REPARAMETRIZATION INVARIANCE AND PARTIAL RESUMMATION OF THE HEAVY-QUARK EXPANSION*

K. Keri Vos

Theoretische Physik 1, Naturwissenschaftlich-Technische Fakultät
Universität Siegen, 57068 Siegen, Germany

(Received May 28, 2018)

Reparametrization invariance (RPI) relates different orders in the heavy-quark expansion. We discuss the implications of RPI for total rates of inclusive decays. The obtained results are manifestly RPI, allowing for a re-summation of higher-order terms in the heavy-quark expansion, which reduces the number of independent parameters.

DOI:10.5506/APhysPolB.49.1393

1. Introduction

The Heavy-Quark Expansion (HQE) has become an indispensable tool to study inclusive decays [1], both for the extraction of the CKM parameters from inclusive semileptonic processes as well as in the search for physics beyond the Standard Model. The HQE has been studied extensively in both the perturbative regime as well as in the non-perturbative regime to achieve the highest possible accuracy on *e.g.* the CKM element V_{cb} [2].

In defining the HQE, a time-like vector v has to be introduced. The choice for v is not unique and the final result should be v-independent. This reparametrization invariance (RPI) has been studied in detail [3–8]. In fact, it has been noted that RPI relates different orders in the HQE and that the coefficients of operators of different order in the $1/m$ expansion are related exactly by RPI [7, 8].

We recently extended these existing works [9], and showed that RPI allows for a re-summation of towers of operators. The final result can be written in terms of matrix elements of operators and states defined in full QCD. Our manifest RPI formalism also reduces the number of independent parameters, which in the standard formulation is only implicitly realized. Here, we summarize this recent work [9].

* Presented at the Cracow Epiphany Conference on Advances in Heavy Flavour Physics, Kraków, Poland, January 9–12, 2018.

2. Reparametrization invariance of the Heavy-Quark Expansion

We start by setting up the Operator Product Expansion (OPE) with fields and states defined in full QCD [10]. The decay rate of a heavy (ground-state) B meson, mediated by an effective Hamiltonian density $\mathcal{H}_{\rm eff}$ is

$$\begin{aligned} \Gamma &\propto \sum_X (2\pi)^4 \delta^4(P_B - P_X) |\langle X|\mathcal{H}_{\rm eff}|B(v)\rangle|^2 \\ &= \int \mathrm{d}^4x \, \langle B(v)|\mathcal{H}_{\rm eff}(x)\mathcal{H}^\dagger_{\rm eff}(0)|B(v)\rangle \\ &= 2\mathrm{Im} \int \mathrm{d}^4x \, \langle B(v)| T\left\{\mathcal{H}_{\rm eff}(x)\mathcal{H}^\dagger_{\rm eff}(0)\right\}|B(v)\rangle \,, \end{aligned} \tag{1}$$

where we used the optical theorem to relate the matrix elements to the time-ordered product. This matrix element still contains the heavy-quark mass, which we make explicit by applying a field-redefinition

$$Q(x) = \exp(-im(v\cdot x))Q_v(x)\,, \tag{2}$$

which splits the heavy-quark momentum into $p_B = m_b v + k$, where the residual momentum $k \sim iD\, Q_v$. Note that $Q_v(x)$ is still the field defined in full QCD. The equations of motion give

$$\begin{aligned} Q_v &= \not{v} Q_v + \frac{i\not{D}}{m_b} Q_v \,, \\ (ivD)Q_v &= -\frac{1}{2m_b}\left(i\not{D}\right)\left(i\not{D}\right)Q_v = -\frac{1}{2m_b}(iD)^2 Q_v - \frac{1}{2m_b}(\sigma\cdot G)Q_v \,, \end{aligned} \tag{3}$$

where $(\sigma\cdot G) \equiv (-i\sigma_{\mu\nu})(iD^\mu)(iD^\nu)$.

Applying this field-redefinition gives

$$\begin{aligned} &\int \mathrm{d}^4x \, \langle B(v)|T\left\{\mathcal{H}_{\rm eff}(x)\mathcal{H}^\dagger_{\rm eff}(0)\right\}|B(v)\rangle \\ &= \int \mathrm{d}^4x \, e^{-im_b v\cdot x} \langle B(v)|T\left\{\widetilde{\mathcal{H}}_{\rm eff}(x)\widetilde{\mathcal{H}}^\dagger_{\rm eff}(0)\right\}|B(v)\rangle \,, \end{aligned} \tag{4}$$

where $\widetilde{\mathcal{H}}_{\rm eff}$ is the effective Hamiltonian for the field $\bar{Q}_v$. The OPE, expressed as a series in inverse powers of m_b, takes the form of

$$R(S) = \int \mathrm{d}^4x e^{-im(S\cdot x)} T\left[O_1(x)O_2(0)\right] = \sum_{n,i} C_i^{(n)}(S)\mathcal{O}_i^{(n)} \,, \tag{5}$$

where $S = v - \frac{q}{m_b}$ and q is the momentum transfer. The $O_{1,2}$ are local renormalized operators and $\mathcal{O}_i^{(n)}$ are local operators ordered by increasing dimension and $C_i^{(n)}$ the corresponding Wilson coefficients.

The total rate is then obtained via the optical theorem by taking a forward matrix element of R with the initial state $|B(p_B)\rangle$

$$2m_B \varGamma = \langle R\rangle \equiv \langle B(p_B)|R|B(p_B)\rangle \,. \tag{6}$$

At tree level, the operators $\mathcal{O}_i^{(n)}$ can be written in terms of the field Q_v and chains of covariant derivates

$$R(S) = \sum_{n=0}^{\infty}\sum_{\varGamma} C^{(n,\varGamma)}_{\mu_1\ldots\mu_n}(S)\otimes \bar{Q}_v(iD_{\mu_1}\ldots iD_{\mu_n})Q_v \,, \tag{7}$$

where the sum $\varGamma$ runs over the 16 Dirac matrices and

$$C^{(n,\varGamma)}_{\mu_1\ldots\mu_n} = \tfrac{1}{4}\mathrm{Tr}\left[C^{(n)}_{\mu_1\ldots\mu_n}\right] . \tag{8}$$

Relation (7) is RPI as long as the full sum is taken into account. A reparametrization transformation δ_{RP}, corresponding to a infinitesimal change $v_\mu \to v_\mu + \delta v_\mu$, gives

$$\begin{aligned}\delta_{\mathrm{RP}} v_\mu &= \delta v_\mu \,,\\ \delta_{\mathrm{RP}} iD_\mu &= -m_b \delta v_\mu \,,\end{aligned} \tag{9}$$

with $v\cdot\delta v = 0$. The reparametrization transformation thus relates subsequent orders in the $1/m_b$ expansion. More explicitly, applying δ_{RP} to $R(S)$, gives

$$\begin{aligned}\delta_{\mathrm{RP}} R(S) &= 0 = \sum_{n=0}^{\infty}\left[\delta_{\mathrm{RP}} C^{(n)}_{\mu_1\ldots\mu_n}\right]\bar{Q}_v(iD^{\mu_1}\ldots iD^{\mu_n})Q_v\\ &\quad + \sum_{n=0}^{\infty} C^{(n)}_{\mu_1\ldots\mu_n}\left[\delta_{\mathrm{RP}}\bar{Q}_v(iD^{\mu_1}\cdots iD^{\mu_n})Q_v\right]\\ &= \sum_{n=0}^{\infty}\left[\delta_{\mathrm{RP}} C^{(n)}_{\mu_1\cdots\mu_n}\right]\bar{Q}_v(iD^{\mu_1}\cdots iD^{\mu_n})Q_v\\ &\quad - m_b\sum_{n=0}^{\infty} C^{(n)}_{\mu_1\ldots\mu_n}\left[\delta v^{\mu_1}\,\bar{Q}_v(iD^{\mu_2})\ldots(iD^{\mu_n})Q_v\right.\\ &\quad \left.\ldots + \delta v^{\mu_n}\,\bar{Q}_v(iD^{\mu_1})\cdots(iD^{\mu_{n-1}})Q_v\right] .\end{aligned} \tag{10}$$

This shows that $R(S)$ can only be an RPI quantity if there is a cancellation between different orders in the OPE. This is achieved only if the coefficients satisfy

$$\delta_{\mathrm{RP}} C^{(n)}_{\mu_1\cdots\mu_n}(S) = m\,\delta v^\alpha\left(C^{(n+1)}_{\alpha\mu_1\ldots\mu_n}(S) + \ldots + C^{(n+1)}_{\mu_1\ldots\mu_n\alpha}(S)\right) . \tag{11}$$

We will explicitly show how we use this remarkable relation, to re-sum the different operators in the HQE.

2.1. Scalar toy-model

To simplify the discussion, we first consider a toy-model with a scalar quark field ϕ_v, which satisfies the equation of motion

$$\left[(iD)^2 - m^2\right] \phi_v = 0\,. \tag{12}$$

We define this scalar quark to decay into a lighter scalar quark and a particle without QCD interactions. We then redefine the scalar field using

$$\phi(x) = \frac{1}{\sqrt{2m_b}} \exp\left[-im_b(vx)\right] \phi_v(x)\,, \tag{13}$$

where we have introduced a normalization factor that makes the mass dimension of ϕ_v different from that of ϕ. Considering now the total decay rate, we find

$$\begin{aligned} R \;=\; & c^{(0)}\phi_v^\dagger\phi_v + c_\mu^{(1)}\,\phi_v^\dagger(iD^\mu)\phi_v + c_{\mu\nu}^{(2)}\,\phi_v^\dagger(iD^\mu)(iD^\nu)\phi_v \\ & + c_{\mu\alpha\nu}^{(3)}\,\phi_v^\dagger(iD^\mu)(iD^\alpha)(iD^\nu)\phi_v + \ldots\,, \end{aligned} \tag{14}$$

where the coefficients $c^{(n)}$ depend only on v. These coefficients can be decomposed into linear combinations of all possible tensor structures containing v_μ and $g_{\mu\nu}$ (since $v^2 = 1$). We write

$$c^{(0)}(v) \;=\; a^{(0)}\,, \tag{15}$$

$$c_\mu^{(1)}(v) \;=\; a^{(1)}v_\mu\,, \tag{16}$$

$$c_{\mu\nu}^{(2)}(v) \;=\; a^{(2)}g_{\mu\nu} + b^{(2)}v_\mu v_\nu\,, \tag{17}$$

$$c_{\mu\alpha\nu}^{(3)}(v) \;=\; x_1^{(3)}v_\alpha g_{\mu\nu} + x_2^{(3)}v_\nu g_{\mu\alpha} + x_3^{(3)}v_\mu g_{\nu\alpha} + x_4^{(3)}v_\mu v_\alpha v_\nu\,. \tag{18}$$

For the leading-order term, we find $\delta_{\rm RP}\, c^{(0)} = 0$, such that the decay rate

$$\Gamma = \frac{1}{2m_B}\langle R\rangle = a^{(0)}\frac{1}{2m_B}\left\langle \phi_v^\dagger\phi_v\right\rangle\,. \tag{19}$$

This leading-order matrix element is normalized using the conserved vector current for scalar quarks as

$$\left\langle \phi_v^\dagger\phi_v\right\rangle \equiv 2M_B\mu_3 = 2M_B\left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) = 2m_b\left\langle \phi^\dagger\phi\right\rangle\,, \tag{20}$$

which re-absorbes the well-known kinetic term μ_π^2 in μ_3. We note that this matrix element is equivalent to the full QCD operator (with the normalization factor following from Eq. (13)).

Applying the RPI equation (11) to the first-order term gives

$$\delta_{\mathrm{RP}} c_\mu^{(1)} = a^{(1)} \delta v_\mu = m_b \, \delta v^\alpha \left(c_{\alpha\mu}^{(2)} + c_{\mu\alpha}^{(2)} \right) = 2 m_b \delta v_\mu a^{(2)} \,, \tag{21}$$

leading to a relation between $a^{(1)}$ and $a^{(2)}$. Combining these terms gives

$$a^{(1)} \phi_v^\dagger \left((ivD) + \frac{1}{2m_b} (iD)^2 \right) \phi_v = 0 \,, \tag{22}$$

which vanishes by the equation of motion. In fact, this specific combination is always RPI, as we shall demonstrate for the higher-order terms. Since this term vanishes, there is no first-order ($1/m_b$ term), thus reproducing the well-known result.

Similarly, for the second-order term, we obtain

$$\begin{aligned} \delta_{\mathrm{RP}} c_{\mu\nu}^{(2)} &= b^{(2)} (\delta v_\mu v_\nu + v_\mu \delta v_\nu) = m_b \, \delta v^\alpha \left(c_{\mu\nu\alpha}^{(3)} + c_{\mu\alpha\nu}^{(3)} + c_{\alpha\mu\nu}^{(3)} \right) \\ &= m_b \left(x_1^{(3)} + 2 x_2^{(3)} \right) \delta v_\mu v_\nu + m_b \left(x_1^{(3)} + 2 x_3^{(3)} \right) \delta v_\nu v_\mu \,. \end{aligned} \tag{23}$$

Collecting the terms proporational to $b^{(2)}$ gives

$$b^{(2)} \phi_v^\dagger \left((ivD) + \frac{1}{2m_b} (iD)^2 \right)^2 \phi_v = 0 \,, \tag{24}$$

which vanishes again by the equation of motion, thus leaving also no second-order term. In fact, this observation allows us to generalize our formalism. The genuine n^{th} order terms can be obtained using

$$\delta v^\alpha \left(\tilde{c}_{\mu_1 \dots \mu_n \alpha}^{(n)} + \dots \tilde{c}_{\alpha \mu_1 \dots \mu_n}^{(n)} \right) = 0 \,, \tag{25}$$

since coefficients that do not satisfy this relation will be related by RPI to lower-order coefficients.

Continuing in a similar way with the higher-order terms, we find that the only new third-order term is the Darwin term

$$\phi_v^\dagger \left[iD_\mu, \left[(ivD), iD^\mu \right] \right] \phi_v \,. \tag{26}$$

In fact, when considering the fourth-order ($1/m_b^4$) terms, we find that (ivD) in this operator gets completed to the RPI combination

$$\left((ivD) + \frac{1}{2m_b} (iD)^2 \right) \,. \tag{27}$$

So we redefine the Darwin term as

$$2M_B\rho_D^3 = \left\langle \phi_v^\dagger \left[iD_\mu, \left[(ivD) + \frac{1}{2m_b}(iD)^2, iD^\mu\right]\right] \phi_v \right\rangle . \tag{28}$$

At $1/m_b^4$, we find only two independent RPI parameters, which we define as [9]

$$\begin{aligned} 2M_B r_G^4 &\equiv \left\langle \phi_v^\dagger \left[iD_\mu, iD_\nu\right]\left[iD^\mu, iD^\nu\right] \phi_v \right\rangle \propto \left\langle \vec{E}^2 - \vec{B}^2 \right\rangle , \\ 2M_B r_E^4 &\equiv \left\langle \phi_v^\dagger \left[(ivD), iD_\mu,\right]\left[(ivD), iD^\mu\right] \phi_v \right\rangle \propto \left\langle \vec{E}^2 \right\rangle . \end{aligned} \tag{29}$$

We have chosen these parameters in such a way that they have a clear physical interpretation in terms of the chromo-electric and magnetic fields. We thus find only four independent parameters μ_3, ρ_D^3 and r_G^4, r_E^4 up to $1/m_b^4$ in our scalar toy-model when considering total decay rates. This is a reduction compared to the 6 parameters $(\mu_\pi^2, \rho_D^3, m_1, m_2, m_3, m_4)$ previously found [11]. The four independent parameters depend in a nontrivial way on the mass m_b and actually contain a re-summation of higher-order terms in the HQE expansion dictated by RPI. The final parameters are RPI, which can be made manifest by re-writing the matrix elements in terms of full QCD states and propagators [9].

2.2. Total rate for real quarks

For real quarks the discussion is similar, except that the coefficients now depend also on the Dirac structure. See [9] for a detailed description. For real quarks, the normalization gets an additional term

$$\langle B|\bar{b}_v b_v|B\rangle \equiv 2M_B\mu_3 = 2M_B\left(1 - \frac{\hat{\mu}_\pi^2 - \hat{\mu}_G^2}{2m_b^2}\right) , \tag{30}$$

where

$$2M_B\mu_G^2 \equiv \frac{1}{2}\langle B|\bar{b}_v\left[iD_\mu, iD_\nu\right](-i\sigma^{\mu\nu})b_v|B\rangle . \tag{31}$$

Including the normalization μ_3, μ_G^2 and ρ_D^3, we find in total 8 independent parameters up to $1/m_b^4$. At $1/m_b^4$, we define besides the r_E^4 and r_G^4 in Eq. (29) [9]

$$\begin{aligned} 2m_B s_B^4 &= \left\langle \bar{Q}_v \left[(iD_\mu), (iD_\alpha)\right]\left[(iD^\mu), (iD_\beta)\right]\left(-i\sigma^{\alpha\beta}\right) Q_v \right\rangle \propto \left\langle \vec{\sigma}\cdot\vec{B}\times\vec{B} \right\rangle , \\ 2m_B s_E^4 &= \left\langle \bar{Q}_v \left[(ivD), (iD_\alpha)\right]\left[(ivD), (iD_\beta)\right]\left(-i\sigma^{\alpha\beta}\right) Q_v \right\rangle \propto \left\langle \vec{\sigma}\cdot\vec{E}\times\vec{E} \right\rangle , \\ 2m_B s_{qB}^4 &= \left\langle \bar{Q}_v \left[iD_\mu, \left[iD^\mu, \left[iD_\alpha, iD_\beta\right]\right]\right]\left(-i\sigma^{\alpha\beta}\right) Q_v \propto \left\langle \Box\, \vec{\sigma}\cdot\vec{B}\right\rangle\right\rangle , \end{aligned} \tag{32}$$

where we also indicated their physical interpretation. Again, comparing with [11] shows a reduction of independent parameters.

2.3. Example: Tree level $B \to X_s\gamma$

To demonstrate our method, we briefly discuss the power corrections to $B \to X_s\gamma$. Focussing on the tree-level contribution only and considering only the contribution from

$$\frac{\lambda}{2}\bar{s}\sigma_{\mu\nu}(1-\gamma_5)bF^{\mu\nu}\,, \tag{33}$$

we find for massless s quarks

$$T = -2\lambda^2\bar{b}_v\left[\sigma_{\mu\alpha}q^\alpha\left(\frac{1}{\not{S}+i\not{D}}\right)\sigma_{\nu\beta}q^\beta\frac{1}{q^2}\right]b_v\,, \tag{34}$$

where $S = p - q$, and q is the photon momentum. Expanding the s quark propagator, we find for the total rate [9]

$$\Gamma = \frac{\lambda^2 m_b^3}{4\pi}\left[\mu_3 - \frac{2\mu_G^2}{m_b^2} - \frac{10\rho_D^3}{3m_b^3} - \frac{1}{3m_b^4}\left(4r_G^4 + 4r_E^4 + \frac{s_{qB}^4}{4} - 2s_E^4\right)\right]\,, \tag{35}$$

which indeed has the expected reduced number of coefficients.

3. Conclusion

We recently studied reparametrization invariance of the heavy-quark expansion, using a manifestly RPI set-up [9]. This set-up allows us to re-sum towers of operators and reduces the number of independent parameters for the total rate. Our results are derived at tree level, but most of the relations hold to all orders in α_s. However, when considering radiative corrections, the OPE in Eq. (7) should be extended to also include four-fermion operators such as $(\bar{b}q\ \bar{q}b)$. In addition, radiative corrections to the color-octet operators (those containing the chromo-electric field) will also introduce additional color structures not present at tree level [13, 14]. Finally, the work on the extension of this set-up to differential decay rates is in progress.

REFERENCES

[1] A textbook presentation can be found in: A.V. Manohar, M.B. Wise, *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.* **10**, 1 (2000).

[2] A. Alberti, P. Gambino, K.J. Healey, S. Nandi, *Phys. Rev. Lett.* **114**, 061802 (2015) [arXiv:1411.6560 [hep-ph]].

[3] M.J. Dugan, M. Golden, B. Grinstein, *Phys. Lett. B* **282**, 142 (1992).

[4] Y.Q. Chen, *Phys. Lett. B* **317**, 421 (1993).

[5] N. Brambilla, D. Gromes, A. Vairo, *Phys. Lett. B* **576**, 314 (2003) [arXiv:hep-ph/0306107].

[6] J. Heinonen, R.J. Hill, M.P. Solon, *Phys. Rev. D* **86**, 094020 (2012) [arXiv:1208.0601 [hep-ph]].

[7] M.E. Luke, A.V. Manohar, *Phys. Lett. B* **286**, 348 (1992) [arXiv:hep-ph/9205228].

[8] A.V. Manohar, *Phys. Rev. D* **82**, 014009 (2010) [arXiv:1005.1952 [hep-ph]].

[9] T. Mannel, K.K. Vos, arXiv:1802.09409 [hep-ph].

[10] I.I.Y. Bigi, M.A. Shifman, N.G. Uraltsev, A.I. Vainshtein, *Phys. Rev. D* **52**, 196 (1995) [arXiv:hep-ph/9405410].

[11] T. Mannel, S. Turczyk, N. Uraltsev, *J. High Energy Phys.* **1011**, 109 (2010) [arXiv:1009.4622 [hep-ph]].

[12] T. Mannel, A.V. Rusov, F. Shahriaran, *Nucl. Phys. B* **921**, 211 (2017) [arXiv:1702.01089 [hep-ph]].

[13] A. Kobach, S. Pal, *Phys. Lett. B* **772**, 225 (2017) [arXiv:1704.00008 [hep-ph]].

[14] A. Gunawardana, G. Paz, *J. High Energy Phys.* **1707**, 137 (2017) [arXiv:1702.08904 [hep-ph]].