GENERALISATION OF BTW MODEL WITH NEXT NEAREST NEIGHBOUR

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The two-dimensional BTW model of self-organised criticality (SOC) with critical height, $z_{\rm c} = 8$, is studied by computer simulation in the following two different cases. When the value of height variable of a particular site reaches the critical value, $z_{\rm c} = 8$, the value of height variable of that site is reduced by eight units: *(i)* by distributing eight particles among the four nearest neighbouring sites and four next nearest neighbouring sites, each receiving one particle at a time; *(ii)* by distributing eight particles among the four nearest neighbouring sites, each receiving two particles at a time. It is observed that in the SOC state, the average (spatial) value of height variable, $\bar{z}$, in the BTW model with next nearest neighbour is less than that in the BTW model with only nearest neighbour. But in the SOC state, the distributions of avalanche sizes and durations are identical in both the cases. The distribution of the size of clusters for different values of height variable have been studied in both the cases of BTW model.

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1. Introduction

The phenomena of self-organised criticality (SOC) is characterised by the spontaneous evolution of an extended-driven dynamical system towards a steady state which shows long-range spatial and temporal correlations. Bak, Tang and Wiesenfeld [1, 2] introduced the concept of SOC in terms of a simple cellular automata model. The steady state dynamics of the model shows a power law behaviour in the probability distributions for the occurrence of the relaxation (avalanches) clusters of a certain size, area, lifetime, *etc.* The BTW model has been solved exactly using the commutative property of the particle addition operator [3]. Several properties of this critical state, *e.g.*, entropy, height correlation, height probabilities, cluster statistics, *etc.* have been studied [4–7]. To study the properties of the model in the

SOC state and to estimate various critical exponents, extensive numerical efforts have also been performed [8–14]. The avalanche exponents were estimated using the renormalization scheme [15, 16]. The BTW model in dilute lattice has also been studied recently [17, 18].

The original BTW model is a so-called 'sandpile' model with deterministic and isotropic toppling rule. After the introduction of a stochastic sandpile model by Manna [19], various studies have been performed on stochastic sandpile model [10, 20, 21]. The critical behaviour of the sandpile model with stochasticity in toppling is different from that of deterministic toppling rules [22]. Recently, continuous transformation of the BTW model to the Manna model has been studied [23] by introducing an 'intermediate model'. The BTW model with probabilistically anisotropic toppling rule has been studied recently [24].

In the original BTW model, when a height variable of a particular site reaches the value $z_{\mathrm{c}} = 4$, that particular site becomes unstable and it topples. As the site topples, the particles of that site are distributed among four of its nearest neighbouring sites. In this paper, we will study the BTW model, when particles of the unstable site are distributed not only among the nearest neighbouring sites but also among the next nearest neighbouring sites. Here, in this work, we have considered the 2D BTW model with critical height $z_{\mathrm{c}} = 8$. Thus, when a particular site becomes unstable (*i.e.*, reaches $z_{\mathrm{c}} = 8$), then it topples and eight particles are distributed equally among eight neighbouring sites which includes four of its nearest neighbouring sites and four next nearest neighbouring sites. Thus, each of the eight neighbouring sites receives one particle. However, to study the effects of inclusion of next nearest neighbour in the BTW model, we have also studied the BTW model considering only nearest neighbour, having the same critical height, $z_{\mathrm{c}} = 8$. In this case, when a particular site becomes unstable, eight particles are equally distributed among its four nearest neighbouring sites only, where each site receives two particles.

Would it be interesting to know whether the inclusion of next nearest neighbours in the BTW model changes the behaviour of the critical state? To address this particular question, we have calculated the average value of $\bar{z}$ in the critical state and studied the different statistics of avalanches. It is important to mention in this context that in [25], Hu and Lin showed that the toppling of waves of BTW model on different 2D lattices (such as square, honeycomb, triangular and random lattices having different number of nearest neighbours) have the same set of critical exponents. Najafi, Saman and Rouhani [26] studied the statistics of waves and avalanche frontiers of the dissipative Abelian sandpile model (massive BTW) on honeycomb lattice (having six nearest neighbour for each site) and showed that the exponents are the same as the square lattice.

This paper is organised as follows. In Section 2, the model and simulation is discussed. In Section 3, the results are described. The paper ends with the conclusion in Section 4.

2. The model and simulation

The BTW model is a lattice automata model of sandpile growth, which evolves spontaneously into a critical state. We consider a two-dimensional square lattice of size $L \times L$. The model is described as follows: Each site (i, j) of the lattice is associated with a variable (so-called height) $z(i, j)$ which can take positive integer values varying from 0 to z_{c}. In every time step, one particle is added to a randomly selected site, which increases the value of the height of that site by unity, *i.e.*,

$$z(i,j) = z(i,j) + 1\,. \tag{1}$$

When the height variable of any site (i, j) exceeds a critical value z_{c} (*i.e.*, if $z(i, j) \geq z_{\mathrm{c}}$), then that site becomes unstable and it relaxes by a toppling. When an unstable site topples, the value of the height variable of that site is reduced by z_{c} units and z_{c} particles are distributed among the neighbouring sites (local conservation). In the original BTW model, the critical height, $z_{\mathrm{c}} = 4$, and when the unstable site topples, the particles are distributed among only the four of its nearest neighbouring sites. In this work, we have considered a BTW model with $z_{\mathrm{c}} = 8$. As the unstable site topples, the particles are distributed in two different ways among the neighbouring sites. In one case, we have considered the distribution of particles among the sites which includes four nearest neighbouring sites and four next nearest neighbouring sites (next nearest neighbour BTW model). In the other case, the particles are distributed only among the four nearest neighbouring sites (nearest neighbour BTW model). Thus, in the case of the next nearest neighbour BTW model, the height variables of unstable site and each of the eight of its neighbouring sites change according to the following rule (local conservation), *i.e.*,

$$\begin{aligned} z(i,j) &= z(i,j) - z_{\mathrm{c}}\,, & (2)\\ z(i,j\pm 1) &= z(i,j\pm 1) + 1\,, & (3)\\ z(i\pm 1,j) &= z(i\pm 1,j) + 1\,, & (4)\\ z(i\pm 1,j\pm 1) &= z(i\pm 1,j\pm 1) + 1 & (5) \end{aligned}$$

for $z(i, j) \geq z_{\mathrm{c}}$. Each boundary site is attached to an additional site which acts as a sink. We use here the open boundary conditions so that the system can dissipate through the boundary.

We have studied the nearest neighbour BTW model, having the same critical height, $z_{\rm c} = 8$, to compare the effect of inclusion of next nearest neighbour. In the nearest neighbour BTW model, each of the four nearest neighbouring sites receives two particles, when the unstable site topples. Thus, in this case, as the unstable site topples, the particles from the unstable site will move towards its nearest neighbouring sites as follows:

$$z(i,j) = z(i,j) - z_{\rm c}\,, \tag{6}$$

$$z(i\pm 1,j) = z(i\pm 1,j) + 2 \qquad \text{and} \qquad z(i,j\pm 1) = z(i,j\pm 1) + 2\,. \tag{7}$$

In this work, we have studied the following observations in the next nearest neighbour BTW model and nearest neighbour BTW model with critical height, $z_{\rm c} = 8$. Here, the system starts to evolve from an initial condition, where all the sites have values, $z = 0$.

1. The time evolution of the average (spatial) value of z, *i.e.*,

$$\bar{z} = (1/N)\sum_{i=1}^{N} z_i\,, \qquad \left(N = L^2\right)\,.$$

2. The distribution of the avalanche size, $D(s)$, avalanche time, τ, and number of distinct sites toppled, $D(N_{\rm ds})$.
3. The fraction of sites, f_z, having the height variable $z = 0, 1, 2, \ldots, z_{\rm c}-1$, in the critical state.
4. The distribution of size of clusters formed by the sites having a particular value of height variable, $z = 0, 1, 2, \ldots, z_{\rm c} - 1$.

3. Results

In this paper, we have studied the two-dimensional next nearest neighbour BTW model and nearest neighbour BTW model with critical height, $z_{\rm c} = 8$. Here, we have considered a square lattice of size of $L = 400$. We first studied the time evolution of the average (spatial) value of z (*i.e.*, $\bar{z}$) for both the forms of the BTW model which are plotted in figure 1. Figure 1 shows that the value of $\bar{z}$ in critical state is 4.5 in the next nearest neighbour BTW model but in the nearest neighbour BTW model, the value of $\bar{z}$ in the critical state is 4.75. It is interesting that due to the distribution of particles among the next nearest neighbouring sites, the value of $\bar{z}$ is reduced. Now, to see whether this difference in $\bar{z}$ is due to the finite size effect or not, the variation of $\bar{z}$ is studied for three different values of the system sizes

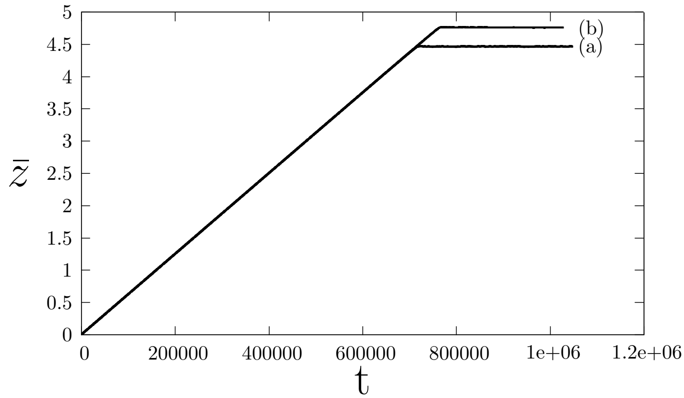

Fig. 1. Plot of time variation of the spatial average of height variable $\bar{z}$ for the distribution of particles from the unstable site among (a) the four nearest and four next nearest neighbouring sites (n.n.n.BTW model) and (b) the four nearest neighbouring sites only (n.n.BTW model).

(*e.g.*, $L = 100$, $L = 200$ and $L = 400$) and plotted in figure 2. The figure shows that the difference in the values of $\bar{z}$ (for two kinds of rules mentioned above) remains unchanged as the system size increases, which reveals that this observation is free from any finite size effect.

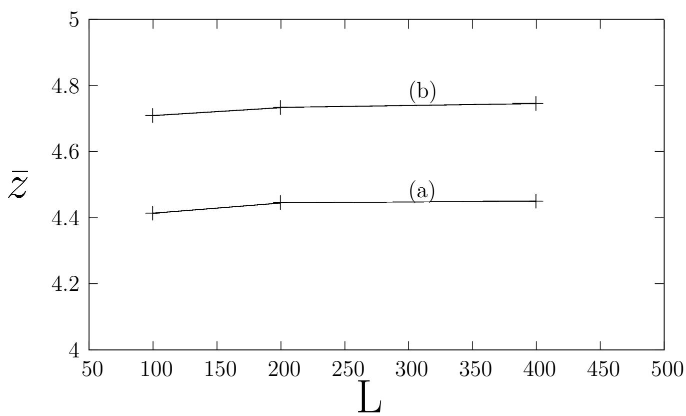

Fig. 2. Plot of variation of the spatial average of height variable $\bar{z}$ with lattice size L for the distribution of particles from the unstable site among (a) the four nearest and four next nearest neighbouring sites (n.n.n.BTW model) and (b) the four nearest neighbouring sites only (n.n.BTW model)

We have also calculated the distributions of duration, τ, and size, s, of the avalanches and the number of distinct sites toppled during avalanches, $D(N_{\mathrm{ds}})$, at the critical state, for both the cases of BTW model with $z_{\mathrm{c}} = 8$.

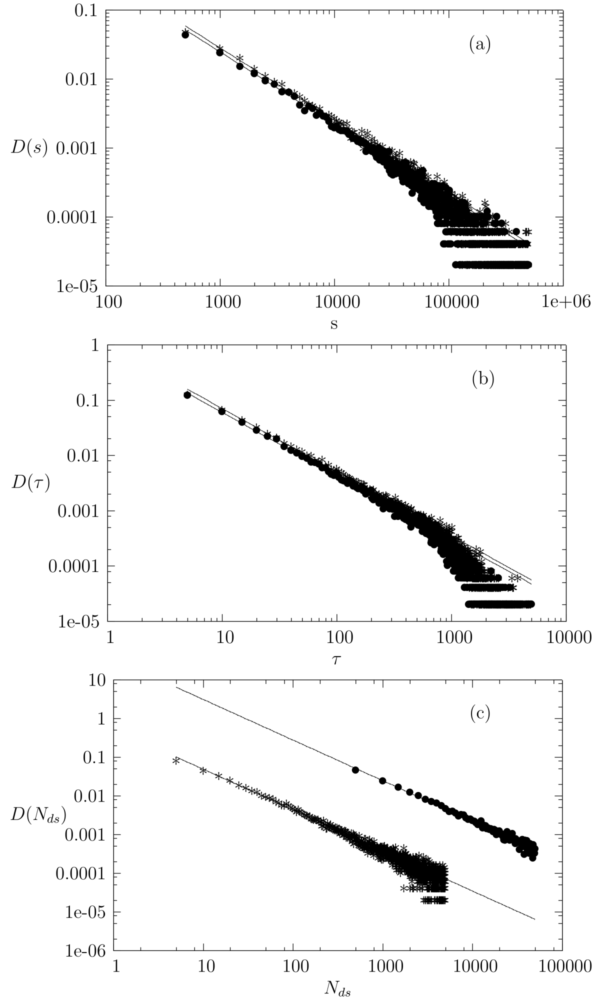

Fig. 3. (a) Log–log plot of distribution of avalanche size (s) for distribution of particles from the unstable site *(i)* among the four nearest neighbours ($*$) and *(ii)* among four nearest neighbours and four next nearest neighbours ($\bullet$). Both the solid lines represent $y \sim x^{-1.05}$. (b) Log–log plot of distribution of avalanche time (τ) for distribution of particles from the unstable site *(i)* among the four nearest neighbours ($*$) and *(ii)* among four nearest neighbours and four next nearest neighbours ($\bullet$). Both the solid lines represent $y \sim x^{-1.15}$. (c) Log–log plot of distribution of number of distinct sites toppled during an avalanche ($N_{\rm ds}$) for distribution of particles from the unstable site *(i)* among the four nearest neighbours ($*$) and *(ii)* among four nearest neighbours and four next nearest neighbours ($\bullet$). Both the solid lines represent $y \sim x^{-1.05}$.

The distributions are obtained for 50 000 number of nonzero avalanches, $L = 400$. The distribution of avalanche size, $D(s)$, is plotted, on a doubly logarithmic scale, in figure 3 (a). Similarly, the distribution of avalanche time, $D(\tau)$, is plotted on a doubly logarithmic scale in figure 3 (b), and the distribution of number of distinct sites toppled during avalanche, $D(N_{\rm ds})$, is plotted on a doubly logarithmic scale in figure 3 (c). We have estimated the value of the exponents within limited accuracy and given by $D(s) \propto s^{-1.05}$, $D(\tau) \propto \tau^{-1.15}$ and $D(N_{\rm ds}) \propto N_{\rm ds}^{-1.05}$. The exponents of the power law, $y \sim x^{-p}$, have been estimated from the slope, $-p$, of linear best fit of $\ln(y) \sim -p\ln(x)$. However, these $y \sim x^{-p}$ are shown in log–log plot using the numerically obtained (from linear best fit) value of p. The power law variations of the distribution of avalanche size, s, and avalanche time, τ, given by $D(s) \propto s^{-1.05}$ and $D(\tau) \propto \tau^{-1.15}$ shows that the exponents are the same as in the original BTW model.

Thus, it is observed that though the average value of height variable in the critical state is different, the variation of different quantities related to avalanches in the critical state are identical and these exponents are identical with those of the original BTW model. To study the effect of inclusion of next nearest neighbour on the behaviour of critical state, various studies related to the structure of the lattice at the critical state have been performed. One such quantity which is used to study the lattice structure in the SOC state, in original BTW model, is the fraction of the sites having different values of the height variable, $z = 0, 1, 2, 3$ [9]. Here, in both the cases of BTW model, in the critical state, the lattice contains the sites having different values of the height variable, $z = 0, 1, 2, \ldots, 7$. In this work, we have calculated the fraction of sites, f_z, for different values of height variable, $z = 0, 1, 2, \ldots, 7$, for both the cases. The variation of f_z with z has been plotted in figure 4. Interestingly, it is observed here, that in the next nearest neighbour BTW model, f_z increases linearly with z, whereas in the nearest neighbour BTW model, $f_0 = f_1$, $f_2 = f_3$, $f_4 = f_5$, and $f_6 = f_7$. Thus, the values of f_z and f_{z+1} are equal for $z = 0, 2, 4, 6$ and the pairwise values of f_z and f_{z+1} increase with z.

The sites having a particular value of height variables form clusters connected via the nearest neighbours. These clusters are of different sizes. The size distribution of the clusters is described by the function $\rho_z(s_z)$, where $\rho_z(s_z)$ denotes the number of clusters (formed via nearest neighbour connection of sites having the same height variable z) of size s_z. In the original BTW model, the cluster size distribution has been obtained [7] using the algorithm described in [27, 28]. In this work, we have also calculated the cluster size distribution, $\rho_z(s_z)$, in the SOC state for both types of BTW model, for different values of height variable, $z = 4, 5, 6, 7$. The normalised cluster size distribution ($n_s = \frac{\rho_z(s_z)}{L^2}$ is the number of clusters of size s per

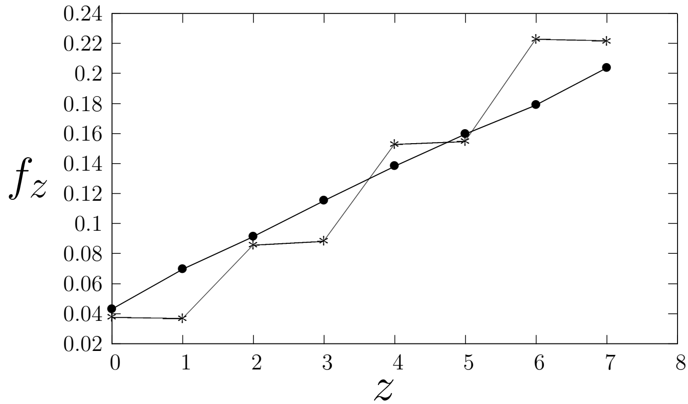

Fig. 4. Plot of f_z against z for distribution of particles from the unstable site among *(i)* the four nearest and four next nearest neighbouring sites ($\bullet$) and *(ii)* the four nearest neighbouring sites ($*$).

lattice site), for different values of height variable z are plotted on a semi-log scale in figure 5 (a) and (b) for next nearest neighbour BTW model and nearest neighbour BTW model, respectively. In both the cases, the cluster size distribution fits with the curve

$$n_s = A * s^{-p} * \exp(-q * s)\,.$$

In the case of next nearest neighbour BTW model, the value of the exponents are $p = 1.1$ for all values of z ($z = 7, 6, 5, 4$) and the values of exponent q are $0.55, 0.57, 0.63$ and 0.76 for $z = 7, 6, 5, 4$, respectively. Thus, we see that the value of the exponent q decreases as the value of height variable increases in the case of the next nearest neighbour BTW model. Whereas in the case of the nearest neighbour BTW model, the value of the exponent p is 1.0 for all values of z ($z = 7, 6, 5, 4$) and the values of exponent q are $0.46, 0.46, 0.68$ and 0.68 for $z = 7, 6, 5, 4$, respectively. Here, the value of the exponent q is same pairwise. The value of $q = 0.46$ for $z = 7$ and 6 and $q = 0.68$ for $z = 5$ and 4. The value of exponent q decreases as the height variable z increases. Thus, the nature of the cluster size distribution is the same as that in the original 2D BTW model [7].

Knowing the size distribution, $\rho_z(s_z)$, for a particular value of height variable z, one can calculate the various statistical quantities, the average size of the cluster (S_z), total number of clusters (N_z), the largest size ($s_z^{\max}$) of the cluster and the fraction of sites having height variable z (f_z). The interesting feature of this study is pairwise identical values of the exponents, q, of the distribution of cluster size in the case of nearest neighbour BTW model. This behaviour also reveals the pairwise equal values of f_zs. The

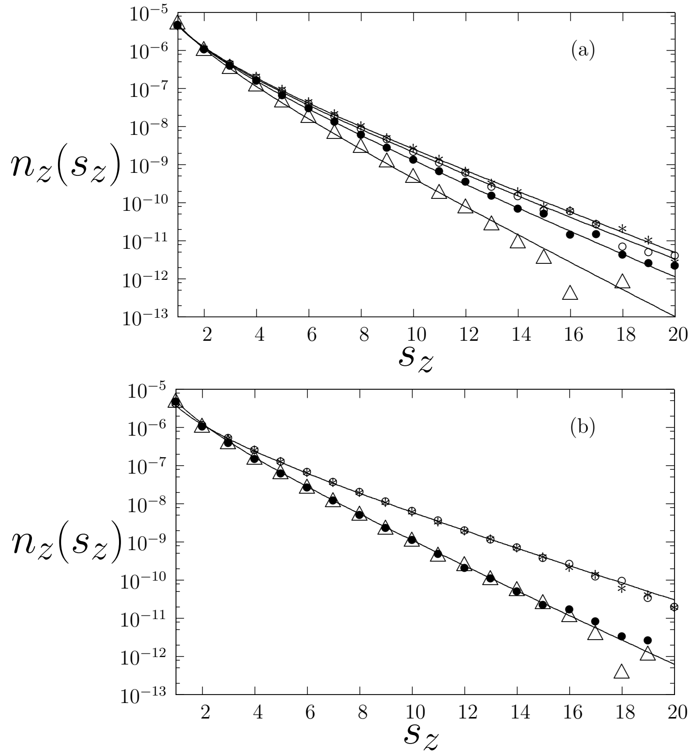

Fig. 5. Plots of normalised cluster size distribution for four values of height variable, $z = 4$ ($\triangle$), 5 ($\bullet$), 6 ($\circ$), 7 ($*$) for distribution of particles from the unstable site among (a) the four nearest and four next nearest neighbouring sites and (b) the four nearest neighbouring sites.

feature of pairwise equality of f_zs in the nearest neighbour BTW model may be due to the fact that the particles are distributed pairwise among the nearest neighbouring sites from the unstable site.

4. Summary

We studied here both the forms of two-dimensional BTW model (nearest neighbour and next nearest neighbour) with critical height $z_c = 8$. It is observed that in the case of nearest neighbour BTW model, the spatial average value of the height variable, $\bar{z}$, reaches a steady value, which is more than the value obtained in the next nearest neighbour BTW model. In both the forms of the BTW model having critical height $z_c = 8$, the exponents of the power law distribution of size, s, duration, τ, of the avalanches and the number of distinct sites toppled during avalanches, N_{ds}, are calculated as $D(s) \propto s^{-1.05}$, $D(\tau) \propto \tau^{-1.15}$ and $D(N_{\rm ds}) \propto N_{\rm ds}^{-1.05}$. Thus, from the present study of avalanche statistics (power law with the same set of values of the exponents as obtained in the original BTW model), it seems that the proposed model with modification of toppling rules (including the next nearest neighbours) does not leave the universality class of original BTW model. The fractions of the lattice sites, f_z, having different height variables, $z = 0, 1, 2, \ldots, 7$, at the critical state have been calculated. The variations of f_z with z in both the cases have also been studied. The normalised cluster size distribution for height variables, $z = 4, 5, 6, 7$, has been obtained for both the cases. It would be interesting to study the dynamical evolution of the cluster, its nucleation and coalescence *etc.*

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