Vol. 49 (2018) ACTA PHYSICA POLONICA B No 7

GENERATION OF CHAOTIC SYSTEMS WITH DIFFERENT NUMBER OF EQUILIBRIA

Long Wang

Department of Scientific Education, Jiangxi University of Technology
Nanchang, 330098, China

Mei Ding†

College of Foreign Languages and International Trade
Jiangxi University of Technology
Nanchang, 330098, China

(Received August 28, 2017; accepted May 14, 2018)

This paper is dedicated to the generation of chaotic systems with different equilibria and different dynamical behaviors. A unified four-dimensional autonomous chaotic system is presented. The number of the equilibria of the system is determined by its nonlinear terms. For different nonlinear terms, the system can generate chaotic attractor with one or two unstable equilibria, multiple coexisting chaotic attractors with three unstable equilibria, multi-scroll hyper-chaotic attractor with multiple equilibria. The dynamical behaviors of the system with different nonlinear terms are analytically and numerically investigated.

DOI:10.5506/APhysPolB.49.1483

1. Introduction

Chaos has been widely discovered in natural systems and artificial systems [1–4]. A nonlinear dynamical system which shows extreme sensitivity to initial conditions on a bounded set is generally considered to be chaotic. The chaotic system has many unique characteristics that can be used for image encryption, fault diagnosis, route planning and weather forecast [5–8]. A lot of scholars have been interested in the study of chaos. After decades of research, the chaos theory and its applications have been developed more and more mature.

Chaos generation is an important issue that has aroused wide concern in scientific community. Since the discovery of the so-called Lorenz attractor, it is generally believed that polynomial autonomous nonlinear differential

† Corresponding author: hyper968@126.com

systems can produce abundant chaos [9]. Until now, a great number of polynomial chaotic systems have been proposed, see [10–15]. The number and stability of equilibria often show important impact on the dynamical behaviors of chaotic systems. An effective way to construct and analyze chaotic systems is to consider the properties of the equilibria. Shilnikov claimed that an autonomous system with one saddle focus and a homoclinic orbit or two saddle foci and a heteroclinic orbit can generate horseshoe-type chaos [16]. The Chen system was proved to be the Shilnikov chaos with heteroclinic orbit [17]. Based on Shilnikov's theorem, Yu *et al.* generated a special kind of chaotic attractors named multi-wing attractors from the Lorenz system family by extending the index-2 saddle foci [18]. Jafari *et al.* [19] gave seventeen examples of non-equilibria systems which generate hidden chaotic attractors. Qiang *et al.* [20] constructed multiple chaotic attractors by increasing the number of saddle foci of the Sprott B system, and presented a new autonomous chaotic system with sign function and various types of coexisting attractors [21]. Danca *et al.* [22] found a chaotic system with five equilibria that yielded a strange attractor with two point attractors, two strange attractors, three limit cycles. The multi-scroll and multi-wing attractors have also been studied by scholars [23–25]. By using the hysteresis series switching approach, Lu *et al.* [26] constructed the multi-scroll attractors from a three-dimensional autonomous system. Bouallegue *et al.* [27] proposed ring-structured multi-scroll and multi-scroll attractors by combining Lorenz attractors through Julia's process. Wang *et al.* [28] derived a type of multi-scroll attractors from a modified Chua's circuit by using the saw-tooth function and studied the circuit realization of the attractors on electronics workbench. Elwakil *et al.* [29] proposed a method of generating multi-scroll chaotic attractors by using the multilevel-logic pulse-excitation. Zhang *et al.* [30] put forward a new chaotic system with multiple-angle sinusoidal nonlinearity generating multi-scroll attractors. In general, the existence of multiple equilibria provides the possibility for the system to have more abundant dynamical behaviors including multiple coexisting attractors, multi-wing or multi-scroll strange attractors, *etc.* It is particularly evident in the high-dimensional neural network which displays multi-stability, multi-periodicity, multi-chaos for its multiple equilibria [31, 32].

Due to the importance of the equilibria in the dynamical behavior of the chaotic system, this paper will present a framework for generating a class of chaotic systems with different number of equilibria by introducing different nonlinear terms. The stability of the equilibria of the systems is investigated. The chaotic systems with one, two, three, and multiple equilibria are presented. The multiple coexisting attractors of the system with three equilibria and the multi-scroll attractor of the system with multiple equilibria are analytically and numerically investigated.

2. Chaotic systems with different number of equilibria

Here, we will consider a unified system which is given by the following four-dimensional autonomous differential equations:

$$\begin{cases} \dot{x} = y\,, \\ \dot{y} = z\,, \\ \dot{z} = -ax - by - z + u\,, \\ \dot{u} = f(x,y,z) - ku \end{cases} \tag{1}$$

with state vector $(x, y, z, u) \in R^4$ and parameter vector $(a, b, k) \in R^3$. The function $f(x, y, z)$ with respect to x, y, z represents the nonlinear term of system (1). In essence, system (1) is an augmented jerk system. The jerk system is expressed by three-dimensional differential equations of the form of $\dddot{x} = J(\ddot{x}, \dot{x}, x)$ [33, 34]. The function $J(\cdot)$ is called the 'jerk' since it portrays the third-time derivative of x and corresponds to the first-time derivative of acceleration in mechanical systems. System (1) is constructed from the jerk system by applying a nonlinear controller u with $\dot{u} = f(x, y, z) - ku$. To a certain extent, system (1) preserves some physical properties of the jerk system for some special forms of the nonlinear function $f(x, y, z)$. If $k > -1$, system (1) is dissipative as its divergence $\nabla V = -(1 + k) < 0$.

The expression of the function $f(x, y, z)$ determines the existence of chaos and the number of equilibria of system (1). Thus, it is important to select a suitable function $f(x, y, z)$ for generating different types of strange attractors from system (1). The following part will investigate the dynamical behaviors of system (1) for given function $f(x, y, z)$ by theoretical and simulation observation.

2.1. One equilibrium with $f(x, y, z) = z^2 - xy$

Suppose that the function $f(x, y, z) = z^2 - xy$, then system (1) has only one equilibrium $P(0, 0, 0, 0)$. The corresponding characteristic equation at the equilibrium can be computed as follows:

$$(\lambda + k)\left(\lambda^3 + \lambda^2 + b\lambda + a\right) = 0\,. \tag{2}$$

By applying the Routh–Hurwitz criterion, we can easily judge that the equilibrium P is asymptotically stable if $k > 0$, $b > a > 0$. If $b = a$, then the roots of Eq. (2) can be calculated as $\lambda_1 = -k$, $\lambda_2 = -1$, $\lambda_{3,4} = \pm\sqrt{b}i$. Obviously, Eq. (2) has a pair of pure imaginary roots. Since the transversality condition

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}a}\right)\Big|_{\lambda=\sqrt{b}i,a=b} = -\frac{b^2 + k^2 b}{2b^3 + (1 + k^2)\,b^2 + k^2 b} < 0$$

is satisfied, then system (1) yields a Hopf bifurcation at P when a increases through a critical value $a_0 = b$. System (1) becomes unstable with the emergence of periodic oscillation.

For parameters $b = 0.5, k = 5$ and initial value $x_0 = (0.1, 0.1, 0.1, 0.1)$, we can plot the bifurcation diagram and maximum Lyapunov exponent (MLE) of system (1) with regard to $a \in [0.1, 1.3]$, as shown in Fig. 1. It is obvious

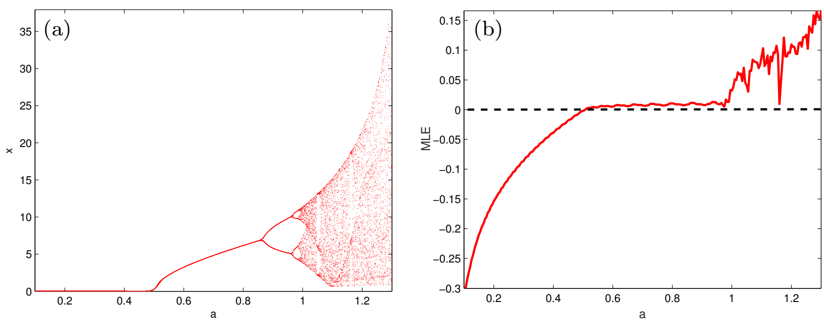

Fig. 1. Bifurcation diagram and maximum Lyapunov exponent of system (1) with $f(x, y, z) = z^2 - xy$ and $a \in [0.1, 1.3]$.

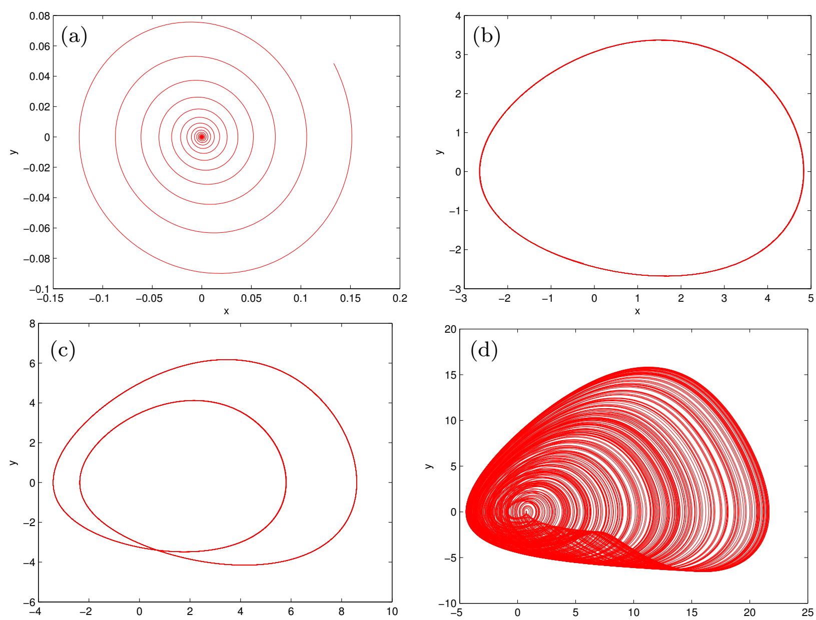

Fig. 2. The projections on phase plane x–y of system (1) with: (a) $a = 0.4$; (b) $a = 0.8$; (c) $a = 0.9$; (d) $a = 1.2$.

that system (1) is stable with $a \in [0.1, 0.5)$, periodic with $a \in [0.5, 1)$, chaotic with $a \in [1, 1.3)$ as its MLE is, respectively, less than zero, equal to zero, greater than zero. System (1) enters the periodic state through the Hopf bifurcation when $a = 0.5$. The chaos in system (1) is generated by period-doubling bifurcation with the variation of a. The projections on phase plane x–y shown in Fig. 2 give a close look at different states of system (1) with $a = 0.4$, $a = 0.8$, $a = 0.9$, $a = 1.2$. It is clear that system (1) is chaotic with $a = 1.2$. The chaotic attractor has a fractal dimension $D = 3.0189$.

2.2. Two equilibria with $f(x, y, z) = x^2$

Suppose that the function $f(x, y, z) = x^2$, then system (1) has two equilibria $P(0, 0, 0, 0)$, $Q(ak, 0, 0, a^2k)$. The eigenvalues λ of the equilibrium P and μ of the equilibrium Q should satisfy the following equations:

$$\begin{aligned} (\lambda + k)\left(\lambda^3 + \lambda^2 + b\lambda + a\right) &= 0\,, \\ (\mu + k)\left(\mu^3 + \mu^2 + b\mu + a\right) - 2ak &= 0\,. \end{aligned}$$

For verifying the stability of P and Q, we should analyze the distribution of the roots of Eq. (3) and Eq. (4). If all the roots of Eq. (3) (or Eq. (4)) are on the left half of the complex plane, then we can say P (or Q) is local stable. Otherwise P and Q are unstable. According to the Routh–Hurwitz criterion, we get that P is stable with $k > 0$, $b > a > 0$ and Q is always unstable with $k > 0$, $a > 0$.

The dynamical evolution of system (1) can be illustrated by plotting its bifurcation diagram and maximum Lyapunov exponent (MLE). Figure 3 shows the bifurcation diagram and maximum Lyapunov exponent with respect to the parameters $b = 1, k = 2, a \in [0.2, 2.1]$ and initial value $x_0 = (0.1, 0.1, 0.1, 0.1)$. From Fig. 3, we know that system (1) is stable with $a \in [0.2, 1)$, periodic with $a \in [1, 1.83)$ and chaotic with $a \in [1.83, 2.1]$. The chaotic attractor in system (1) with $a = 2$ is observed in Fig. 4. We also can use the electronic circuit to demonstrate the physical existence of the chaos in system (1) with $f(x, y, z) = x^2$ and $a = 2, b = 1, k = 2$. A circuit diagram is designed on the basis of the system equations, as shown in Fig. 5. By fixing C1 = C2 = C3 = C4 = 10 nF, R9 = R10 = R11 = R14 = 100 kΩ, R1 = R2 = R3 = R4 = R5 = R6 = R7 = R8 = R12 = R13 = R16 = R17 = R18 = R19 = R20 = R21 = R22 = R24 = 10 kΩ, R15 = R23 = 5 kΩ and running the circuit, we can observe the chaotic attractor of system (1) in the oscilloscope as shown in Fig. 6. It is obvious that Fig. 6 (a), (b) is consistent with Fig. 4 (a), (b). It implies that the chaos of system (1) is physically verified.

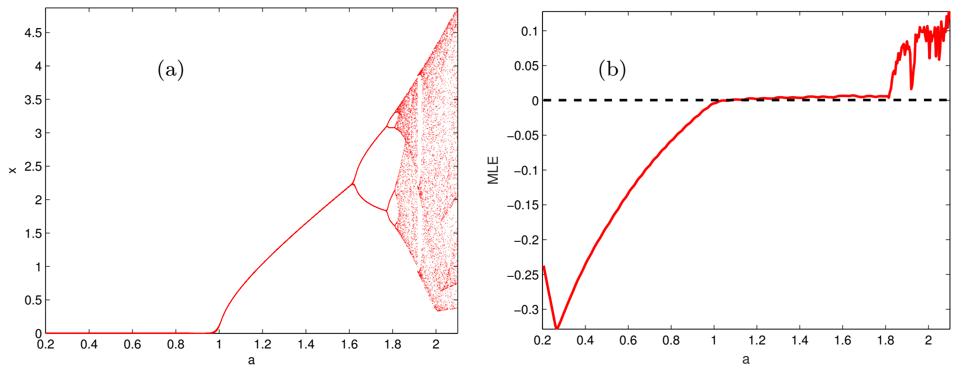

Fig. 3. Bifurcation diagram and maximum Lyapunov exponent of system (1) with $f(x, y, z) = x^2$ and $a \in [0.2, 2.1]$.

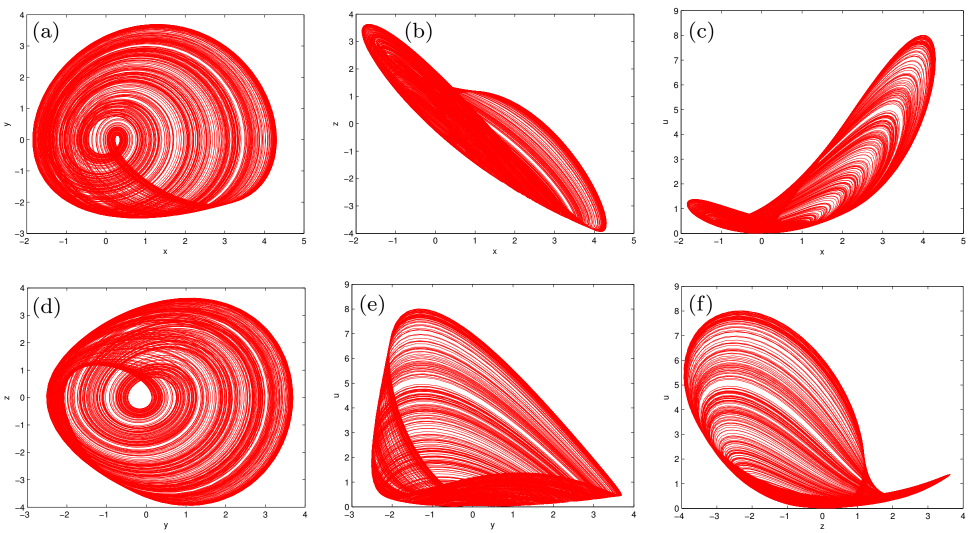

Fig. 4. Chaotic attractor of system (1) with $f(x, y, z) = x^2$ and $a = 2$, $b = 1$, $k = 2$.

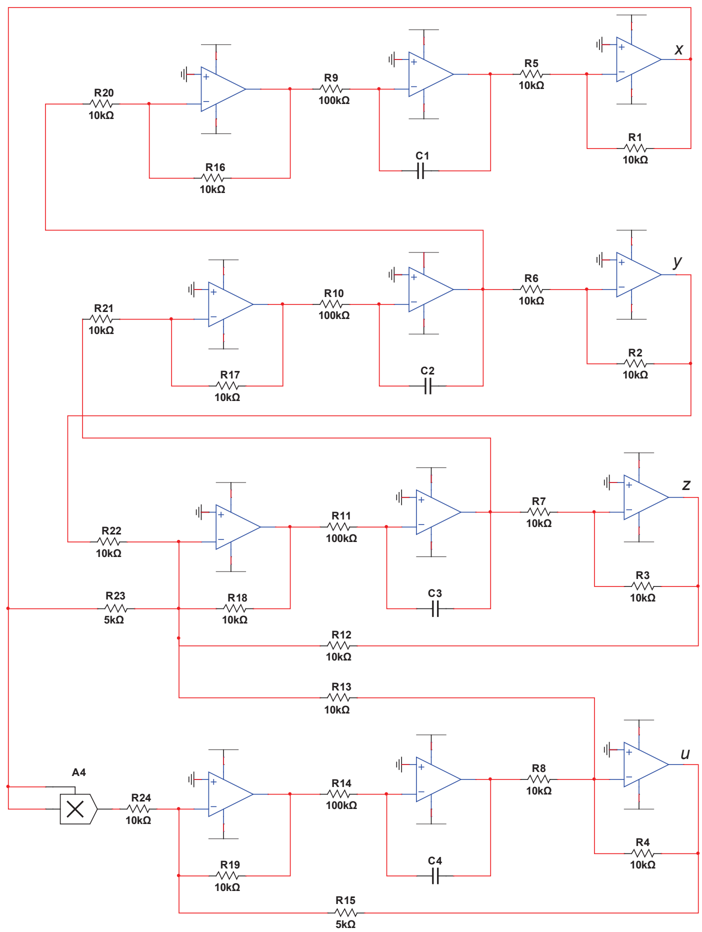

Fig. 5. The circuit diagram of system (1).

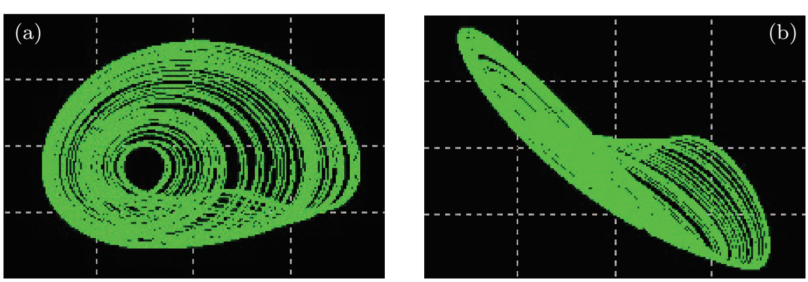

Fig. 6. The phase portraits of system (1) in oscilloscope: (a) x–y; (b) x–z.

2.3. *Three equilibria with* $f(x, y, z) = x^3$

Suppose that the function $f(x, y, z) = x^3$, then system (1) has three equilibria $P(0,0,0,0)$, $Q_{1,2}(\pm\sqrt{ak}, 0, 0, \pm a\sqrt{ak})$. Similarly, we can obtain that P is stable with $k > 0$, $b > a > 0$. The eigenvalues μ of the equilibria $Q_{1,2}$ should satisfy the following equation:

$$(\mu + k)\left(\mu^3 + \mu^2 + b\mu + a\right) - 3ak = 0\,. \tag{3}$$

Thus, $Q_{1,2}$ is always unstable with $k > 0$, $a > 0$. Let $a = 3.7, b = 1, k = 4$, the equilibria and their eigenvalues of system (1) are expressed as follows:

$$\begin{aligned}
&P(0,0,0,0):\\
&\lambda_1 = -4\,, \qquad \lambda_2 = -1.6963\,, \qquad \lambda_{3,4} = 0.3482 \pm 1.4353i\,,\\
&Q_{1,2}(\pm 3.8471, 0, 0, \pm 14.2343):\\
&\mu_1 = -3.5262\,, \qquad \mu_2 = -2.699\,, \qquad \mu_{3,4} = 0.6126 \pm 1.6537i\,.
\end{aligned}$$

Evidently, P and $Q_{1,2}$ are all index-2 saddle foci. Importantly, system (1) generates a pair of strange attractors under the parameters $a = 3.7, b = 1$, $k = 4$, as shown in Fig. 7. The dark gray (red colored) and light gray (green colored) attractors are, respectively, yielded from initial values $x_0 = (0.1, 0.1, 0.1, 0.1)$, $x'_0 = (-0.1, -0.1, -0.1, -0.1)$. Their Lyapunov exponents are $l_1 = 0.1308$, $l_2 = -0.0028$, $l_3 = -0.8523$, $l_4 = -4.2757$. The Lyapunov dimension is fractal as $D = 3.0255$.

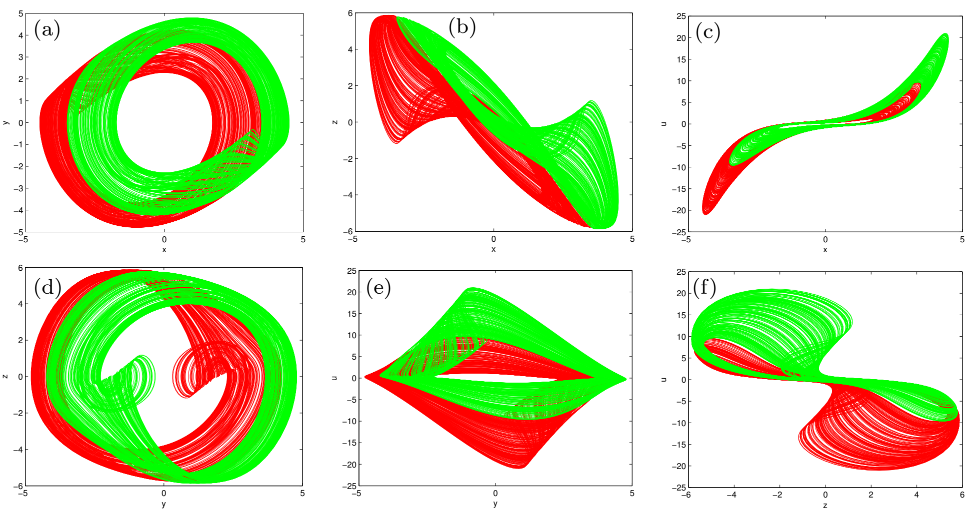

Fig. 7. (Color online) Two strange attractors of system (1) with $f(x, y, z) = x^3$, $a = 3.7, b = 1, k = 4$ and initial values x_0 (dark gray/red), x'_0 (light gray/green).

The coexisting attractors of system (1) can be verified by using the bifurcation diagram. Figure 8 (a) and 8 (b), respectively, show the bifurcation diagrams of system (1) with $a \in [0.1, 3.7]$, $b = 1$, $k = 4$ and $a = 3.2, b = 1, k \in [1.6, 7.6]$. The red and green branches are generated from initial values x_0 and x'_0. In Fig. 8, the separated branches indicate the

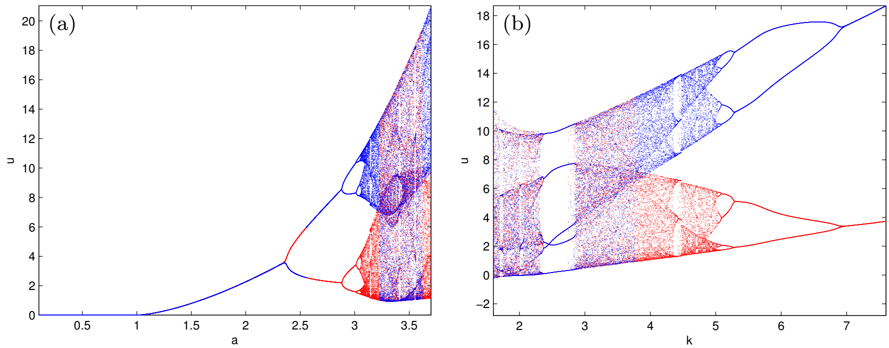

Fig. 8. (Color online) Bifurcation diagrams of system (1) with $f(x, y, z) = x^3$ and: (a) $a \in [0.1, 3.7]$; (b) $k \in [1.6, 7.6]$.

coexistence of two attractors in system (1). Clearly, Fig. 8 (a) shows that system (1) is mono-stable with $a \in [0.1, 1)$, mono-periodic with $a \in [1, 2.37)$, multi-periodic with $a \in [2.37, 3.06)$, multi-chaotic with $a \in [3.06, 3.23)$ and $a \in [3.62, 3.7]$, mono-chaotic with $a \in [3.23, 3.62)$. It can be illustrated by

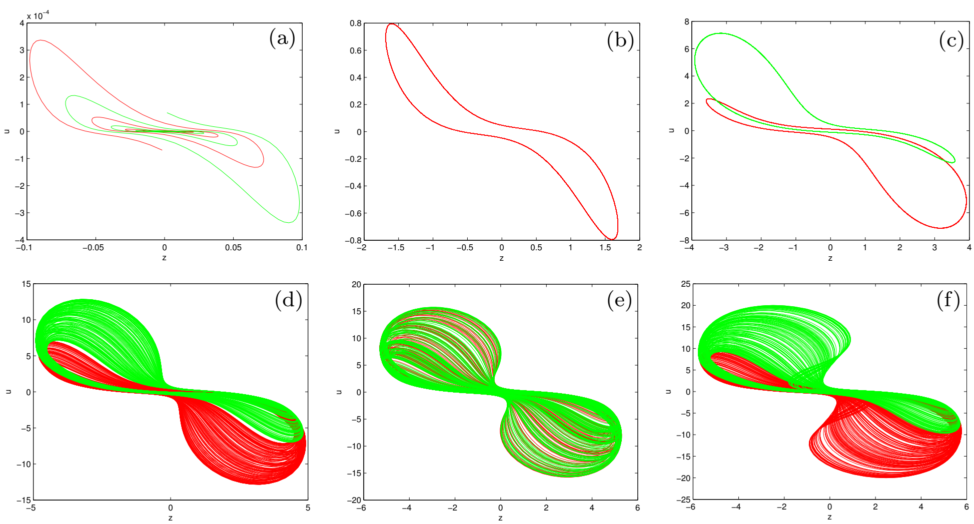

Fig. 9. Different states of system (1): (a) mono-stable with $a = 0.7$; (b) mono-periodic with $a = 1.5$; (c) multi-periodic with $a = 2.7$; (d) mono-chaotic with $a = 3.2$; (e) multi-chaotic with $a = 3.4$; (f) multi-chaotic with $a = 3.65$.

phase portraits in Fig. 9 with parameter values $a = 0.7, 1.5, 2.7, 3.2, 3.4, 3.65$. Figure 8 (b) shows that system (1) has a pair of chaotic attractors with $k \in [3.9, 5.1)$ and a pair of periodic attractors with $k \in [5.1, 7.6]$. By selecting $k = 4.5, 6.5$, we can observe the coexisting attractors of system (1) as shown in Fig. 10.

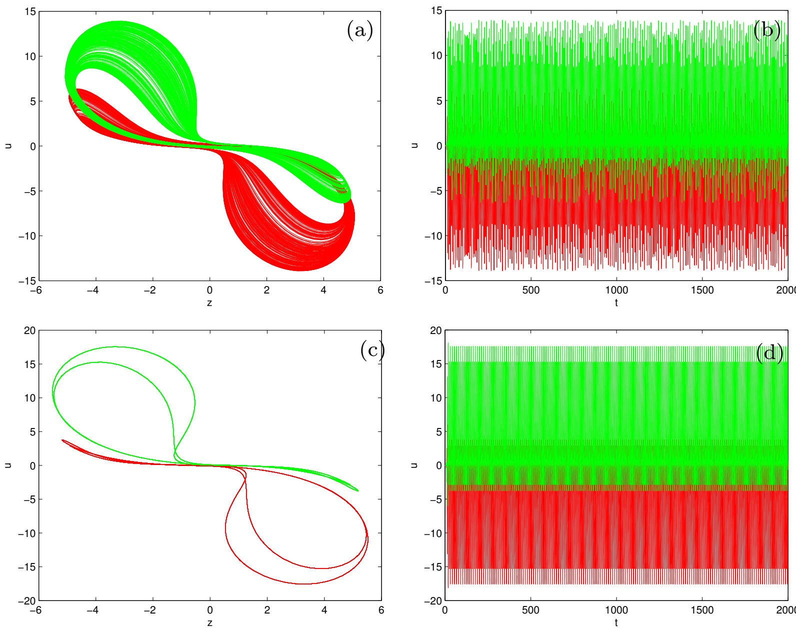

Fig. 10. The coexisting attractors of system (1) with $f(x, y, z) = x^3$ and $k = 4.5, 6.5$.

2.4. *Multiple equilibria with $f(x, y, z) = \sum_{i=1}^{n} sgn(x + s_i)$*

Suppose that the function $f(x, y, z) = \sum_{i=1}^{n} \mathrm{sgn}(x + s_i)$, where $i = 1, 2, \cdots, n$, s_i are real numbers. It is clear that the number of the equilibria is determined by n and s_i. Assume $M(x^*, y^*, z^*, u^*)$ is an equilibrium of system (1), where $x^* = f(x^*)/ak$, $y^* = z^* = 0$, $u^* = ax^*$, then the characteristic equation at M can be obtained as follows:

$$(\lambda + k)\left(\lambda^3 + \lambda^2 + b\lambda + a\right) = 0\,. \tag{4}$$

Thus, M is stable with $k > 0$, $b > a > 0$. Obviously, the stability of M is only determined by a, b, k and has nothing to do with the function $f(x, y, z)$. Owing to the existence of multiple equilibria, system (1) can generate multi-scroll chaotic attractor. We will use the numerical simulations for finding the multi-scroll attractor of system (1). In the following discussion, we fix the parameter values at $a = 1, b = 0.5, k = 2$. Therefore, all the equilibria of system (1) are index-2 saddle foci with eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1.2442$, $\lambda_{3,4} = 0.1221 \pm 0.8882i$.

For the function $f(x, y, z) = \mathrm{sgn}(x)$, system (1) has two equilibria $M_{1,2}$ $(\pm0.5, 0, 0, \pm0.5)$ and generates two-scroll attractor as shown in Fig. 11 (a). For $f(x, y, z) = \mathrm{sgn}(x) + \mathrm{sgn}(x + 1) + \mathrm{sgn}(x - 1)$, system (1) has four equilibria $M_{1,2}(\pm0.5, 0, 0, \pm0.5)$, $M_{3,4}(\pm1.5, 0, 0, \pm1.5)$ and generates four-scroll attractor as shown in Fig. 11 (b). For $f(x, y, z) = \mathrm{sgn}(x) + \mathrm{sgn}(x + 1) + \mathrm{sgn}(x - 1) + \mathrm{sgn}(x + 2) + \mathrm{sgn}(x - 2)$, system (1) has six equilibria $M_{1,2}(\pm0.5, 0, 0, \pm0.5)$, $M_{3,4}(\pm1.5, 0, 0, \pm1.5)$, $M_{5,6}(\pm2.5, 0, 0, \pm2.5)$ and generates six-scroll attractor as shown in Fig. 11 (c). For $f(x, y, z) = \mathrm{sgn}(x)+\mathrm{sgn}(x+1)+\mathrm{sgn}(x-1)+\mathrm{sgn}(x+2)+\mathrm{sgn}(x-2)+\mathrm{sgn}(x+3)+\mathrm{sgn}(x-3)$, system (1) has eight equilibria $M_{1,2}(\pm0.5, 0, 0, \pm0.5)$, $M_{3,4}(\pm1.5, 0, 0, \pm1.5)$, $M_{5,6}(\pm2.5, 0, 0, \pm2.5)$, $M_{7,8}(\pm3.5, 0, 0, \pm3.5)$ and generates eight-scroll attractor as shown in Fig. 11 (d). These multi-scroll attractors has the same Lyapunov exponents $l_1 = 0.1220$, $l_2 = 0.1212$, $l_3 = -1.2432$, $l_1 = -2$. Since there exist two positive Lyapunov exponents, they are all hyper-chaotic. As to different values of n, we can get different scroll attractors. If n approaches to infinity, then infinite scroll attractor can be obtained.

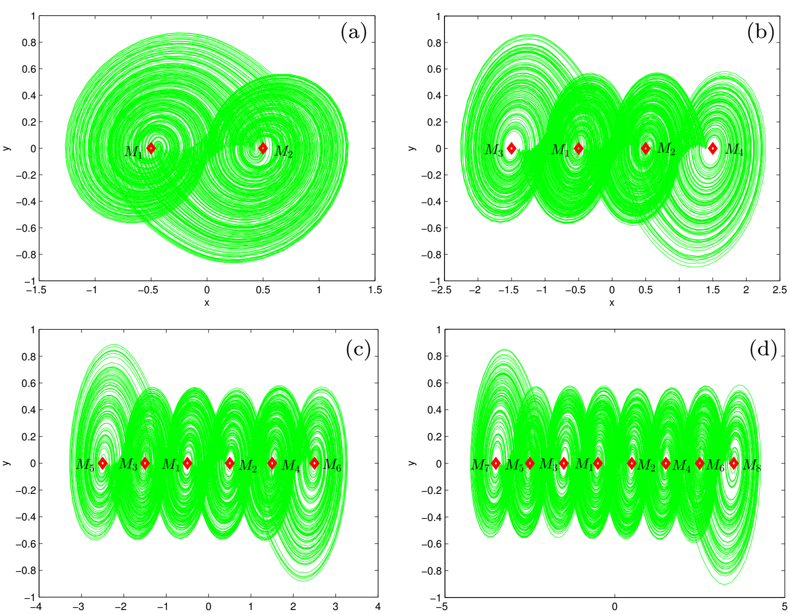

Fig. 11. The multi-scroll attractors of system (1) with $f(x, y, z) = \sum_{i=1}^{n} \mathrm{sgn}(x + s_i)$.

3. Conclusions

In this paper, we have considered the problem of the construction of a framework for generating chaotic systems with different number of equilibria. For accomplishing this proposition, a four-dimensional autonomous chaotic system has been presented. The nonlinear term of the system determines its equilibria and dynamical behaviors. Interestingly, the system yields a pair of chaotic attractors or periodic attractors with nonlinear term $f(x,y,z) = x^3$ and multi-scroll hyper-chaotic attractors with nonlinear term $f(x,y,z) = \sum_{i=1}^{n} \mathrm{sgn}(x+s_i)$. This work shows that the number and stability of the equilibria usually determine the types of the attractors of the system.

REFERENCES

[1] M. Somenath *et al.*, *Chaos Solitons Fract.* **96**, 23 (2017).

[2] R.T. Fotsa, P. Woafo, *Chaos Solitons Fract.* **93**, 48 (2016).

[3] H. Jalili, H. Salarieh, G. Vossoughi, *Nonlinear Dyn.* **82**, 1355 (2015).

[4] F. Pittorino *et al.*, *Phys. Rev. Lett.* **118**, 098102 (2017).

[5] T. Hu, Y. Liu, L. Gong, C. Ouyang, *Nonlinear Dyn.* **87**, 51 (2017).

[6] G. Zhou *et al.*, *Neurocomputing* **169**, 150 (2015).

[7] Y. Du, T.A. Duever, H. Budman, *Comput. Chem. Eng.* **76**, 63 (2015).

[8] A. Trevisan, L. Palatella, *Int. J. Bifur. Chaos* **21**, 3389 (2011).

[9] E.N. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963).

[10] G. Chen, T. Ueta, *Int. J. Bifurc. Chaos* **9**, 1465 (1999).

[11] C. Li, H. Li, Y. Tong, *Optik* **124**, 1516 (2013).

[12] W.B. Liu, G. Chen, *Int. J. Bifur. Chaos* **14**, 971 (2004).

[13] Q. Lai, S. Chen, *Optik* **127**, 3000 (2016).

[14] A. Akgul, S. Hussain, I. Pehlivan, *Optik* **127**, 7062 (2016).

[15] Q. Lai, T. Nestor, J. Kengne, X.W. Zhao, *Chaos Solitons Fract.* **107**, 92 (2018).

[16] C.P. Silva, *IEEE Trans. Circ. Syst. I: Fund. Theory Appl.* **40**, 675 (1993).

[17] T. Zhou, Y. Tang, G. Chen, *Int. J. Bifur. Chaos* **14**, 3167 (2004).

[18] S. Yu, W.S. Tang, J. Lu, G. Chen, *Int. J. Bifur. Chaos* **20**, 29 (2010).

[19] S. Jafari, J.C. Sprott, S. Mohammad Reza Hashemi Golpayegani, *Phys. Lett. A* **377**, 699 (2013).

[20] Q. Lai, S. Chen, *Int. J. Bifur. Chaos* **26**, 1650177 (2016).

[21] Q. Lai, A. Akgul, X.W. Zhao, H. Pei, *Int. J. Bifur. Chaos* **27**, 1750142 (2017).

[22] M.F. Danca, M. Feckan, N. Kuznetsov, G. Chen, *Int. J. Bifur. Chaos* **26**, 1650038 (2016).

[23] K. Bouallegue, *Int. J. Bifur. Chaos* **25**, 1530002 (2015).
[24] C. Zhang, S. Yu, *Int. J. Circ. Theory Appl.* **41**, 221 (2013).
[25] Y. Guo, G. Qi, Y. Hamam, *Nonlinear Dyn.* **85**, 2765 (2016).
[26] J. Lu, F. Han, X. Yu, G. Chen, *Automatica* **40**, 1677 (2004).
[27] K. Bouallegue, A. Chaari, A. Toumi, *Chaos Solitons Fract.* **44**, 79 (2011).
[28] F.Q. Wang, C.X. Liu, *Int. J. Mod. Phys. B* **22**, 2399 (2008).
[29] A.S. Elwakil, S. Ozoguz, *IEEE Trans. Circ. Syst. II: Express Briefs* **53**, 862 (2006).
[30] C. Zang, W.S. Tang, S. Yu, *Int. J. Bifur. Chaos* **19**, 2073 (2009).
[31] E.B. Haghighi, G. Palm, M. Rahmati, M.J. Yazdanpanah, *Neural Networks* **65**, 53 (2015).
[32] X. Nie, W.X. Zheng, *IEEE Trans. Cyber.* **46**, 679 (2016).
[33] J.C. Sprott, *Am. J. Phys.* **65**, 537 (1997).
[34] J.C. Sprott, *Am. J. Phys.* **68**, 758 (2000).