{\infty}\frac{e^{-t}}{t}\,{\rm d}t\,. \tag{39}$$

Using our standard set of parameters (see Ref. [10]), we integrate (38) numerically and find

$$\mathcal{F}_{78}^{(c)}\in\frac{1}{m_b}\,\frac{C_{8g}(\mu)C_{7\gamma}(\mu)}{C_{\rm OPE}}\,4\pi\alpha_s(\mu)\,e_{\rm spec}\,[0.058\ {\rm GeV},0.068\ {\rm GeV}]\,. \tag{40}$$

We note that this estimate does not include any uncertainty due to the use of the VIA in Eq. (36). We can again express our numbers in percentages

$$\mathcal{F}_{78}^{(c)}\in[-0.2,-0.1]\%\,. \tag{41}$$

Within the interference of $\mathcal{O}_{8g}$ with $\mathcal{O}_{8g}$, the shape function $\bar{g}_{88}$ is more complicated than the ones in the previous cases, because not much is known about it. But from the explicit form and PT invariance, one

can conclude that $\bar{g}_{88}$ is real. Moreover, one can also show in that the convolution with the hard-collinear function is real (see Ref. [14]). With $\bar{h}_{88} := \int \mathrm{d}\omega \bar{g}_{88}(\omega, \omega_1, \omega_2, \mu)$, we find for the convolution integral

$$\mathcal{F}_{88} = \frac{1}{m_b} \frac{C_{8g}(\mu) C_{8g}(\mu)}{C_{\mathrm{OPE}}} 4\pi\alpha_{\mathrm{s}}(\mu)\, e_{d,s}^2\, \mathrm{Re} \int\limits_{\frac{q^2_{\min}}{M_B}}^{\frac{q^2_{\max}}{M_B}} \frac{\mathrm{d}\bar{n}\cdot q}{\bar{n}\cdot q} \times \int \frac{\mathrm{d}\omega_1}{\omega_1 + \bar{n}\cdot q + i\epsilon} \frac{\mathrm{d}\omega_2}{\omega_2 + \bar{n}\cdot q - i\epsilon} 2\bar{h}_{88}(\omega_1, \omega_2, \mu)\,. \tag{42}$$

We cannot arrive at any stricter estimation from the convolution, however, we have been able to separate factors like $e_{d,s}^2$ *etc.* Thus, we estimate

$$\Lambda(\mu) = \mathrm{Re} \int\limits_{\frac{q^2_{\min}}{M_B}}^{\frac{q^2_{\max}}{M_B}} \frac{\mathrm{d}\bar{n}\cdot q}{\bar{n}\cdot q} \int \frac{\mathrm{d}\omega_1}{\omega_1 + \bar{n}\cdot q + i\epsilon} \frac{\mathrm{d}\omega_2}{\omega_2 + \bar{n}\cdot q - i\epsilon} 2\bar{h}_{88}(\omega_1, \omega_2, \mu) \tag{43}$$

to be of $\mathcal{O}(\Lambda_{\mathrm{QCD}})$. So we assume $0\ \mathrm{GeV} < \Lambda(\mu) < 1\ \mathrm{GeV}$. Compared to the estimates found in Eqs. (25) and (40), this leads to a rather conservative estimate of the convolution integral

$$\mathcal{F}_{88} \in [0, 0.5]\%\,. \tag{44}$$

Our final estimates of the resolved contributions to the leading order,

$$\begin{aligned} \mathcal{F}_{17}^{s} &\in [-0.5, +3.4]\%\,, & \mathcal{F}_{17}^{d} &\in [-0.6, +4.1]\%\,, \\ \mathcal{F}_{78}^{d,s} &\in [-0.2, -0.1]\%\,, & \mathcal{F}_{88}^{d,s} &\in [0, 0.5]\% \end{aligned}$$

can now be summed up using the scanning method. Our final results are

$$\mathcal{F}_{1/m_b}^{d} \in [-0.8, +4.5]\,, \qquad \mathcal{F}_{1/m_b}^{s} \in [-0.7, +3.8]\,. \tag{45}$$

As discussed, this estimate of the resolved contributions represents an irreducible theoretical uncertainty of the total rate of the inclusive decays $\bar{B} \to X_d \ell^+ \ell^-$ and $\bar{B} \to X_d \ell^+ \ell^-$.

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