INFLUENCE OF RELATIVISTIC VISCOSITY IN THE THERMODYNAMICAL QUANTITIES IN THE ACCRETION DISKS AROUND THE ROTATING BLACK HOLES

Mahboobe Moeen

Department of Sciences, Kosar University of Bojnord, Bojnord, Iran
Dr.moeen@kub.ac.ir

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In this paper, we study the relativistic, steady state, optically thin, advection-dominated accretion disks around the rotating black holes. We study axisymmetric and vertically averaged disks. For shear stress viscosity, the exact relations of the four-velocity with no approximation are derived. We effort to derive the general and analytic relation for density, relativistic enthalpy, temperature, pressure and inertial energy. We use the radial model for the radial component of four-velocity. In the radial model, the figures of density *etc.* are plotted. The influences of shear and bulk coefficients and spin of the black hole *etc.* are studied.

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1. Introduction

The relativistic accretion disks around the black holes were studied by many authors. Viscosity is important for energy distribution, so many authors used the different way to study viscosity and its influences. Abramowicz *et al.* [1] used the non-relativistic α model viscosity in the study of accretion disks in the Kerr metric. Gammie and Popham [2] used non-relativistic and relativistic casual viscosity to study the thin ADAF disks in the Kerr metric. Takahashi [3] studied the transonic accretion disks around the rotating black holes by using relativistic and non-relativistic causal viscosity in the relativistic accretion disks in the Kerr–Schild coordinate. Moeen [4] studied the relativistic accretion disks in the Schwarzschild metric with the radial model for four-velocity and calculated the components of shear stress tensor and some important variables of disks. Moeen [5] calculated all components of shear, bulk, and shear stress tensor in the equatorial plane for relativistic accretion disks in the Kerr metric.

We concentrate on the optically thin, advection-dominated accretion flows around the rotating black holes. In these disks, most of the heat generated by relativistic viscosity is advected to the black hole, so this disk is of low luminosity. In the relativistic accretion disks, viscosity has the important role in generating and distributing energy. In the relativistic viscosity, two important components are the rt and $r\phi$ components which were calculated with different methods. In this paper, we use the relations of Moeen [5] and derive a simplified and global relation for these components which includes relativistic bulk and shear viscosity. We also use the relativistic equation of state to calculate some basic variables such as density, relativistic enthalpy, temperature, pressure and inertial energy analytically. In the previous paper, we see the radial form for radial components of four-velocity, so the radial model of Moeen [4] is used and then the relationship of density, relativistic enthalpy, temperature, pressure and inertial energy and also the influences of relativistic viscosity are seen. Space-time and references frame are discussed in Sec. 2. Basic equations of the relativistic disks are given in Sec. 3. Relativistic relations of shear stress viscosity, shear and bulk viscosity are studied in Sec. 4. In Sec. 5, the rt and rϕ components of viscosity are calculated. Density, relativistic enthalpy temperature, pressure, inertial energy and sound velocity are derived in Secs. 6 and 7. Sample solution is presented in Sec. 8. Summary and conclusions are given in Sec. 9.

2. Space-time and reference frames

We study the relativistic, steady state, axisymmetric accretion disks around the rotating black holes with the zero magnetic field. Spherical coordinate system (t, r, θ, ϕ) is used. All calculations are done in the Boyer–Lindquist coordinates (BLF), so the components of the Kerr metric $g_{\mu\nu}$, and its inverse, $g^{\mu\nu}$ in the BLF are:

$$g_{tt} = -\left(1 - \frac{2mr}{\Sigma}\right), \qquad g_{rr} = \frac{\Sigma}{\Delta}, \qquad g_{\theta\theta} = \Sigma,$$
$$g_{\phi\phi} = \frac{A\sin^2\theta}{\Sigma}, \qquad g_{t\phi} = -\frac{2mar\sin^2\theta}{\Sigma}, \tag{1}$$
$$g^{tt} = -\left(\frac{A}{\Delta\Sigma}\right), \qquad g^{rr} = \frac{\Delta}{\Sigma}, \qquad g^{\theta\theta} = \frac{1}{\Sigma},$$
$$g^{\phi\phi} = \frac{1}{\Delta\sin^2\theta}\left(1 - \frac{2mr}{\Sigma}\right), \qquad g^{t\phi} = -\frac{2mar}{\Sigma\Delta}, \tag{2}$$

where $m = GM/c^2$ is the geometric mass, M is the black hole mass, G is the gravitational constant and c is the speed of light, $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$ and $A = \Sigma\Delta + 2mr(r^2 + a^2)$. The angular momentum of the black hole, J, is

$$a = Jc/GM^2\,, \tag{3}$$

where $-1 < a < 1$. For basic scaling in this paper, we set $G = M = c = 1$.

3. Basic equation

We use the basic equations of relativistic accretion disks around the rotating black holes that respect mass conservation, radial momentum conservation and angular momentum conservation [6]:

$$-4\pi H_\theta \rho u^r r^2 \;=\; 1\,, \tag{4}$$

$$\dot{M}\eta u_\phi - 4\pi H_\theta r^2 t^r_\phi \;=\; \dot{M} j\,, \tag{5}$$

$$4\pi H_\theta r^2\left((P+\rho+u)\,u_t u^r + t^r_t\right) \;=\; \dot{E}\,. \tag{6}$$

In those equations, ρ is the density, H_θ is the half thickness, $\dot{M}$ is the accretion rate, $\eta = \frac{\rho+P+u}{\rho}$ is the relativistic enthalpy, $\dot{M}j$ is the total inward flux of the angular momentum, $\dot{E}$ is the actual rate of change of the black hole mass, P is the pressure and u is the internal energy. Similar to Gammie and Popham [2], we assume $\dot{E} \approx \dot{M} = 1$. $u^\mu = (u^t, u^r, u^\theta, u^\phi)$ are components of four-velocity, $u_\mu = (u_t, u_r, u_\theta, u_\phi)$ are components of contravariant four-velocity, (in this paper, we ignore u^θ ($u^\theta = 0$)), a^μ is four acceleration of the fluid, $t^{\mu\nu}$ are components of shear stress viscosity which will be studied.

4. Shear and bulk viscosity and shear stress viscosity

4.1. Bulk and shear tensor

The relativistic bulk tensor ($b^{\mu\nu}$) in the relativistic Navier–Stokes flow is [7]

$$b^{\mu\nu} = \Theta h^{\mu\nu}\,, \tag{7}$$

where $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor and the expansion of the fluid world line ($\Theta = u^\gamma_{;\gamma}$) in this paper is [4]

$$\Theta = u^\gamma_{;\gamma} = \frac{\partial u^\gamma}{\partial x^\gamma} + \Gamma^\nu_{\gamma\nu} u^\gamma = \frac{\partial u^\gamma}{\partial x^\gamma} + \frac{2}{r} u^r = u^r_{,r} + \frac{2u^r}{r}\,. \tag{8}$$

The shear rate, $\sigma_{\alpha\beta}$, is [3]

$$\begin{aligned} \sigma_{\alpha\beta} &= \tfrac{1}{2}\left(u_{\alpha;\gamma} h^\gamma_\beta + u_{\beta;\gamma} h^\gamma_\alpha\right) - \tfrac{1}{3}\Theta h_{\mu\nu} \\ &= \tfrac{1}{2}\left(u_{\mu;\nu} + u_{\nu;\mu} + a_\mu u_\nu + a_\nu u_\mu\right) - \tfrac{1}{3}\Theta h_{\mu\nu}\,, \end{aligned} \tag{9}$$

where $a_\mu = u_{\mu;\gamma} u^\gamma$ is the contravariant of the four acceleration. Therefore, the relativistic shear tensor ($\sigma^{\mu\nu}$) of the fluid is

$$\sigma^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta}(\sigma_{\alpha\beta}) = \tfrac{1}{2}\left(u^\mu_{;\gamma} h^{\gamma\nu} + u^\nu_{;\gamma} h^{\gamma\mu}\right) - \tfrac{1}{3}\Theta h^{\mu\nu}\,. \tag{10}$$

4.2. Shear stress viscosity

In the relativistic Navier–Stokes equations, the shear stress viscosity is written as [7]

$$t^{\mu\nu} = -2\lambda\sigma^{\mu\nu} - \zeta\Theta h^{\mu\nu}\,, \tag{11}$$

where λ is the coefficient of the dynamical viscosity, ζ is the coefficient of the bulk viscosity.

5. The rt and $r\phi$ components of viscosity

5.1. Simplified relations of rt and $r\phi$ components of shear tensor

All components of shear and bulk tensor in the equatorial plan are calculated in [5], but in most papers, two components of shear tensor rt and $r\phi$ are used more often. We use the relation of these two components of Moeen [5] and we convert them to the facile relations.

The rt and $r\phi$ components of shear tensor are [5]:

$$\begin{aligned}
\sigma^{tr} &= \sigma^{rt} = \tfrac{1}{2}\Big[\left(u^t_{,r} + \Gamma^t_{rt}u^t + \Gamma^t_{r\phi}u^\phi\right)h^{rr} + \Gamma^t_{tr}u^r h^{rt} + \Gamma^t_{\phi r}u^r h^{r\phi} \\
&\quad + \left(u^r_{,r} + \Gamma^r_{rr}u^r\right)h^{rt} + \left(\Gamma^r_{tt}u^t + \Gamma^r_{t\phi}u^\phi\right)h^{tt} + \left(\Gamma^r_{\phi t}u^t + \Gamma^r_{\phi\phi}u^\phi\right)h^{t\phi}\Big] \\
&\quad - \tfrac{1}{3}\left(u^r_{,r} + \tfrac{2u^r}{r}\right)h^{rt}\,, \\
\sigma^{r\phi} &= \sigma^{\phi r} = \tfrac{1}{2}\Big[\left(u^r_{,r} + \Gamma^r_{rr}u^r\right)h^{r\phi} + \left(\Gamma^r_{tt}u^t + \Gamma^r_{t\phi}u^\phi\right)h^{t\phi} \\
&\quad + \left(\Gamma^r_{\phi t}u^t + \Gamma^r_{\phi\phi}u^\phi\right)h^{\phi\phi} + (u^\phi_{,r} + \Gamma^\phi_{rt}u^t + \Gamma^\phi_{r\phi}u^\phi)h^{rr} \\
&\quad + \Gamma^\phi_{tr}u^r h^{rt} + \Gamma^\phi_{\phi r}u^r h^{r\phi}\Big] - \tfrac{1}{3}\left(u^r_{,r} + \tfrac{2u^r}{r}\right)h^{r\phi}\,,
\end{aligned} \tag{12}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols (see Appendix A) and in the equatorial plan ($\theta = \frac{\pi}{2}$), $u^\mu = (u^t, u^r, 0, u^\phi)$ are the components of the four-velocity. After some calculations, the rt and $r\phi$ components of shear tensor in the Kerr metric are:

$$\begin{aligned}
\sigma^{tr} = \sigma^{rt} &= \frac{1}{6r^4\Delta}\Big[9(u^r)^2u^tr^4 + 6(u^r)^2u^tr^2a^2 + 3r^2\Delta^2u^t_{,r} + 3r^4\Delta u^t_{,r}(u^r)^2 \\
&-18r^4au^\phi(u^r)^2 - 6a^3u^\phi(u^r)^2r^2 + 3\left(u^t\right)^3\Delta^2 - 6au^\phi\left(u^t\right)^2\Delta^2 + \Delta(u^r)^2u^tr^3 \\
&+\Delta u^tu^rr^4u^r_{,r} - 3\left(u^\phi\right)^2r^3u^t\Delta^2 + 3\left(u^\phi\right)^2a^2u^t\Delta^2 - 3u^t(u^r)^2r^5 \\
&-2\Delta u^t(u^r)^2r^7\Big]\,,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\sigma^{r\phi} = \sigma^{\phi r} &= \frac{1}{6r^4\Delta}\Big[3\left(u^t\right)^2u^\phi\Delta^2 - 6a\left(u^\phi\right)^2u^t\Delta^2 + \Delta(u^r)^2u^\phi r^3 \\
&+\Delta u^ru^\phi r^4u^r_{,r} - 3\left(u^\phi\right)^3r^3\Delta^2 + 3\left(u^\phi\right)^3a^2\Delta^2 + 6a(u^r)^2u^tr^2
\end{aligned}$$

$$+3u^{\phi}_{,r}r^2\Delta^2+3u^{\phi}_{,r}r^4\Delta(u^r)^2+3u^{\phi}r^5(u^r)^2-6u^{\phi}a^2(u^r)^2r^2$$
$$-9(u^r)^2u^{\phi}r^4-2\Delta u^{\phi}(u^r)^2r^7\Big]\,. \tag{14}$$

5.2. *The rt and rφ components of shear stress viscosity*

In the Kerr metric, the rt and $r\phi$ components of bulk tensor are [5]:

$$\begin{aligned} b^{tr} &= b^{rt}=\left(u^r_{,r}+\frac{2u^r}{r}\right)h^{rt}=\left(u^r_{,r}+\frac{2u^r}{r}\right)\left(u^ru^t\right)\,,\\ b^{r\phi} &= b^{\phi r}=\left(u^r_{,r}+\frac{2u^r}{r}\right)h^{r\phi}=\left(u^r_{,r}+\frac{2u^r}{r}\right)\left(u^ru^{\phi}\right)\,. \end{aligned}\tag{15}$$

With Eqs. (11), (13), (14) and (15), the rt and $r\phi$ components of shear stress viscosity are:

$$\begin{aligned} t^{tr}=t^{rt}=-\frac{1}{3r^4\Delta}\Big[&9\lambda(u^r)^2u^tr^4+6\lambda(u^r)^2u^tr^2a^2+3\lambda\Delta^2r^2u^t_{,r}\\ &+3\lambda r^4\Delta u^t_{,r}(u^r)^2-18\lambda r^4au^{\phi}(u^r)^2-6\lambda a^3u^{\phi}(u^r)^2r^2+3\lambda\left(u^t\right)^3\Delta^2\\ &-6\lambda au^{\phi}\left(u^t\right)^2\Delta^2+\lambda\Delta(u^r)^2u^tr^3+\lambda\Delta u^tu^rr^4u^r_{,r}-3\lambda\left(u^{\phi}\right)^2r^3u^t\Delta^2\\ &+3\lambda\left(u^{\phi}\right)^2a^2u^t\Delta^2-3\lambda u^t(u^r)^2r^5-2\lambda\Delta u^t(u^r)^2r^7+3\zeta u^t(u^r)^2r^3\Delta\\ &+3\zeta u^tu^rr^4\Delta u^r_{,r}+3\zeta u^t(u^r)^2r^7\Delta\Big]\,, \end{aligned}\tag{16}$$

$$\begin{aligned} t^{r\phi}=t^{\phi r}=-\frac{1}{3r^4\Delta}\Big[&3\lambda\left(u^t\right)^2u^{\phi}\Delta^2-6\lambda a\left(u^{\phi}\right)^2u^t\Delta^2+\lambda\Delta(u^r)^2u^{\phi}r^3\\ &+\lambda\Delta u^ru^{\phi}r^4u^r_{,r}-3\lambda\left(u^{\phi}\right)^3r^3\Delta^2+3\lambda\left(u^{\phi}\right)^3a^2\Delta^2+6\lambda a(u^r)^2u^tr^2\\ &+3\lambda u^{\phi}_{,r}r^2\Delta^2+3\lambda u^{\phi}_{,r}r^4\Delta(u^r)^2+3\lambda u^{\phi}r^5(u^r)^2-6\lambda u^{\phi}a^2(u^r)^2r^2\\ &-9\lambda(u^r)^2u^{\phi}r^4-2\lambda\Delta u^{\phi}(u^r)^2r^7-3\zeta(u^r)^2u^{\phi}r^4+3\zeta(u^r)^2u^{\phi}r^3\Delta\\ &+3\zeta u^ru^{\phi}r^4\Delta u^r_{,r}+3\zeta(u^r)^2u^{\phi}r^7\Big]\,. \end{aligned}\tag{17}$$

By similar calculation, the t_{rt} and $t_{r\phi}$ components in the Kerr metric are:

$$\begin{aligned} t_{tr}\;=\;t_{rt}=-\frac{1}{3r^3\Delta^2}\Big[&3\lambda\Delta^2u_{t,r}r^3-6\lambda\Delta r^3u_t-6\lambda\Delta a^2ru_t-6\lambda\Delta rau_{\phi}\\ &+3\lambda\Delta^3ru_{t,r}u_r^2r+3\lambda r^5u_t^3+6\lambda r^3a^2u_t^3-12\lambda a^2r^2u_t^3-24\lambda r^2au_t^2u_{\phi}\\ &+3\lambda ra^4u_t^3+18\lambda r^3au_t^2u_{\phi}+6\lambda ra^3u_{\phi}u_t^2-3\lambda\Delta^3u_tu_r^2+\lambda\Delta^3ru_tu_ru_{r,r}\\ &-3\lambda r^4u_tu_{\phi}^2+12\lambda r^3u_tu_{\phi}^2-12\lambda r^2u_tu_{\phi}^2+3\lambda ra^2u_tu_{\phi}^2+\lambda\Delta^2ru_tu_r^2\\ &-\lambda\Delta^2r^2u_tu_r^2-6\zeta\Delta^2ru_tu_r^2+3\zeta\Delta^3ru_tu_ru_{r,r}+6\zeta\Delta^2r^2u_tu_r^2\Big]\,, \end{aligned}\tag{18}$$

$$
\begin{aligned}
t_{r\phi} \;=\; t_{\phi r} = -\frac{1}{3r^3\Delta^2}\Big[& -6\lambda\Delta r^4 u_\phi - \lambda\Delta^2 r^2 u_\phi u_r^2 + \lambda\Delta^2 r u_r^2 u_\phi \\
& +3\lambda\Delta^3 r u_{\phi,r} u_r^2 + 3\lambda\Delta^2 r^3 u_{\phi,r} - 3\lambda\Delta^3 u_r^2 u_\phi + 3\lambda r a^2 u_\phi^3 + 3\lambda r^5 u_t^2 u_\phi \\
& +3\lambda r a^4 u_t^2 u_\phi + 18\lambda\Delta a r^3 u_t + 6\lambda\Delta r a^3 u_t + 6\lambda\Delta r a^2 u_\phi + 12\lambda\Delta r^3 u_\phi \\
& -3\lambda r^4 u_\phi^3 + 12\lambda r^3 u_\phi^3 - 12\lambda r^2 u_\phi^3 + 6\lambda u_t^2 u_\phi r^3 a^2 - 12\lambda u_t^2 u_\phi r^2 a^2 \\
& -18\lambda u_t u_\phi^2 r^3 a + 6\lambda u_t u_\phi^2 r a^3 - 24\lambda u_t u_\phi^2 r^2 a + 6\zeta\Delta^2 r^2 u_r^2 u_\phi \\
& -6\zeta\Delta^2 r u_r^2 u_\phi + 3\zeta\Delta^3 r u_r u_{r,r} u_\phi\Big] . \qquad (19)
\end{aligned}
$$

6. Density and relativistic enthalpy

In [6], density and relativistic enthalpy are derived as

$$
\rho \;=\; \frac{u_t t^r_\phi - u_\phi t^r_t}{u^r(j u_t + u_\phi)} , \qquad (20)
$$

$$
\eta \;=\; -\frac{1}{u_t} - \frac{t^r_t}{\rho u^r u_t} , \qquad (21)
$$

where t^r_ϕ and t^r_t are

$$
t^r_\phi = g^{rr} t_{r\phi} , \qquad t^r_t = g^{rr} t_{rt} . \qquad (22)
$$

We put Eqs. (18), (19) and (22) into Eq. (20), so the density is derived as

$$
\begin{aligned}
\rho = \frac{\lambda}{r^2\Delta u_r (j u_t + u_\phi)}\Big(& 2u_t u_\phi r^3 - 6u_t r^2 u_\phi - u_t r^2 \Delta u_{\phi,r} - u_t u_{\phi,r}\Delta^2 u_r^2 \\
& -6ar^2 u_t^2 - 2a^3 u_t^2 - 4u_t u_\phi a^2 + u_\phi r^2 \Delta u_{t,r} - 2a u_\phi^2 + u_\phi u_r^2 u_{t,r}\Delta^2\Big) . \qquad (23)
\end{aligned}
$$

From Eqs. (18), (20) and (23) in Eq. (21), η is calculated as

$$
\begin{aligned}
\eta = -\frac{1}{u_t} - (j u_t + u_\phi)\Big[& 3\lambda\Delta^2 r^3 u_{t,r} - 6\lambda\Delta r^3 u_t - 6\lambda\Delta a^2 r u_t - 6\lambda\Delta r a u_\phi \\
& +3\lambda\Delta^3 r u_r^2 u_{t,r} + 3\lambda r^5 u_t^3 + 6\lambda r^3 a^2 u_t^3 - 12\lambda r^2 a^2 u_t^3 - 24\lambda r^2 a u_t^2 u_\phi + 3\lambda r a^4 u_t^3 \\
& +18\lambda r^3 a u_\phi u_t^2 + 6\lambda r a^3 u_t^2 u_\phi - 3\lambda\Delta^3 u_r^2 u_t + \lambda\Delta^3 r u_t u_r u_{r,r} - 3\lambda r^4 u_t u_\phi^2 \\
& +12\lambda r^3 u_t u_\phi^2 - 12\lambda r^2 u_t u_\phi^2 + 3\lambda r a^2 u_t u_\phi^2 + 2\lambda\Delta^2 r u_t u_r^2 - 2\lambda\Delta^2 r^2 u_t u_r^2 \\
& -6\zeta\Delta^2 r u_t u_r^2 + 3\zeta\Delta^3 r u_t u_r u_{r,r} + 6\zeta\Delta^2 r^2 u_t u_r^2\Big] \Big/ \Big[3\Delta\lambda r u_t\Big(2u_t u_\phi r^3 \\
& -6u_t r^2 u_\phi - u_t r^2\Delta u_{\phi,r} - u_t u_{\phi,r}\Delta^2 u_r^2 - 6ar^2 u_t^2 - 2a^3 u_t^2 - 4u_t u_\phi a^2 \\
& +u_\phi r^2 \Delta u_{t,r} - 2a u_\phi^2 + u_\phi u_r^2 u_{t,r}\Delta^2\Big)\Big] . \qquad (24)
\end{aligned}
$$

7. Temperature, pressure and inertial energy

In the relativistic accretion disks, the generated heat by viscosity is [8]

$$Q^{+}_{\rm vis} = \frac{3}{2r^{\frac{3}{2}}} T_{r\phi} D C^{-1} \,, \tag{25}$$

where $T_{r\phi} = 2t_{r\phi}H_\theta$ is the vertically integrated viscous stress of $r\phi$ component and D and C in [9] are:

$$C = 1 - \frac{3}{r} + \frac{2a}{r^{\frac{3}{2}}} \,, \qquad D = 1 - \frac{2}{r} + \frac{r^2}{a^2} = \frac{\Delta}{r^2} \,. \tag{26}$$

Moreover, the radiation cooling (Q^-) is [9]

$$Q^{-}_{\rm rad} = 2F = \frac{4b_{\rm r}T^4}{3k_{\rm es}\Sigma} \,, \tag{27}$$

where F is the flux of the radiation from each face of the disk, $b_{\rm r}$ is the radiation constant, T is the temperature of the disk, $k_{\rm es}$ is the electron-scattering opacity and $\Sigma = 2\rho H_\theta$ is the vertical density. From energy equation, we have

$$Q^{+}_{\rm vis} - Q^{-}_{\rm rad} = Q_{\rm adv} = f Q^{+}_{\rm vis} \,, \tag{28}$$

where $Q_{\rm adv}$ is advected energy, $f = \frac{Q_{\rm adv}}{Q^{+}}$ shows the relative important of advection energy. With Eqs. (25)–(28), after some calculation, we have

$$\frac{3\Delta H_\theta t_{r\phi}(1-f)}{r^2\left(r^{\frac{3}{2}} - 3r^{\frac{1}{2}} + 2a\right)} = \frac{b_{\rm r}T^4}{3k_{\rm es}\rho H_\theta} \,. \tag{29}$$

We use Eq. (4) to eliminate H_θ, therefore T is calculated as

$$T = \left(\frac{9k_{\rm es}\Delta t_{r\phi}}{32\pi^2(1-f)b_{\rm r}r^6\rho\left(u^r\right)^2\left(r^{\frac{3}{2}} - 3r^{\frac{1}{2}} + 2a\right)}\right)^{\frac{1}{4}} \,. \tag{30}$$

We apply the relativistic equation of state, so the relativistic pressure P is [10]

$$P = \rho T \,. \tag{31}$$

Furthermore, we can calculate the internal energy u and sound velocity $c_{\rm s}$ from [2]

$$\begin{aligned} g(T) &= \frac{45T^2 + 45T + 12}{15T^2 + 20T + 8} \,, \\ u &= \rho T g(T) \,, \\ c_{\rm s} &= \frac{T}{1 + T[1 + g(T)]} \,. \end{aligned} \tag{32}$$

8. Sample solution

In this section, we use the radial model for the radial component of the four-velocity for Keplerian angular momentum of Moeen [5] to see a sample solution of ADAF disks relationship

$$
\begin{aligned}
u^{\hat{r}}_{\mathrm{LNRF}} &= -\frac{\beta r}{r^n} \Rightarrow u^r_{\mathrm{BLF}} = -\frac{\beta\sqrt{\Delta}}{r^{n+1}}\,,\\
\Omega &= \Omega_k^+ = \frac{u^\phi}{u^t} = \frac{1}{r^{\frac{3}{2}}+a}\,,
\end{aligned}
\tag{33}
$$

where β and n are positive and constant, $u^{\hat{r}}_{\mathrm{LNRF}}$ and u^r_{BLF} are the radial components of four-velocity in LNRF and BLF, so the four-velocity in BLF is [5]

$$
u^\mu = \left(\frac{r^{\frac{3}{2}}+a}{r^{n+1}}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^2-3r+2ar^{\frac{1}{2}}}}, -\frac{\beta\sqrt{\Delta}}{r^{n+1}}, 0, \frac{1}{r^{n+1}}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^2-3r+2ar^{\frac{1}{2}}}}\right). \tag{34}
$$

Moreover, u_μ are calculated with metric components as

$$
\begin{aligned}
u_\mu = &\left(-\frac{r^{\frac{3}{2}}-2r^{\frac{1}{2}}+a}{r^n}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^2-3r+2ar^{\frac{1}{2}}}}, -\frac{\beta r}{r^n\sqrt{\Delta}}, 0,\right.\\
&\left.\frac{r^2-2ar^{\frac{1}{2}}+a^2}{r^n}\sqrt{\frac{r^{2n+1}+r\beta^2}{r^2-3r+2ar^{\frac{1}{2}}}}\right).
\end{aligned}
\tag{35}
$$

We use the components of the four-velocity and covariant components of four-velocity to derive the density, relativistic enthalpy, temperature, inertial energy and sound velocity.

Figures 1 and 2 show the influences of the coefficients of bulk and shear viscosity on some important thermodynamic quantities. In Fig. 3, the effect of the spin of the black hole (a) on the thermodynamical quantities is seen. The influence of n parameter in the components of the four-velocity and thermodynamic quantities is presented in Figs. 4 and 5. The influence of β parameter on the covariant components of the four-velocity and thermodynamic quantities is seen in Figs. 6 and 7.

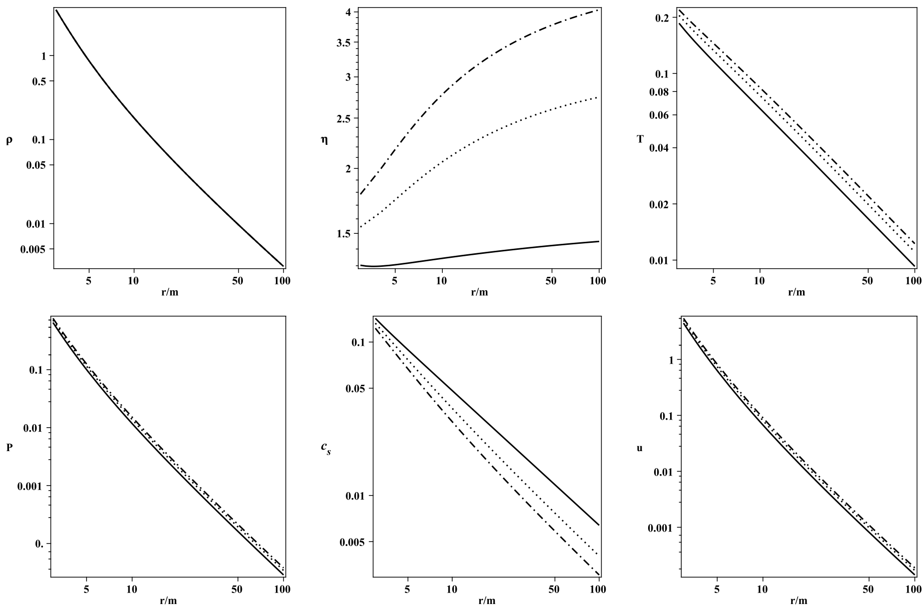

Fig. 1. Influence of the coefficient of the bulk viscosity in $\beta = 1$, $n = \frac{1}{2}$, $a = .9$ and $\lambda = 2$. Solid curves $\zeta = 2$, dotted curves $\zeta = 4$ and dash-dotted curves $\zeta = 6$.

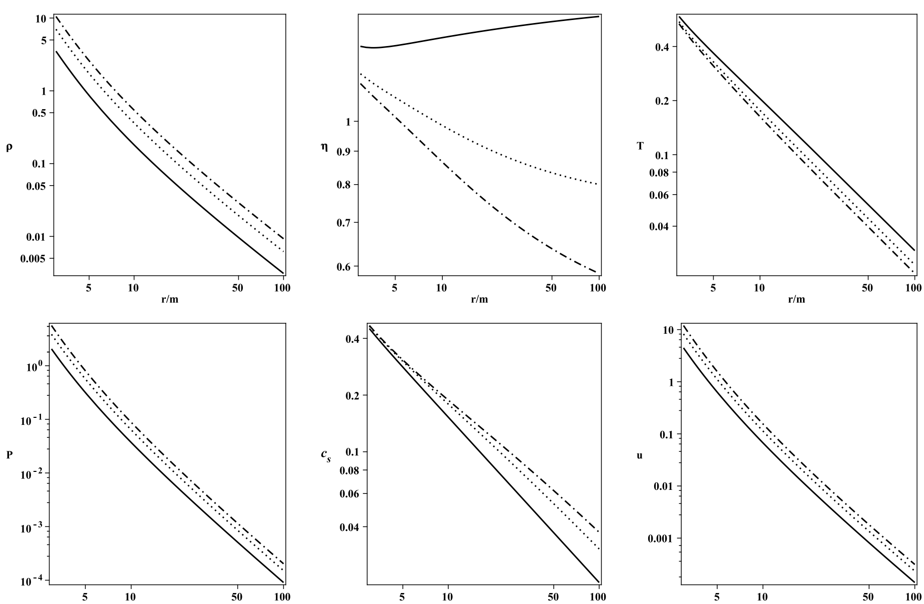

Fig. 2. Influence of the coefficient of the shear viscosity in $\beta = 1$, $n = \frac{1}{2}$, $a = .9$ and $\zeta = 2$. Solid curves $\lambda = 2$, dotted curves $\lambda = 4$ and dash-dotted curves $\lambda = 6$.

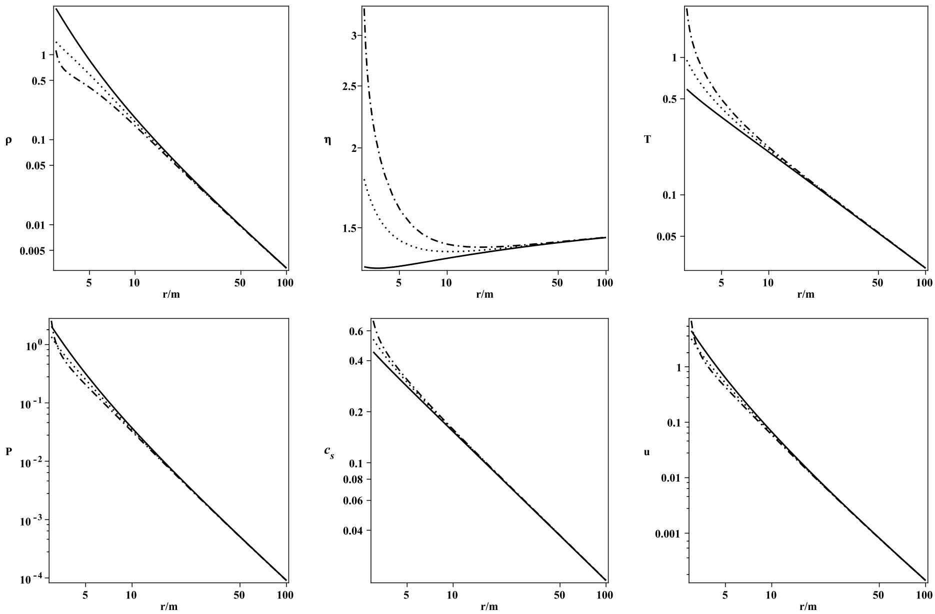

Fig. 3. Influence of spin of black hole (a) in $\beta = 1$, $n = \frac{1}{2}$, $\lambda = 2$ and $\zeta = 2$. Solid curves $a = .9$, dotted curves $a = .5$ and dash-dotted curves $a = .1$.

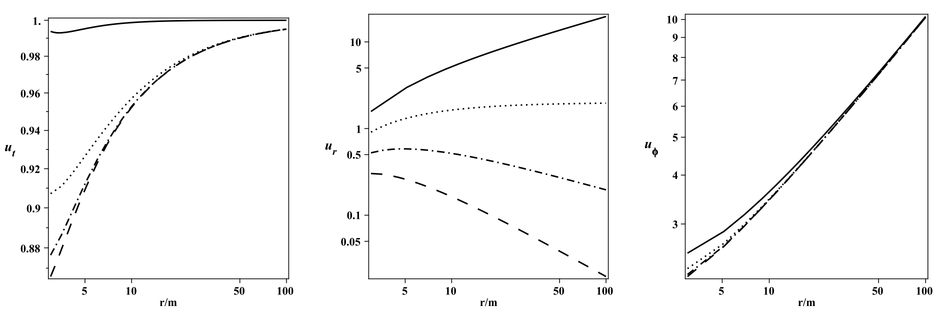

Fig. 4. Influence of the n parameter in the components of four-velocity in $\beta = 1$, $a = .9$. Solid curves $n = \frac{1}{2}$, dotted curves $n = 1$, dash-dotted curves $n = \frac{3}{2}$ and dashed curves $n = 1$.

9. Summary and conclusions

Components of shear and bulk tensor are calculated in [5], but in this paper, we derive the simplified and exact relations for the two important components (rt and $r\phi$) with no approximation which will be useful in the future calculations. In σ^{tr}, all components of the four-velocity are seen but the sentences with u^r and u^t are seen more often. Moreover, in $\sigma^{r\phi}$, the

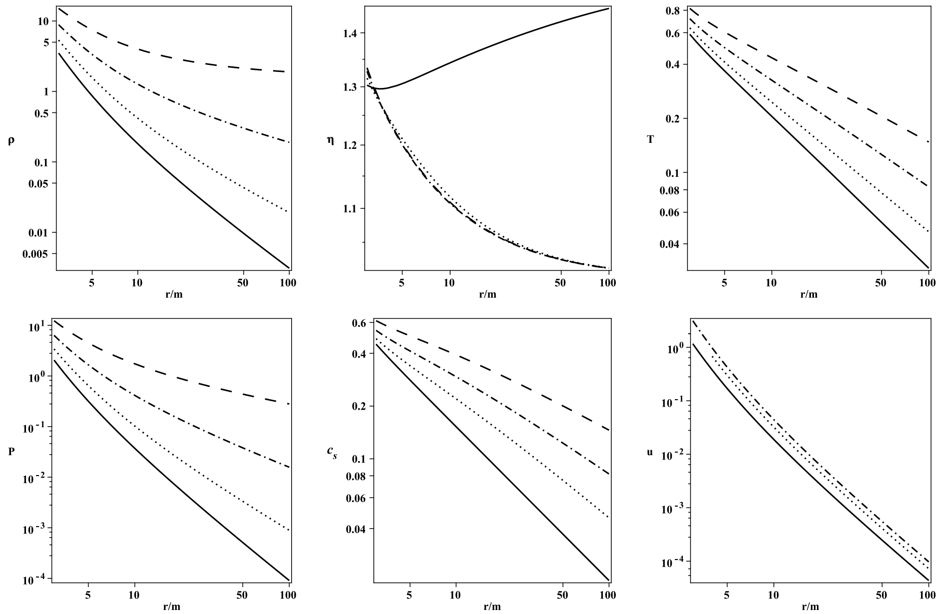

Fig. 5. Influence of the n parameter (power of r in u^r) in $\beta = 1$, $a = .9$, $\lambda = 2$ and $\zeta = 2$. Solid curves $n = \frac{1}{2}$, dotted curves $n = 1$, dash-dotted curves $n = \frac{3}{2}$ and dashed curves $n = 1$.

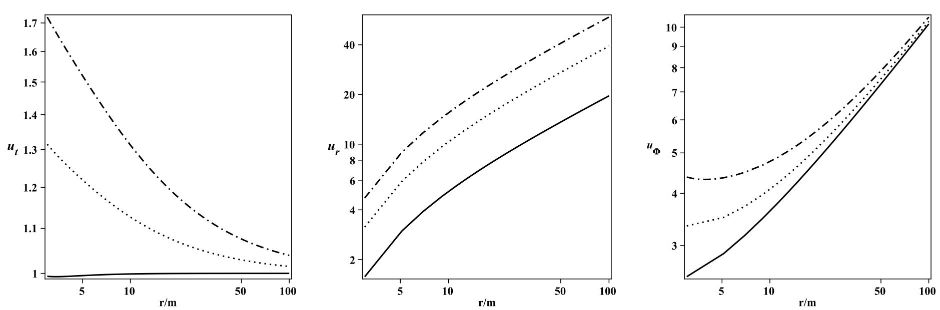

Fig. 6. Influence of β parameter in the components of four-velocity in $n = \frac{1}{2}$, $a = .9$. Solid curves $\beta = 1$, dotted curves $\beta = 2$, and dash-dotted curves $\beta = 4$.

sentences with u^r and u^ϕ are seen more often. The t^{rt}, $t^{r\phi}$, t_{rt} and $t_{r\phi}$ are calculated and we can see that the coefficient of dynamical viscosity (coefficient of shear viscosity) and coefficient of bulk viscosity are both effective. We use the relativistic equations of state to obtain the relations of ρ and η with the four-velocity and coefficient of shear and bulk viscosity. We use the energy equation to derive temperature, pressure, inertial energy and sound velocity. For more detailed discussion, we use the radial model of

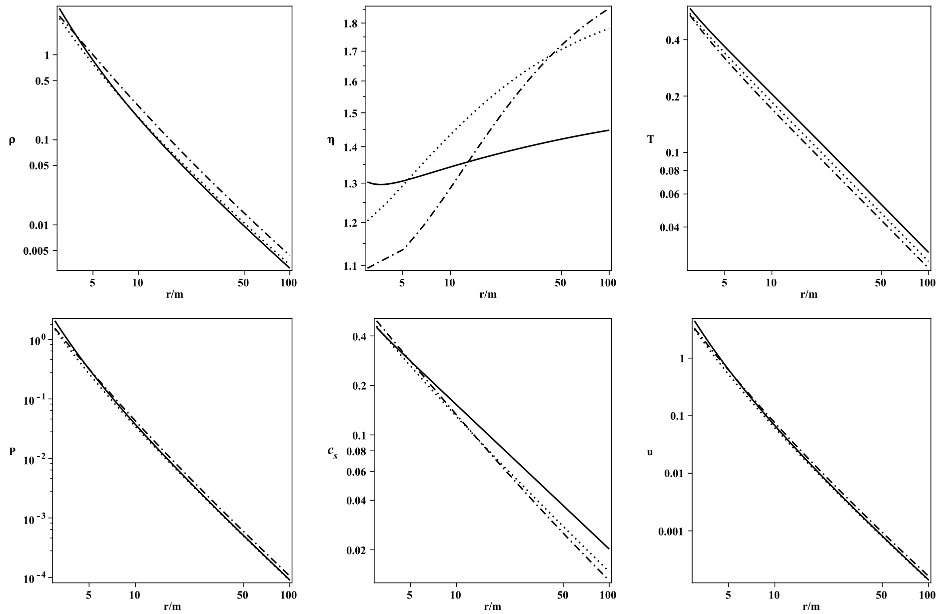

Fig. 7. Influence of β parameter in $n = \frac{1}{2}$, $a = .9$ $\lambda = 2$ and $\zeta = 2$. Solid curves $\beta = 1$, dotted curves $\beta = 2$, and dash-dotted curves $\beta = 4$.

four-velocity of [4]. Figures 1–7 show the treatment of ρ, η, P, T, c_s and u with changing in various variables. In Fig. 1, we see that increasing in the bulk coefficient, ζ causes increasing in η, T, P, u and decreasing in c_s, but it has no influence in ρ. Figure 2 shows increasing in the coefficient of shear viscosity, λ causes increasing in ρ, P, c_s, u and decreasing in η and T. Figure 3 shows the influence of specific angular momentum of black holes. In this figure, we see that the influence of a parameter is important in the inner radii, especially it is important in η. In Fig. 4, we see that increasing in n leads decreasing in the values of covariant components of the four-velocity. In Fig. 5, we see that increasing in n is due to increasing in ρ, T, c_s, P, u and decreasing in η. In Fig. 6, we see that increasing in β leads to increasing in the values of covariant components of the four-velocity. Figure 7 shows that increasing in β is due to increasing in ρ, P, u and decreasing in c_s, T, but for η is due to decreasing in the inner radii and increasing in the outer radii. This sample solution shows that $\eta = 1$ is not a good approximation in all cases. Furthermore, we see that the coefficients of shear and bulk viscosity are effective. These figures show that the coefficients of shear viscosity have greater influence and these two coefficients have the clear effects on η.

Appendix A

The Christoffel symbols in the BLF

The Christoffel symbols ($\Gamma^{\alpha}_{\beta\gamma}$) in the equatorial plan and in our scaling of the BLF are [5]:

$$
\begin{aligned}
\Gamma^{t}_{tt} &= 0\,, \qquad \Gamma^{t}_{rt}=\Gamma^{t}_{tr}=\frac{a^2+r^2}{r^2\left(a^2+r^2-2r\right)}\,, \qquad \Gamma^{t}_{t\theta}=\Gamma^{t}_{\theta t}=0\,,\\
\Gamma^{t}_{t\phi} &= \Gamma^{t}_{\phi t}=0\,, \qquad \Gamma^{t}_{rr}=0\,, \qquad \Gamma^{t}_{r\theta}=\Gamma^{t}_{\theta r}=0\,,\\
\Gamma^{t}_{r\theta} &= \Gamma^{t}_{\theta r}=0\,, \qquad \Gamma^{t}_{r\phi}=\Gamma^{t}_{\phi r}=-\frac{a\left(a^2+3r^2\right)}{r^2\left(a^2+r^2-2r\right)}\,, \qquad \Gamma^{t}_{\theta\theta}=0\,,\\
\Gamma^{t}_{\theta\phi} &= \Gamma^{t}_{\phi\theta}=0\,, \qquad \Gamma^{t}_{\phi\phi}=0\,, \qquad \Gamma^{r}_{tt}=\frac{a^2+r^2-2r}{r^4}\,,\\
\Gamma^{r}_{tr} &= \Gamma^{r}_{rt}=0\,, \qquad \Gamma^{r}_{t\theta}=\Gamma^{r}_{\theta t}=0\,, \qquad \Gamma^{r}_{t\phi}=\Gamma^{r}_{\phi t}=-\frac{a\left(a^2+r^2-2r\right)}{r^4}\,,\\
\Gamma^{r}_{rr} &= \frac{a^2-r}{r\left(a^2+r^2-2r\right)}\,, \qquad \Gamma^{r}_{r\theta}=\Gamma^{r}_{\theta r}=0\,, \qquad \Gamma^{r}_{r\phi}=\Gamma^{r}_{\phi r}=0\,,\\
\Gamma^{r}_{\theta\theta} &= -\frac{a^2+r^2-2r}{r}\,, \qquad \Gamma^{r}_{\theta\phi}=\Gamma^{r}_{\phi\theta}=0\,,\\
\Gamma^{r}_{\phi\phi} &= \frac{\left(a^2+r^2-2r\right)\left(-r^3+a^2\right)}{r^4}\,, \qquad \Gamma^{\theta}_{tt}=0\,, \qquad \Gamma^{\theta}_{tr}=\Gamma^{\theta}_{rt}=0\,,\\
\Gamma^{\theta}_{t\theta} &= \Gamma^{\theta}_{\theta t}=0\,, \qquad \Gamma^{\theta}_{t\phi}=\Gamma^{\theta}_{\phi t}=0\,, \qquad \Gamma^{\theta}_{rr}=0\,,\\
\Gamma^{\theta}_{r\theta} &= \Gamma^{\theta}_{\theta r}=\frac{1}{r}\,, \qquad \Gamma^{\theta}_{r\phi}=\Gamma^{\theta}_{\phi r}=0\,, \qquad \Gamma^{\theta}_{\theta\theta}=0\,,\\
\Gamma^{\theta}_{\theta\phi} &= \Gamma^{\theta}_{\phi\theta}=0\,, \qquad \Gamma^{\theta}_{\phi\phi}=0\Gamma^{\phi}_{tt}=0\,, \qquad \Gamma^{\phi}_{tr}=\Gamma^{\phi}_{rt}=\frac{a}{r^2\left(a^2+r^2-2r\right)}\,,\\
\Gamma^{\phi}_{t\theta} &= \Gamma^{\phi}_{\theta t}=0\,, \qquad \Gamma^{\phi}_{t\phi}=\Gamma^{\phi}_{\phi t}=0\,, \qquad \Gamma^{\phi}_{rr}=0\,,\\
\Gamma^{\phi}_{r\theta} &= \Gamma^{\phi}_{\theta r}=0\,, \qquad \Gamma^{\phi}_{r\phi}=\Gamma^{\phi}_{\phi r}=\frac{r^3-a^2-2r^2}{r^2\left(a^2+r^2-2r\right)}\,,\\
\Gamma^{\phi}_{\theta\theta} &= 0\,, \qquad \Gamma^{\phi}_{\theta\phi}=\Gamma^{\phi}_{\phi\theta}=0\,, \qquad \Gamma^{\phi}_{\phi\phi}=0\,.
\end{aligned}
\tag{A.1}
$$

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