CHAOS SYNCHRONIZATION OF CANONICALLY AND LIE-ALGEBRAICALLY DEFORMED HENON–HEILES SYSTEMS BY ACTIVE CONTROL

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(Received October 6, 2017; accepted May 7, 2018)

Recently, there has been provided two chaotic models based on the twist-deformation of classical Henon–Heiles system. First of them has been constructed on the well-known, canonical space-time noncommutativity, while the second one on the Lie-algebraically type of quantum space, with two spatial directions commuting to classical time. In this article, we find the direct link between mentioned above systems, by synchronization both of them in the framework of active control method. Particularly, we derive at the canonical phase-space level the corresponding active controllers as well as we perform (as an example) the numerical synchronization of analyzed models.

DOI:10.5506/APhysPolB.49.1459

1. Introduction

Since Edward Lorenz proposed his widely-known "model of weather", there have appeared a lot of papers dealing with so-called chaotic models, whose dynamics is described by strongly sensitive with respect to initial conditions nonlinear differential equations. The most popular of them are: Lorenz system [1], Roessler system [2], Rayleigh–Benard system [3], Henon–Heiles system [4], jerk equation [5], Duffing equation [6], Lotka–Volter system [7], Liu system [8], Chen system [9], and Sprott system [10]. A lot of them have been applied in various fields of industrial and scientific divisions, such as, for example: Physics, Chemistry, Biology, Microbiology, Economics, Electronics, Engineering, Computer Science, Secure Communications, Image Processing and Robotics.

The one of the most interesting among the above models seems to be the so-called Henon–Heiles system, which has been provided in pure astrophysical context. It concerns the problem of nonlinear motion of a star around a galactic center, where the motion is restricted to a plane. It is defined by the following Hamiltonian function:

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{2}\left(p_i^2 + x_i^2\right) + x_1^2 x_2 - \frac{1}{3}x_2^3\,, \tag{1}$$

which in Cartesian coordinates x_1 and x_2 describes the set of two nonlinearly coupled harmonic oscillators. In polar coordinates r and θ, it corresponds to the particle moving in noncentral potential of the form of

$$V(r,\varphi) = \frac{r^2}{2} + \frac{r^3}{3}\sin\left(3\varphi\right)\,, \tag{2}$$

with $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$. The above model has been inspired by observations indicating that star moving in a weakly perturbated central potential should have apart of total energy $E_{\rm tot}$ constant in time, also the second conserved physical quantity I[1]. It has been demonstrated with the use of so-called Poincaré section method, that such a situation appears in the case of Henon–Heiles system only for the values of control parameter $E_{\rm tot}$ below the threshold $E_{\rm th} = 1/6$. For higher energies, the trajectories in phase space become chaotic and the quantity I does not exist (see *e.g.* [11, 12]).

Recently, in articles [13] and [14] there have been proposed two noncommutative counterparts of the above-mentioned Henon–Heiles system. They have been defined respectively on the following canonically as well as Lie-algebraically deformed Galilei space-times [15–17][2,3]

$$[t,\hat{x}_i] = 0\,, \qquad [\hat{x}_i,\hat{x}_j] = i\theta_{ij}\,, \tag{3}$$

and

$$[t,\hat{x}_i] = 0\,, \qquad [\hat{x}_i,\hat{x}_j] = \frac{i}{\kappa}t\epsilon_{ij}\,, \tag{4}$$

[1] The quantity I plays the role of additional constant of motion, which leads to the regular trajectories of a particle.

[2] The canonically and Lie-algebraically noncommutative space-times have been defined as the quantum representation spaces, so-called Hopf modules (see *e.g.* [15, 16]), for the twist-deformed quantum Galilei Hopf algebras $\mathcal{U}_\theta(\mathcal{G})$ and $\mathcal{U}_\kappa(\mathcal{G})$ respectively.

[3] It should be noted that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [18, 19]), apart from canonical [15–17] space-time noncommutativity, there also exist Lie-algebraic [17–22] and quadratic [17, 22–24] types of quantum spaces.

with constant deformation parameters $\theta_{ij} = -\theta_{ji}$ and κ. Particularly, there have been provided the Hamiltonian functions of the models as well as the corresponding canonical equations of motion. Besides, it has been demonstrated that for proper values of deformation parameters θ and κ, and for proper values of control parameters, there appears (much more intensively) chaos in both systems. Consequently, in such a way, it has been shown the impact of the above noncommutative space-times on the basic dynamical properties of this important classical chaotic model. It should be noted that such deformed constructions are inspired by investigations dealing with noncommutative classical and quantum mechanics (see *e.g.* [25–28]) as well as with field theoretical systems (see *e.g.* [29–31]), in which the quantum space-time is not classical. Such models follow (particularly) from formal arguments based mainly on Quantum Gravity [32, 33] and String Theory [34, 35], indicating that space-time at Planck scale becomes noncommutative.

One of the most important problems of the chaos theory concerns so-called chaos synchronization phenomena. Since Pecora and Caroll [36] introduced a method to synchronize two identical chaotic systems, the chaos synchronization has received increasing attention due to great potential of applications in many scientific discipline. Generally, there are known several methods of chaos synchronization such as: OGY method [37], active control method [38, 39], adaptive control method [40, 41], backstepping method [42, 43], sampled-data feedback synchronization method [44], time-delay feedback method [45] and sliding mode control method [46, 47]. The mentioned methods have been applied to the synchronization of many identical as well as different chaotic models, such as, for example, Sprott, Lorenz and Roessler systems respectively [48, 49].

In this article, we synchronize by active control scheme the canonically deformed Henon–Heiles (master) system [13] with its Lie-algebraically noncommutative (slave) partner [14]. In this aim, we establish the proper so-called active controllers with the use of the Lyapunov stabilization theory [50]. Additionally, we illustrate the obtained results by numerical calculations performed for particular values of deformation parameters θ_{ij} and κ.

The paper is organized as follows. In Section 2, we recall chaotic canonically and Lie-algebraically deformed Henon–Heiles models proposed in articles [13] and [14] respectively. In Section 3, we remind the basic concepts of active synchronization method, while in Section 4, we find the active controllers which synchronize both noncommutative systems. The conclusions and final remarks are discussed in Section 5.

2. The noncommutative Henon–Heiles models

In this section, we very shortly remind the basic facts concerning two chaotic Henon–Heiles models defined on noncommutative Galilei space-times (3) and (4) respectively. As it was mentioned in Introduction, first of them has been provided in paper [13], while the second one in article [14].

2.1. Classical Henon–Heiles system on canonically deformed space-time

In accordance with [13], the dynamics of the model is given by the following Hamiltonian function:

$$H\left(\hat{p},\hat{x}\right) = \frac{1}{2}\sum_{i=1}^{2}\left(\hat{p}_i^2 + \hat{x}_i^2\right) + \hat{x}_1^2\hat{x}_2 - \frac{1}{3}\hat{x}_2^3 \tag{5}$$

defined on the canonically deformed phase space of the form of[4]

$$\{\,\hat{x}_1,\hat{x}_2\,\} = 2\theta\,, \qquad \{\,\hat{p}_1,\hat{p}_2\,\} = \{\,\hat{x}_i,\hat{p}_j\,\} = 0 \tag{6}$$

with constant parameter $\theta = \theta_{12} = -\theta_{21}$. In terms of commutative canonical variables (x_i, p_i), the Hamiltonian looks as follows:

$$\begin{aligned} H(p,x) \;=\; & \frac{1}{2M(\theta)}\left(p_1^2+p_2^2\right) + \frac{1}{2}M(\theta)\Omega^2(\theta)\left(x_1^2+x_2^2\right) - S(\theta)L \\ & + \left(x_1-\theta p_2\right)^2\left(x_2+\theta p_1\right) - \frac{1}{3}\left(x_2+\theta p_1\right)^3\,, \end{aligned} \tag{7}$$

where

$$L \;=\; x_1p_2 - x_2p_1\,, \tag{8}$$

$$1/M(\theta) \;=\; 1+\theta^2\,, \tag{9}$$

$$\Omega(\theta) \;=\; \sqrt{(1+\theta^2)}\,, \tag{10}$$

and

$$S(\theta) = \theta\,. \tag{11}$$

Due to the form of the above energy function, the symbols $M(\theta)$ and $\Omega(\theta)$ denote the new, deformed mass and frequency of particle, respectively. Obviously, quantity L plays the role of the angular momentum vector, while $S(\theta)$ can be interpreted as the present in third term of the Hamiltonian, the new θ-dependent coefficient. It should be also noted that two last, nonlinear

[4] The correspondence relations are $\{\,\cdot,\cdot\,\} = \frac{1}{i}\,[\,\cdot,\cdot\,]$.

members of formula (7) remain responsible for chaotic behavior of the system, while the corresponding to $H(p,x)$ canonical equations of motion are given by

$$\dot{x}_1 = [1/M(\theta)]\,p_1 + S(\theta)x_2 + \left[(x_1-\theta p_2)^2 - (x_2+\theta p_1)^2\right]\theta\,, \tag{12}$$

$$\dot{x}_2 = [1/M(\theta)]\,p_2 - S(\theta)x_1 - 2(x_2+\theta p_1)\,(x_1-\theta p_2)\,\theta\,, \tag{13}$$

$$\dot{p}_1 = -M(\theta)\Omega^2(\theta)x_1 + S(\theta)p_2 - 2\,(x_2+\theta p_1)\,(x_1-\theta p_2)\,, \tag{14}$$

$$\dot{p}_2 = -M(\theta)\Omega^2(\theta)x_2 - S(\theta)p_1 - (x_1-\theta p_2)^2 + (x_2+\theta p_1)^2\;. \tag{15}$$

Of course, for deformation parameter θ approaching zero, the above system becomes classical.

2.2. Classical Henon–Heiles system on Lie-algebraically deformed space-time

The model is defined by the Hamiltonian function (5) given on the following Lie-algebraically deformed phase space:

$$\{\,\hat{x}_1,\hat{x}_2\,\} = \frac{2t}{\kappa}\,, \qquad \{\,\hat{p}_i,\hat{p}_j\,\} = 0\,, \qquad \{\,\hat{x}_i,\hat{p}_j\,\} = \delta_{ij} \tag{16}$$

with constant, mass-like parameter κ[5]. In terms of commutative variables, the above Hamiltonian takes the form of

$$H(p,x,t) = \frac{1}{2M\left(\frac{t}{\kappa}\right)}\left(p_1^2+p_2^2\right) + \frac{1}{2}M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)\left(x_1^2+x_2^2\right) - S\left(\frac{t}{\kappa}\right)L$$
$$+\left(x_1-\frac{t}{\kappa}p_2\right)^2\left(x_2+\frac{t}{\kappa}p_1\right) - \frac{1}{3}\left(x_2+\frac{t}{\kappa}p_1\right)^3\,, \tag{17}$$

where

$$L = x_1p_2 - x_2p_1\,, \tag{18}$$

$$\frac{1}{M\left(\frac{t}{\kappa}\right)} = 1+\left(\frac{t}{\kappa}\right)^2\,, \tag{19}$$

$$\Omega\left(\frac{t}{\kappa}\right) = \sqrt{\left(1+\left(\frac{t}{\kappa}\right)^2\right)} \tag{20}$$

[5] One can check that $[\kappa] = \mathrm{kg}$.

and

$$S\left(\frac{t}{\kappa}\right) = \frac{t}{\kappa}\,. \tag{21}$$

It is worth to notice that due to the similar form of energy functions (7) and (17), the all coefficients $M\left(\frac{t}{\kappa}\right)$, $\Omega\left(\frac{t}{\kappa}\right)$ as well as $S\left(\frac{t}{\kappa}\right)$ can be interpreted in the same manner as their θ-deformed counterparts (10)–(11). However, contrary to the pervious case, the Lie-algebraically modified quantities (20)–(21) are time-dependent, and the corresponding canonical equations of motion look as follows:

$$\dot{x}_1 = 1/M\left(\frac{t}{\kappa}\right)p_1 + S\left(\frac{t}{\kappa}\right)x_2 + \left[\left(x_1 - \frac{t}{\kappa}p_2\right)^2 - \left(x_2 + \frac{t}{\kappa}p_1\right)^2\right]\frac{t}{\kappa}\,, \tag{22}$$

$$\dot{x}_2 = 1/M\left(\frac{t}{\kappa}\right)p_2 - S\left(\frac{t}{\kappa}\right)x_1 - 2\left[x_2 + \frac{t}{\kappa}p_1\right]\left[x_1 - \frac{t}{\kappa}p_2\right]\frac{t}{\kappa}\,, \tag{23}$$

$$\dot{p}_1 = -M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)x_1 + S\left(\frac{t}{\kappa}\right)p_2 - 2\left[x_2 + \frac{t}{\kappa}p_1\right]\left[x_1 - \frac{t}{\kappa}p_2\right]\,, \tag{24}$$

$$\dot{p}_2 = -M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)x_2 - S\left(\frac{t}{\kappa}\right)p_1 - \left[x_1 - \frac{t}{\kappa}p_2\right]^2 + \left[x_2 + \frac{t}{\kappa}p_1\right]^2\,. \tag{25}$$

Obviously, for deformation parameter κ running to infinity, the above model becomes commutative.

3. Chaos synchronization by active control — general prescription

In this section, we remind the general scheme of chaos synchronization of two systems by the so-called active control procedure [38, 39]. Let us start with the following master model[6]:

$$\dot{x} = Ax + F(x)\,, \tag{26}$$

where $x = [x_1, x_2, \ldots, x_n]$ is the state of the system, A denotes the $n \times n$ matrix of the system parameters and $F(x)$ plays the role of the nonlinear part of the differential equation (26). The slave model dynamics is described by

$$\dot{y} = By + G(y) + u\,, \tag{27}$$

with $y = [y_1, y_2, \ldots, y_n]$ being the state of the system, B denoting the n-dimensional quadratic matrix of the system, $G(y)$ playing the role of nonlinearity of equation (27) and $u = [u_1, u_2, \ldots, u_n]$ being the active controller

[6] $\frac{\mathrm{d}o}{\mathrm{d}t} = \dot{o}$.

of the slave model. Besides, it should be mentioned that for matrices $A = B$ and functions $F = G$, the states x and y describe two identical chaotic systems. In the case of $A \neq B$ or $F \neq G$, they correspond to the two different chaotic models.

Let us now provide the following synchronization error vector:

$$e = y - x\,, \tag{28}$$

which in accordance with (26) and (27) obeys

$$\dot{e} = By - Ax + G(y) - F(x) + u\,. \tag{29}$$

In active control method, we try to find such a controller u, which synchronizes the state of the master system (26) with the state of the slave system (27) for any initial condition $x_0 = x(0)$ and $y_0 = y(0)$. In other words, we design a controller u in such a way that for system (29), we have

$$\lim_{t\to\infty} ||e(t)|| = 0\,, \tag{30}$$

for all initial conditions $e_0 = e(0)$. In order to establish the synchronization (29), we use the Lyapunov stabilization theory [50]. It means that if we take as a candidate the Lyapunov function of the form of

$$V(e) = e^T PV(e)e\,, \tag{31}$$

with P being a positive $n \times n$ matrix, then we wish to find the active controller u so that

$$\dot{V}(e) = -e^T QV(e)e\,, \tag{32}$$

where Q is a positive definite $n \times n$ matrix as well. Then systems (26) and (27) remain synchronized.

4. Chaos synchronization of the models

The described in previous section algorithm can be used to the synchronization of the two above-reminded noncommutative Henon–Heiles systems. In our treatment, the canonically deformed model [13] plays the role of master system

$$\dot{x}_1 \;=\; [1/M(\theta)]\,p_1 + S(\theta)x_2 + \left[(x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2\right]\theta\,, \tag{33}$$

$$\dot{x}_2 \;=\; [1/M(\theta)]\,p_2 - S(\theta)x_1 - 2(x_2 + \theta p_1)\,(x_1 - \theta p_2)\,\theta\,, \tag{34}$$

$$\dot{p}_1 \;=\; -M(\theta)\Omega^2(\theta)x_1 + S(\theta)p_2 - 2\,(x_2 + \theta p_1)\,(x_1 - \theta p_2)\,, \tag{35}$$

$$\dot{p}_2 \;=\; -M(\theta)\Omega^2(\theta)x_2 - S(\theta)p_1 - (x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2\,. \tag{36}$$

while its slave partner is given by the Lie-algebraically noncommutative model [14]

$$\begin{aligned}\dot{y}_1 &= 1/M\left(\frac{t}{\kappa}\right)\pi_1 + S\left(\frac{t}{\kappa}\right)y_2 \\ &\quad + \left[\left(y_1 - \frac{t}{\kappa}\pi_2\right)^2 - \left(y_2 + \frac{t}{\kappa}\pi_1\right)^2\right]\frac{t}{\kappa} + u_{y_1}\,, \end{aligned}\tag{37}$$

$$\begin{aligned}\dot{y}_2 &= 1/M\left(\frac{t}{\kappa}\right)\pi_2 - S\left(\frac{t}{\kappa}\right)y_1 \\ &\quad -2\left[y_2 + \frac{t}{\kappa}\pi_1\right]\left[y_1 - \frac{t}{\kappa}\pi_2\right]\frac{t}{\kappa} + u_{y_2}\,, \end{aligned}\tag{38}$$

$$\begin{aligned}\dot{\pi}_1 &= -M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)y_1 + S\left(\frac{t}{\kappa}\right)\pi_2 \\ &\quad -2\left[y_2 + \frac{t}{\kappa}\pi_1\right]\left[y_1 - \frac{t}{\kappa}\pi_2\right] + u_{\pi_1}\,, \end{aligned}\tag{39}$$

$$\begin{aligned}\dot{\pi}_2 &= -M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)y_2 - S\left(\frac{t}{\kappa}\right)\pi_1 + \\ &\quad -\left[y_1 - \frac{t}{\kappa}\pi_2\right]^2 + \left[y_2 + \frac{t}{\kappa}\pi_1\right]^2 + u_{\pi_2} \end{aligned}\tag{40}$$

with active controllers u_{y_1}, u_{y_2}, u_{π_1} and u_{π_2}, respectively.

Using the above equations of motion, one can check that the dynamics of synchronization errors $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ is obtained as[7]

$$\begin{aligned}\dot{e}_{y_1} &= 1/M\left(\frac{t}{\kappa}\right)\pi_1 + S\left(\frac{t}{\kappa}\right)y_2 + \left[\left(y_1 - \frac{t}{\kappa}\pi_2\right)^2 - \left(y_2 + \frac{t}{\kappa}\pi_1\right)^2\right]\frac{t}{\kappa} \\ &\quad -\frac{1}{M(\theta)}p_1 - S(\theta)x_2 - \left[(x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2\right]\theta + u_{y_1}\,, \end{aligned}\tag{41}$$

$$\begin{aligned}\dot{e}_{y_2} &= 1/M\left(\frac{t}{\kappa}\right)\pi_2 - S\left(\frac{t}{\kappa}\right)y_1 - 2\left[y_2 + \frac{t}{\kappa}\pi_1\right]\left[y_1 - \frac{t}{\kappa}\pi_2\right]\frac{t}{\kappa} \\ &\quad -\frac{1}{M(\theta)}p_2 + S(\theta)x_1 + 2(x_2 + \theta p_1)\,(x_1 - \theta p_2)\,\theta + u_{y_2}\,, \end{aligned}\tag{42}$$

[7] See also formula (29).

$$\dot{e}_{\pi_1} = -M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)y_1 + S\left(\frac{t}{\kappa}\right)\pi_2 - 2\left[y_2 + \frac{t}{\kappa}\pi_1\right]\left[y_1 - \frac{t}{\kappa}\pi_2\right]$$
$$+M(\theta)\Omega^2(\theta)x_1 - S(\theta)p_2 + 2\left(x_2 + \theta p_1\right)\left(x_1 - \theta p_2\right) + u_{\pi_1}\,, \quad (43)$$

$$\dot{e}_{\pi_2} = -M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)y_2 - S\left(\frac{t}{\kappa}\right)\pi_1 - \left[y_1 - \frac{t}{\kappa}\pi_2\right]^2 + \left[y_2 + \frac{t}{\kappa}\pi_1\right]^2$$
$$+M(\theta)\Omega^2(\theta)x_2 + S(\theta)p_1 + \left(x_1 - \theta p_2\right)^2 - \left(x_2 + \theta p_1\right)^2 + u_{\pi_2}\,. \quad (44)$$

Besides, if we define the positive Lyapunov function by[8]

$$V(e) = \frac{1}{2}\left(e_{y_1}^2 + e_{y_2}^2 + e_{\pi_1}^2 + e_{\pi_2}^2\right)\,, \quad (45)$$

then for the following choice of control functions:

$$u_{y_1} = \left[1/M(\theta)\right]p_1 + S(\theta)x_2 + \left[\left(x_1 - \theta p_2\right)^2 - \left(x_2 + \theta p_1\right)^2\right]\theta$$
$$-1/M\left(\frac{t}{\kappa}\right)\pi_1 - S\left(\frac{t}{\kappa}\right)y_2 - \left[\left(y_1 - \frac{t}{\kappa}\pi_2\right)^2 + \left(y_2 + \frac{t}{\kappa}\pi_1\right)^2\right]\frac{t}{\kappa} - e_{y_1}\,, \quad (46)$$

$$u_{y_2} = \left[1/M(\theta)\right]p_2 - S(\theta)x_1 - 2(x_2 + \theta p_1)\left(x_1 - \theta p_2\right)\theta$$
$$-1/M\left(\frac{t}{\kappa}\right)\pi_2 + S\left(\frac{t}{\kappa}\right)y_1 + 2\left[y_2 + \frac{t}{\kappa}\pi_1\right]\left[y_1 - \frac{t}{\kappa}\pi_2\right]\frac{t}{\kappa} - e_{y_2}\,, \quad (47)$$

$$u_{\pi_1} = -M(\theta)\Omega^2(\theta)x_1 + S(\theta)p_2 - 2\left(x_2 + \theta p_1\right)\left(x_1 - \theta p_2\right) +$$
$$+M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)y_1 - S\left(\frac{t}{\kappa}\right)\pi_2 + 2\left[y_2 + \frac{t}{\kappa}\pi_1\right]\left[y_1 - \frac{t}{\kappa}\pi_2\right] - e_{\pi_1}\,, \quad (48)$$

$$u_{\pi_2} = -M(\theta)\Omega^2(\theta)x_2 - S(\theta)p_1 - \left(x_1 - \theta p_2\right)^2 + \left(x_2 + \theta p_1\right)^2 +$$
$$+M\left(\frac{t}{\kappa}\right)\Omega^2\left(\frac{t}{\kappa}\right)y_2 + S\left(\frac{t}{\kappa}\right)\pi_1 + \left[y_1 - \frac{t}{\kappa}\pi_2\right]^2 + \left[y_2 + \frac{t}{\kappa}\pi_1\right]^2 - e_{\pi_2}\,, \quad (49)$$

we have[9]

$$\dot{V}(e) = -\left(e_{y_1}^2 + e_{y_2}^2 + e_{\pi_1}^2 + e_{\pi_2}^2\right)\,. \quad (50)$$

Such a result means (see general prescription) that the canonically (see (33)–(36)) and Lie-algebraically (see (37)–(40)) Henon–Heiles systems are synchronized for all initial conditions with active controllers (46)–(49).

Let us now illustrate the above considerations by the proper numerical calculations.

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[8] The matrix $P = 1$ in formula (31).

[9] The matrix $Q = 1$ in formula (32).

First of all, we solve canonically deformed system (33)–(36) with $\theta = 1$ as well as we integrate the Lie-algebraically model (37)–(40) for $\kappa = 1$ and without active controllers u_{y_1}, u_{y_2}, u_{π_1} and u_{π_2}, for two different sets of initial conditions

$$(x_{01}, x_{02}; p_{01}, p_{02}) = (0.01, -0.01; 0, 0) \tag{51}$$

and

$$(y_{01}, y_{02}; \pi_{01}, \pi_{02}) = (0, 0; -0.02, 0.02)\,, \tag{52}$$

respectively. The results are presented in figure 1 — one can see that there exist (in fact) the divergences between both phase-space trajectories. Next, we find the solutions for the master system (33)–(36) (the (x, p)-trajectory) and for its slave partner (37)–(40) with active controllers (46)–(49) (the (y, π)-trajectory) for initial data (51) and (52) respectively. Now, we see that the corresponding phase-space trajectories become synchronized — the vanishing in time error functions e_{y_i} and e_{π_i} are presented in figure 2. Additionally, we repeat the above numerical procedure for two another sets of initial data: $(x_0; p_0) = (0, 0; 0, 0)$ and $(y_0; \pi_0) = (0.02, -0.02; 0, 0)$; the obtained results are presented in figures 3 and 4, respectively.

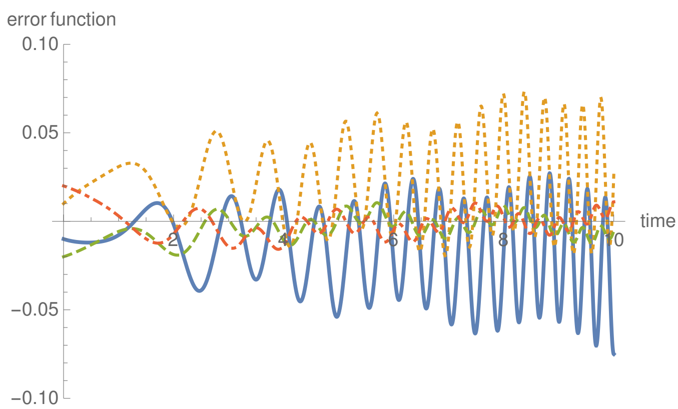

Fig. 1. (Color online) The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for the canonically deformed Henon–Heiles system with initial conditions (51) (the (x, p)-trajectory), and for the Lie-algebraically noncommutative Henon–Heiles model without correlation functions u_{y_i}, u_{π_i} for the initial conditions (52) (the (y, π)-trajectory). The solid blue line corresponds to the e_{y_1}-error function, the dotted orange one — to e_{y_2}, the dashed green one — to e_{π_1}, and the dot-dashed red one — to e_{π_2}, respectively.

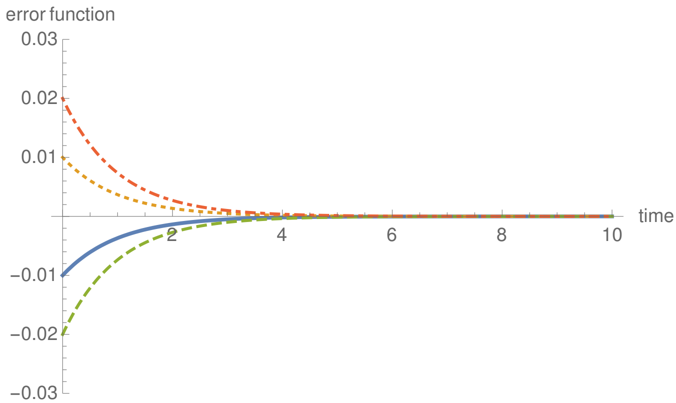

Fig. 2. (Color online) The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for the canonically deformed Henon–Heiles model defined by the master system (33)–(36) with the initial conditions (51) (the (x, p)-trajectory), and for the slave Lie-algebraically noncommutative Henon–Heiles system (37)–(40) with the initial conditions (52) (the (y, π)-trajectory). The solid blue line corresponds to the e_{y_1}-error function, the dotted orange one — to e_{y_2}, the dashed green one — to e_{π_1}, and the dot-dashed red one — to e_{π_2}, respectively.

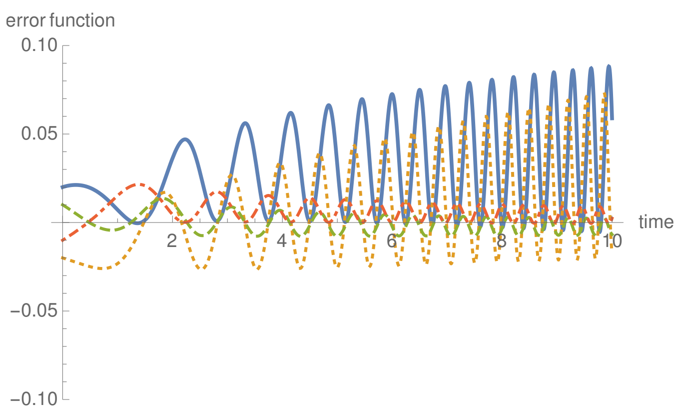

Fig. 3. (Color online) The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for the canonically deformed Henon–Heiles model with the initial conditions $(x_0; p_0) = (0, 0; 0, 0)$ (the (x, p)-trajectory), and for the Lie-algebraically noncommutative Henon–Heiles model without correlation functions u_{y_i}, u_{π_i} for the initial conditions $(y_0, \pi_0) = (0.02, -0.02; 0.01, -0.01)$ (the (y, π)-trajectory). The solid blue line corresponds to the e_{y_1}-error function, the dotted orange one — to e_{y_2}, the dashed green one — to e_{π_1}, and the dot-dashed red one — to e_{π_2}, respectively.

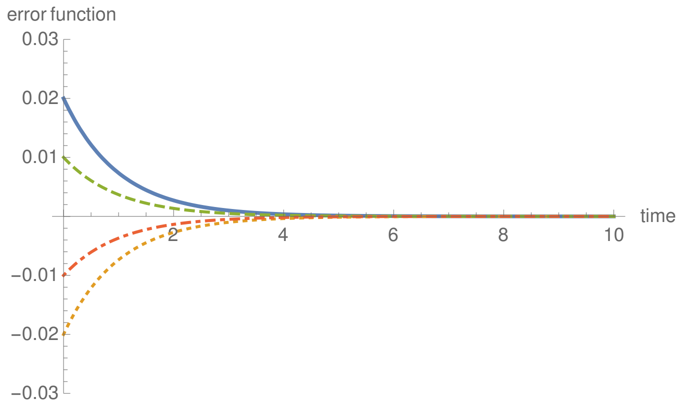

Fig. 4. (Color online) The error functions $e_{y_i} = y_i - x_i$ and $e_{\pi_i} = \pi_i - p_i$ for the canonically deformed Henon–Heiles model defined by the master system (33)–(36) with the initial conditions $(x_0; p_0) = (0, 0; 0, 0)$ (the (x, p)-trajectory), and for the slave Lie-algebraically noncommutative Henon–Heiles system (37)–(40) with the initial conditions $(y_0, \pi_0) = (0.02, -0.02; 0.01, -0.01)$ (the (y, π)-trajectory). The solid blue line corresponds to the e_{y_1}-error function, the dotted orange one — to e_{y_2}, the dashed green one — to e_{π_1}, and the dot-dashed red one — to e_{π_2}, respectively.

5. Final remarks

In this article, we synchronize two noncommutative Henon–Heiles models with the use of active control method. Particularly, we find the proper active controllers (46)–(49) as well as perform numerical synchronization of the systems for fixed values of deformation parameters θ and κ.

In our opinion, the obtained result seems to be quite interesting at least due to the two reasons. Firstly, it finds the direct dynamical link between two models defined on the completely different noncommutative space-times — the canonically twisted space and the Lie-algebraically deformed space-time respectively. Such a connection suggests that there may exist another, more fundamental (for example taken at the kinematical level) link between both systems considered here. Secondly, it combines in quite matured way two disparate scientific fields, such as the elements of Quantum Group Theory with the techniques typical for the Classical Chaos domain.

Finally, it should be noted that the presented investigations can be extended in various ways. For example, one may consider synchronization of the noncommutative Henon–Heiles models with the use of other methods mentioned in Introduction. Obviously, the works in this direction already started and are in progress.

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