LETTERS TO THE EDITOR

LAGRANGIANS FOR MASSIVE, ARBITRARY SPIN FIELDS

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Explicit Lagrangians for massive, arbitrary spin fields are developed. These Lagrangians lead, by the usual variation, to the field equations and subsidiary conditions for fields describing massive particles of arbitrary spin.

A question in field theory which has been discussed for many years is [1]: is it possible to write a Lagrangian for arbitrary spin fields which, upon variation, leads to the field equations and subsidiary conditions. Fierz and Pauli [1, 2] studied the problem for spin k by introducing auxiliary fields describing particles of spin k^1 $(< k)$ into the Lagrange densities. Kawakami [3] has developed a Lagrange density for spin 5/2 by introducing additional terms containing γ matrices or derivatives into the Lagrange density, but this approach leads to second order derivatives in the Lagrange density. In this note Lagrange densities are given for arbitrary integer and half-integer spin fields describing particles of mass m which lead, by variation, to the correct field equations and subsidiary conditions. These Lagrange densities contain no auxiliary fields or derivatives of second order or higher. In this paper the theory is confined to flat space-time. The difficulties encoutered in the generalization to curved space-time are not discussed [4].

The Lagrange densities considered here contain Hermitian terms which will lead to the field equations and symmetry conditions and anti-Hermitian terms containing the remaining subsidiary conditions. Once the field equations and subsidiary conditions are satisfied the Lagrange densities and energy momentum tensor are Hermitian.

Consider first the case for integer spin k. The field variables are k-th rank tensors. The Lagrange density for this case is

$$L = (-1)^k\{\partial_\varrho \psi^{\dagger\mu_1\cdots\mu_k}\partial^\varrho \psi_{(\mu_1\cdots\mu_k)} - m^2\psi^{\dagger\mu_1\cdots\mu_k}\psi_{\mu_1\cdots\mu_k} + $$
$$+ i\partial_\beta \psi^{\dagger\beta\mu_2\cdots\mu_k}\partial^\varrho \psi_{\varrho\mu_2\cdots\mu_k} + im^2 g_{\alpha\beta}\psi^{\dagger\alpha\beta\mu_3\cdots\mu_k} g^{\varrho\sigma}\psi_{\varrho\sigma\mu_3\cdots\mu_k}\}, \tag{1}$$

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where $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and

$$\psi_{(\mu_1 \cdots \mu_k)} = \frac{1}{k!}(\psi_{\mu_1 \cdots \mu_k} + \psi_{\mu_2 \mu_1 \cdots \mu_k} + \ldots). \tag{2}$$

Proof:

The Euler-Lagrange equations resulting from variation with respect to $\psi^{\dagger \mu_1 \ldots \mu_k}$ are

$$(-1)^k\{-\partial_\varrho \partial^\varrho \psi_{(\mu_1 \cdots \mu_k)} - m^2 \psi_{\mu_1 \cdots \mu_k}\} + im^2 g_{\mu_1 \mu_2} g^{\varrho\sigma} \psi_{\varrho\sigma\mu_3 \cdots \mu_k} - i\partial_{\mu_1} \partial^\varrho \psi_{\varrho\mu_2 \cdots \mu_k} = 0. \tag{3}$$

Variation of L with respect to $\psi^{\mu_1 \cdots \mu_k}$ gives the same equations with $\psi_{\mu_1 \ldots \mu_k}$ replaced by $\psi^\dagger_{\mu_1 \ldots \mu_k}$. Taking the Hermitian conjugate of the equation for $\psi^\dagger_{\mu_1 \ldots \mu_k}$ and adding to Eq. (3) gives

$$\partial_\varrho \partial^\varrho \psi_{(\mu_1 \cdots \mu_k)} + m^2 \psi_{\mu_1 \cdots \mu_k} = 0 \tag{4}$$

and

$$m^2 g_{\mu_1 \mu_2} g^{\alpha\beta} \psi_{\alpha\beta\mu_3 \cdots \mu_k} - \partial_{\mu_1} \partial^\beta \psi_{\beta\mu_2 \cdots \mu_k} = 0. \tag{5}$$

Interchanging successively all indices in Eq. (4) and subtracting each one from Eq. (4) gives the symmetry condition,

$$\psi_{\mu_1 \cdots \mu_k} = \psi_{(\mu_1 \cdots \mu_k)}. \tag{6}$$

Using this in Eq. (4) gives the field equation,

$$(\partial_\varrho \partial^\varrho + m^2) \psi_{\mu_1 \cdots \mu_k} = 0. \tag{7}$$

Multiplying $g^{\mu_1 \mu_2}$ into Eq. (5) gives

$$4m^2 g^{\alpha\beta} \psi_{\alpha\beta\mu_3 \cdots \mu_k} = \partial^\alpha \partial^\beta \psi_{\alpha\beta\mu_3 \cdots \mu_k}. \tag{8}$$

Subtracting ∂^{μ_2} of Eq. (5) from ∂_{μ_1} of Eq. (8) yields

$$\partial_{\mu_1} g^{\alpha\beta} \psi_{\alpha\beta\mu_3 \cdots \mu_k} = 0. \tag{9}$$

Using ∂^{μ_1} of Eq. (9) and $g^{\mu_1 \mu_2}$ of the field equation gives

$$g^{\alpha\beta} \psi_{\alpha\beta\mu_3 \cdots \mu_k} = 0. \tag{10}$$

Now, using Eq. (10) in Eq. (5) and ∂^{μ_1} of the field equation gives

$$\partial_\alpha \psi^{\alpha\mu_2 \cdots \mu_k} = 0. \tag{11}$$

Eqs (6), (7), (10) and (11) are the complete set of field equations and subsidiary conditions [2]. The above procedure can also be used for fields that are not complex. In this case the field equation resulting from variation of the Lagrangian with respect to $\psi^{\mu_1 \ldots \mu_k}$ is added to its complex conjugate, resulting in Eqs (4) and (5). Thus by applying the variational principle to our Lagrangian we can derive all the equations necessary to describe fields of integer spin k.

Next, consider the case of half integer spin fields. Particles of spin $k+1/2$ can be described by 4 component spinor tensors of rank k. The total Lagrange density for half integer spin fields is

$$L = (-1)^k\{\bar{\psi}^{\mu_1\cdots\mu_k} i\partial_\varrho\gamma^\varrho\psi_{(\mu_1\cdots\mu_k)} - m\bar{\psi}^{\mu_1\cdots\mu_k}\psi_{\mu_1\cdots\mu_k}\} + im\bar{\psi}^{\mu_1\cdots\mu_k}\gamma_{\mu_1}\gamma^\alpha\psi_{\alpha\mu_2\cdots\mu_k}. \tag{12}$$

Proof:

Variation of L with respect to $\bar{\psi}^{\mu_1\cdots\mu_k}$ gives

$$(-1)^k\{i\partial_\varrho\gamma^\varrho\psi_{(\mu_1\cdots\mu_k)} - m\psi_{\mu_1\cdots\mu_k}\} + im\gamma_{\mu_1}\gamma^\alpha\psi_{\alpha\mu_2\cdots\mu_k} = 0. \tag{13}$$

The variation with respect to $\psi^{\mu_1\cdots\mu_k}$ is

$$(-1)^k\{-i\partial_\varrho\bar{\psi}_{(\mu_1\cdots\mu_k)}\gamma^\varrho - m\bar{\psi}_{\mu_1\cdots\mu_k}\} + im\bar{\psi}_{\alpha\mu_2\cdots\mu_k}\gamma^\alpha\gamma_{\mu_1} = 0. \tag{14}$$

Taking the adjoint of Eq. (14) and adding it to Eq. (13) gives

$$i\partial_\varrho\gamma^\varrho\psi_{(\mu_1\cdots\mu_k)} - m\psi_{\mu_1\cdots\mu_k} = 0 \tag{15}$$

and

$$i\gamma_{\mu_1}\gamma^\alpha\psi_{\alpha\mu_2\cdots\mu_k} = 0. \tag{16}$$

Again, interchanging tensor indices in Eq. (15) and subtracting from Eq. (15) gives the symmetry condition

$$\psi_{\mu_1\cdots\mu_k} = \psi_{(\mu_1\cdots\mu_k)} \tag{17}$$

from which Eq. (15) becomes the Dirac type field equation

$$(i\partial_\varrho\gamma^\varrho - m)\psi_{\mu_1\cdots\mu_k} = 0. \tag{18}$$

Multiplying Eq (16) by γ^{μ_1} gives the supplementary condition

$$\gamma^{\mu_1}\psi_{\mu_1\cdots\mu_k} = 0. \tag{19}$$

Finally, from Eqs (18) and (19) we obtain

$$\partial^\alpha\psi_{\alpha\mu_1\cdots\mu_k} = 0. \tag{20}$$

Eqs (17)–(20) are the field equations and supplementary conditions for spin $k+1/2$ [5].

We have shown that it is possible to write explicit Lagrangians for arbitrary spin fields. These Lagrangians are immediately applicable to the description of k or $k+1/2$ spin fields without further modification. It is unnecessary to include derivatives of order higher than unity or auxiliary fields in these Lagrangians. The quantization of these fields and the inclusion of interactions will be discussed in a subsequent publication.

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