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THE STATISTICAL DESCRIPTION OF MULTIHADRON PRODUCTION PROCESSES*

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The statistical approach to multihadron production processes assumes that the relative probability for different inclusive final states is determined by the corresponding level densities. The proposed statistical descriptions can be divided into two classes: purely statistical models (Fermi model, Pomeranchuk model, uncorrelated jet model, statistical bootstrap model), and hybrid models, which combine non-statistical cluster formation with statistical cluster decay (two- and multi-center models, thermodynamical model, nova and multinova models, diffraction fragmentation models).

After a survey of the various models with comments on their interrelation, we discuss in particular asymptotic phase space behaviour, the solution of the statistical bootstrap equation, the dynamical interpretation of this solution in terms of resonance interactions, and the connection between the statistical bootstrap approach and aspects of the dual resonance model.

1. Introduction

The increasing number of secondaries produced with increasing energy in hadron-hadron collisions leads quite naturally to the hope that from a certain energy on statistical descriptions of particle production may prove useful. Before studying various proposals of this type, their interrelations and their connections to dynamical models, let us state more clearly what we want to say when we refer to something as a statistical description.

In classical statistical mechanics, each macroscopic (thermodynamic) state, characterized *e. g.* by total internal energy E, volume V and particle number N, is compatible with many microscopic states, namely all configurations

$$\{\vec{p}_i, \vec{x}_i;\ i = 1, 2, \ldots, N\} \tag{1.1}$$

with momenta $\vec{p}_i$ and coordinates $\vec{x}_i$ obeying energy conservation and volume restrictions. The fundamental postulate of statistical mechanics [1] now states that the relative proba-

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bility of two given macroscopic states is determined by the relative number of compatible microscopic states: denoting by $P(E_\alpha, V_\alpha, N_\alpha)$ the probability of the macroscopic state α and by $\tau(E_\alpha, V_\alpha, N_\alpha)$ the number of microscopic states associated with α, we have

$$\frac{P(E_\alpha, V_\alpha, N_\alpha)}{P(E_\beta, V_\beta, N_\beta)} = \frac{\tau(E_\alpha, V_\alpha, N_\alpha)}{\tau(E_\beta, V_\beta, N_\beta)} . \tag{1.2}$$

Equivalently we could postulate the same weight for each allowed microscopic state — in this language, (1.2) is the assumption of equal *a priori* probabilities. As the standard method to calculate the "level density" τ_α in statistical mechanics is through calculation of the corresponding phase space volume, we can obtain yet another formulation by stating that phase space integrals with constant weights determine the relative probabilities of macroscopic final states.

In hadronic production processes one encounters a situation quite analogous to the one just described. Consider the single particle momentum distribution

$$F_c(\vec{p}, s) \equiv 2p_0 \frac{d^3\sigma_c}{d^3p} (\vec{p}, s) \tag{1.3}$$

for the process $a+b \to c+$anything, with $\vec{p}$ denoting the CMS momentum of particle c and $\sqrt{s}$ the incident CMS energy. To such an inclusive distribution there correspond many possible exclusive configurations

$$\{\vec{p}_i;\ i = 1, 2, \ldots, N;\ N = 2, 3, \ldots, N_{\max}(s)\} \tag{1.4}$$

subject, of course, to energy-momentum conservation. We shall therefore speak of statistical models or of a statistical description for such a process if the relative probability of two inclusive final states is, as in statistical mechanics, determined by the corresponding level densities.

Let us emphasize here that in general we do not want to imply by level density that of an ideal gas — besides energy-momentum conservation there can be other non-kinematical constraints forbidding certain states or modifying their distribution. So if we think of level densities in terms of phase space volumes, the latter may well contain "geometrized" dynamics, such as the transverse momentum bound in the uncorrelated jet model. In fact, if we could reformulate the essential features of production dynamics in terms of phase space geometry such that inclusive distributions become determined by volumes of a suitably modified phase space, then we would have obtained the statistical limit or closure [2] of production dynamics.

We now want to indicate briefly general limitations and possibilities of statistical descriptions. No statistical model can, without further input, provide absolute predictions for *e. g.* total or differential cross-sections: the overall normalization of the production amplitude does not enter in (1.2) and hence is outside the scope of statistical considerations. The same applies to the phase of a production amplitude, which is of relevance *e. g.* in the calculation of the elastic differential cross-section *via* unitarity ("overlap problem" [3]): as statistical descriptions always refer to physical transition rates and thus to squared matrix elements, nothing is said about the phase of the corresponding amplitude. On the other hand, a statistical description can, in general, give definite (though perhaps incorrect) answers to all questions about the relative weights of different final states: branching ratios for various particle numbers, decomposition of σ_{in} into the various N particle number distributions, spectra (p_L and p_T distributions), correlations — for all these quantities, statistical models provide predictions subject to experimental test.

The material to be presented in these lectures will be organized as follows: in Section 2 we shall give a review of the main statistical schemes of multihadron production proposed up to now[1]. One of the things to be seen from this survey is to what extent statistical concepts have become common, if not essential, in present day high energy hadron physics.

In the following sections we shall then discuss particular aspects in more detail: in Section 3 the high energy behaviour of the level density given by conventional phase space is investigated, while in 4 we discuss the level density obtained in the statistical bootstrap approach. In Section 5 we then formulate the latter in terms of phase space, in order to show the essential dynamical input of the approach: resonance structure governed by linear Regge trajectories. Concluding, we discuss in Section 6 the relation between statistical bootstrap approach and dual resonance description, to illustrate on hand of a dynamical model the lind of approximation made in the bootstrap picture.

2. Survey of statistical descriptions

As starting point, let us take the perfect gas analogue for particle production, the FERMI MODEL [7]. If we don't know anything about the production dynamics determining the momentum space distribution, the simplest thing to assume is an equidistribution [7, 8]

$$\sigma_N(s) \sim \frac{1}{N!}\int \prod_1^N \frac{d^3p_i}{2p_{i0}}\,\delta^{(4)}\left(\sum_1^N p_i - Q\right)|\langle p_1 \ldots p_N|S|q_1q_2\rangle|^2 \tag{2.1}$$

$$\sim \frac{\kappa^N}{N!}\int \prod_1^N \frac{d^3p_i}{2p_{i0}}\,\delta^{(4)}\left(\sum_1^N p_i - Q\right) \equiv \frac{\kappa^N}{N!}\,\Omega_N(s) \tag{2.2}$$

with $Q = q_1+q_2$, $s = Q^2$, as initial state, and for the case of identical particles of mass m, obeying Boltzmann statistics. If we write $\kappa = 2m\,V_0$, then V_0 has the dimensions of a volume, which we can interpret as expression of the finite interaction region in hadron collisions:

$$V_0 \sim \left(\frac{1}{m}\right)^3. \tag{2.3}$$

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[1] Unfortunately we cannot include here a discussion of Landau's hydrodynamical model [4]; for a recent review of this and further references to it, see Ref. [5] and [6].

The quantity $V_0^N(2m)^N \Omega_N(s)$ moreover is a generalization of N-particle phase space

$$\Gamma_N(E) \sim \int \prod_1^N d^3p_i d^3x_i \delta(\sum_1^N w_i(x_i, p_i) - E), \tag{2.4}$$

where w_i, p_i, x_i denote energy, momentum and space coordinate of the i-th particle; it thus counts the number of allowed states for given CMS energy $\sqrt{s}$ and N free hadrons inside the given interaction volume V_0. As in statistical mechanics of non-interacting gases the probability of a given final state in the Fermi model is thus simply determined by the number of kinematically allowed number of final states (equal *a priori* probabilities). The asymptotic predictions of this model, and of others to be discussed subsequently, are summarized in Table I.

TABLE I

The asymptotic results of various statistical decay schemes; M denotes the cluster mass, w the center of mass energy of the observed secondary

	Multiplicity	Average secondary energy $\bar{w}$	Single particle spectrum $F(w, M)$
Statistical model, covariant	$M^{2/3}$	$M^{1/3}$	$e^{-\text{const.}(mV_0/M)^{1/3}w}$
Statistical model, Fermi	$M^{3/4}$	$M^{1/4}$	$e^{-\text{const.}(V_0/M)^{1/4}w}$
Uncorrelated jet model	$\ln M$	$\dfrac{M}{\ln M}$	$(1-x)^{\text{const.}}$ $x = 2w/M$
Statistical bootstrap model	M	const	$e^{-\text{const.}w}$

Theoretically, there was an almost immediate objection to the Fermi model: Pomeranchuk [9] argued that the emitted hadrons could be considered free only once their mutual separation had exceeded the range of hadronic forces. This led him to propose an overall coordinate space volume increasing linearly with particle number, so that

$$\kappa = 2mV; \quad V = NV_0 \sim N\left(\frac{1}{m}\right)^3 \tag{2.5}$$

in (2.2). This POMERANCHUK MODEL, which subsequently received rather little attention [10], in fact provides an alternative approach [11] to the solution of the statistical bootstrap condition; we therefore shall briefly return to its implications in Section 4.

Empirically, the momentum space equidistribution (2.2) soon became untenable, mainly on three accounts: jet structure, resonance production and leading particle effects. Let us concentrate for the moment on the first two. The transverse momentum bound giving rise to the jet structure is easily accommodated (though not explained!) by modifying the momentum space measure

$$\frac{d^3p}{2p_0} \rightarrow \frac{d^3p}{2p_0} f(p_T) \tag{2.6}$$

to obtain the UNCORRELATED JET MODEL [3, 12]; typically

$$f(p_T) \sim e^{-a|\vec{p}_T|}. \tag{2.7}$$

The essential effect of the modification (2.6) is to render the single particle momentum space one- instead of threedimensional [13]; the bounded p_T amounts practically to a mass renormalization.

The asymptotic results of the uncorrelated jet model are also summarized in Table I; it should be emphasized that scaling and logarithmic multiplicity increase are closely connected and (for bounded p_T) both a consequence of the asymptotically dimensionless longitudinal phase space measure $dp_L/2p_L$. It should also be noted that significant differences between inclusive distribution from the Fermi model and from the uncorrelated jet model can only be expected at energies high enough so that the average secondary energy ($\sim \sqrt{s}/N$) is clearly larger than $\frac{3}{2}$ of the average transverse momentum — which in practice requires $P_{Lab} \gtrsim 100$ GeV/c [14].

The copious, perhaps dominant multiparticle production through intermediate strongly decaying resonances led to the THERMODYNAMICAL MODEL [15], which proposes in a two-step picture the production and subsequent decay of excited centers of hadronic matter at rest — so-called fireballs, considered as extension of the resonance concept towards higher masses. While the formation process of fireballs is taken as (non-statistical) input, their decay is described statistically. The formation process initiated by the hadron-hadron collision is in the thermodynamical model assumed to give no transverse motion to the created fireballs ("no turbulence"), so that transverse spectra yield direct information on fireball decay. Different decay modes are again compared by the relative number of allowed microscopic final states, but the density of states $\tau(M)$ of a fireball of mass $M = \sqrt{p^2}$ is now determined by the STATISTICAL BOOTSTRAP CONDITION [15, 16]: any fireball at any step of the cascade is described by the same function $\tau(x)$. The resulting bootstrap equation [15, 16, 17]

$$\tau(P^2) = \Theta(P_0)\delta(P^2 - m^2) + \\ + \sum_{N=2}^{\infty} \left[\frac{B^{N-1}}{N!} \int \prod_{1}^{N} \{d^4k_i \tau(k_i^2)\} \delta^{(4)} \left(\sum_{1}^{N} k_i - P \right) \right] \tag{2.8}$$

leads to a density of states

$$\tau(M^2) \sim M^{-3} e^{M/T_0};\ T_0 = T_0(B) \tag{2.9}$$

increasing linearly exponential with fireball mass. It is this strong increase (stronger, as we shall see, than any conventional phase space volume) which results in secondaries of bounded energy in the fireball CMS and thus leads to the transverse momentum bound: the inclusive single particle momentum distribution at 90° and for incident CMS energy $M = \sqrt{p^2}$ becomes

$$[F(\vec{p}, M)]_{90^\circ} \sim \frac{\tau([P-p]^2)}{\tau(P^2)} \sim e^{-\frac{1}{T_0}\sqrt{p^2{}_T + m^2}} \tag{2.10}$$

showing the direct relation between level density and p_T-bound.

The thermodynamical model provides an example of a "hybrid" model, in which one combines non-statistical formation with statistical decay; the peripherality of hadronic production is in such a desc iption introduced through the formation mechanism, while the statistical decay of the fireball is supposed to lead in its CMS to a more or less isotropic momentum distribution (fully isotropic for the thermodynamical model). Earlier examples of a hybrid type were the TWO- or MULTI-CENTER MODELS [18, 19], in which the decay was described by a level density obtained from conventional phase space rather than from a bootstrap condition. Later hybrid attempts are found in the nova and the diffraction fragmentation models (see below), which have the same decay pattern as the thermodynamical model, but propose a different form of fireball production process.

In the longitudinal direction (*i. e.*, along the beam), the thermodynamical model proposes the formation of many fireballs in relative motion, from those at rest in the target CMS continuously to those at rest in the projectile CMS. The superposition of fireballs is governed by a velocity distribution not determined by the model, but instead fitted by experiment. Hence the essential assumptions of the model are: (i) no turbulence (no transverse motion of fireballs) and (ii) the bootstrap condition to determine $\tau(x)$; the essential prediction gives the inclusive transverse momentum distribution of the secondaries.

Before continuing, let us comment briefly on the concept of fireball. We shall call cluster any hadronic system "at rest", more specifically, any system whose intrinsic spin j is much less than its rest mass M

$$j \ll M. \tag{2.11}$$

Of particular interest is often the case $M \to \infty$, $j/M \to 0$. If a cluster in this limit decays in its CMS into secondaries of bounded energy, we call the cluster a fireball. This definition essentially coincides with that proposed by Mięsowicz [19] and also agrees with the terminology of the statistical bootstrap approach, where fireballs are hadronic systems decaying isotropically into secondaries of bounded energy. The case $j \sim M$ would lead to considerable difficulties: the separation between formation and decay then becomes more or less a matter of operational definition.

A picture quite similar to the thermodynamical model is proposed in the NOVA [20] and the DIFFRACTION FRAGMENTATION [21] MODEL. The decay scheme is that of the statistical bootstrap approach (fireball = nova). The formation process, however, is somewhat modified. Let us consider the nova model in the case of proton-proton interactions (*cf.* Fig. 1) at incident CMS energy $\sqrt{s}$; denote with m the proton mass. The differential cross-section for the production of a fireball (nova) of mass M at fixed proton-proton momentum transfer $t = (k_p - k_{p'})^2$ is then written

$$\frac{d\sigma}{dtdM} \sim \beta(t, M, m) g(M) \beta(t, m, m) S^{2[\alpha_{\mathscr{P}}(t)-1]}. \tag{2.12}$$

Here $\beta(t, M, m)$ and $\beta(t, m, m)$ denote the squared proton-fireball-Pomeron and proton-proton-Pomeron form factors, respectively, $g(M)$ the fireball production distribution in

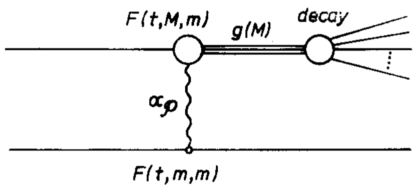


Fig. 1. Nova Model

mass and $\alpha_{\mathscr{P}}(t)$ the Pomeron trajectory. If we allow fireball excitation of both incident protons, we obtain analogously the diffraction fragmentation model.

The essential differences between the thermodynamical model and the nova or diffraction fragmentation picture are the following: in the latter we always have a two "body" final state, so that the rest system of a fireball is determined by its mass; in the thermodynamical model any number of fireballs can be produced, allowing — apart from energy — momentum conservation — any mass fireball in any reference system between target and projectile rest frame. On the other hand, the "no turbulence" assumption is weakened in the nova/diffraction fragmentation picture: with $\beta(t, M, m)$ as determined *e. g.* from exclusive data ($\sim \exp A(M)t$), fireballs with transverse motion are suppressed, but not excluded.

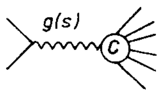
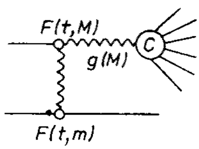
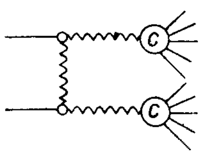
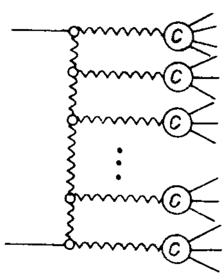
A more general model can thus be obtained by allowing the production of many fireballs in a multiperipheral picture, *i. e.* with Regge propagators and form factors instead of the no turbulence assumption. Such considerations were in fact proposed [22] before the advent of the nova and the diffraction fragmentation model and have recently attracted renewed interest [23].

For all hybrid models large transverse momenta become problematic: the "no turbulence" assumption as well as all form factor pictures are abstractions from the world of small t (or close poles) and should not be expected to work indefinitely (deviations in fact seem arleady indicated by present CERN-ISR data [24.]) A possible modification [25] can be obtained by considering the analogous case in deep inelastic e–p scattering, where orthodox vector meson dominance (disjoint t and M dependence) also has to be replaced by scaling vertices (functions of t/M^2).

Concluding our survey, we give in Table I a summary of the most important results from the different statistical decay schemes (some of these results will be derived in subsequent sections), and in Table II a schematic compendium of various possible statistical descriptions of multiparticle production.

TABLE II

Schematic display of the main statistical approaches to multihadron production

purely statistical models		C = Phase Space (PS):: Fermi Model = Longitud. PS (LPS): Uncorrelated Jet Model = Fireball (FB): Statistical Bootstrap Model
hybrid models		C = PS: One Center Model (with leading particle) = LPS: Longitudinal Cluster Model = FB: Nova Model
		C = PS: Two Center Model = LPS: Longitudinal Diffraction Excitation Model = FB: Diffraction Fragmentation Model
		C = PS: Multi-Center Model PS+cut in cluster mass: Chan-Łoskiewicz-Allison Model FB: Multi-Nova Model FB+(t = 0) condition: Thermodynamical Model

3. *The calculation of phase space volumes*

This section will constitute an interlude of a more technical nature: the evaluation of N particle phase space integrals and of the sum of such integrals over all N. Our aim here will be to provide an intuitive idea of how such calculations are performed; details and more rigorous argumentation can be found in an extensive literature [26].

We begin with a greatly simplified case, which however contains already most significant features of phase space volumes. Consider the integral

$$\Omega_N(E) \equiv \int_0^\infty \prod_1^N dq_i \delta\left(\sum_1^N q_i - E\right) \qquad (3.1)$$

which can be interpreted as the momentum space of one-dimensional mass zero particles with only energy conservation. It is easily solvable in closed form by writing

$$\Omega_N(E) = \frac{e^{\alpha E}}{2\pi} \int_{-\infty}^{\infty} d\gamma e^{i\gamma E}[\varphi(\alpha + i\gamma)]^N, \tag{3.2}$$

$$\varphi(z) = \int_0^{\infty} dq e^{-izq} = \frac{1}{z} \; (\mathrm{Im}\, z < 0), \tag{3.3}$$

which yields

$$\Omega_N(E) = \frac{e^{\alpha E}}{2\pi} \int_{-\infty}^{\infty} d\gamma e^{i\gamma E}(\alpha + i\gamma)^{-N} = E^{N-1}/(N-1)!. \tag{3.4}$$

The corresponding sum over all N

$$G(E) \equiv \sum_2^{\infty} \frac{C^N}{N!} \Omega_N(E) \tag{3.5}$$

can also easily be performed and gives

$$G(E) = \sqrt{\frac{C}{E}} [I_1(2\sqrt{CE}) - 1] \tag{3.6}$$

which behaves for $E \to \infty$ as

$$G(E) \simeq \sqrt{\frac{C}{E}} \frac{1}{\sqrt{4\pi\sqrt{CE}}} e^{2\sqrt{CE}}. \tag{3.7}$$

Thus the total phase space volume (assuming Boltzmann statistics) in this simplified case increases exponentially in $E^{1/2}$ [27].

Let us rewrite the above results in a somewhat different form. Applying the Stirling formula to (3.4) gives together with (3.3)

$$\Omega_N(E) \simeq \frac{1}{E} \sqrt{\frac{N}{2\pi}} e^{\alpha E} [\varphi(\alpha)]^N \tag{3.8}$$

if α is fixed through

$$-\frac{\partial \ln \varphi(\alpha)}{\partial \alpha} = \frac{E}{N}. \tag{3.9}$$

Formula (3.9) gives, once (3.3) is substituted,

$$E = N\alpha^{-1} \tag{3.10}$$

which with $\alpha^{-1} = kT$ represents the Stefan-Boltzmann equation for a system defined through (3.1). Thus (3.8) essentially is the canonical approximation of the microcanonical form (3.1), expressed in terms of a temperature $\alpha^{-1} = kT$ defined by (3.9). It can be shown by applying techniques from probability theory [28] that these results in fact are very general and can be used for the evaluation of most phase space integrals [28, 29, 30]. We shall not attempt to prove this here, but rather illustrate the point by looking at another example solvable in closed form, the covariant N-particle phase space [8] for zero mass particles

$$\Omega_N(W) = \int \sum_1^N \frac{d^3 p_i}{2p_{i0}} \delta^{(4)}\left(\sum_1^N p_i - P\right) = \left(\frac{\pi}{2}\right)^{N-1} \frac{W^{2N-4}}{(N-1)!(N-2)!}, \tag{3.11}$$

$$W^2 = P^2.$$

In this case we have

$$\varphi(\alpha) = \int \frac{d^3 p}{2p_0} e^{-\alpha p} = \frac{2\pi}{\alpha^2} \tag{3.12}$$

which after substitution into (3.8) and (3.9) gives the Stirling approximation of (3.11). If $m \neq 0$, one finds

$$\varphi(\alpha) = \frac{2\pi m}{\alpha} K_1(m\alpha) \tag{3.13}$$

which requires a numerical solution of (3.9) to evaluate (3.11) *via* the approximation (3.8) [29].

Consider now the general form

$$\Omega_N^{(\nu)}(W) \equiv \int \prod_1^N \left\{\frac{d^3 p_i}{2p_{i0}} p_{i0}^{\nu}\right\} \delta^{(4)}\left(\sum_1^N p_i - P\right), \quad \nu > -2, \tag{3.14}$$

which gives for $m = 0$

$$\varphi_\nu(\alpha) = \int \frac{d^3 p}{2p_0} e^{-\alpha p} p_0^{\nu} = \frac{C_0}{\alpha^{\nu+2}}, \quad C_0 = C_0(\nu). \tag{3.15}$$

Substituting this in (3.8) gives us, up to correction terms of order Stirling approximation

$$\Omega_N^{(\nu)}(W) \sim \left[\frac{eW}{(\nu+2)N}\right]^{(\nu+2)N} \sim \frac{W^{(\nu+2)N}}{(N!)^{\nu+2}} \tag{3.16}$$

which of course reduces to (3.11) for $\nu = 0$. To obtain an idea of the behaviour of the resulting sum over N

$$G^{(\nu)}(W) = \sum_2^\infty \frac{C^{(\nu+2)N}}{N!} \Omega_N^{(\nu)}(W) \tag{3.17}$$

we write, using (2.16),

$$G^{(\nu)}(W) \sim \sum_N \left[\frac{e(CW)^{\frac{\nu+2}{\nu+3}}}{(\nu+3)N}\right]^{(\nu+3)N} \tag{3.18}$$

which brings us to expect

$$G^{(\nu)}(W) \sim e^{[CW]^{(\nu+2)/(\nu+3)}}. \tag{3.19}$$

The argumentation leading up to (3.19) can be made rigorous, again by using techniques from probability theory [27, 31]. We note here in particular the special cases $\nu = 0$ (covariant statistical model) and $\nu = 1$ (Fermi model), which lead to level densities increasing exponentially in $W^{2/3}$ and $W^{3/4}$ respectively. As evident fróm (3.19), the level density for any $\nu > -2$ increases exponentially but less than linear in W. The resulting average particle number

$$\bar{N}^{(\nu)}(W) = \frac{\sum_2^\infty N \frac{C^{(\nu+2)N}}{N!} \Omega_N^{(\nu)}(W)}{\sum_2^\infty \frac{C^{(\nu+2)N}}{N!} \Omega_N^{(\nu)}(W)} = \tag{3.20}$$

$$= \frac{\partial \log G^{(\nu)}(W)}{\partial [C^{(\nu+2)}]} \sim W^{1-\frac{1}{\nu+3}}, \tag{3.21}$$

hence also increases always less than linear in W.

The case $\nu = -2$ in equation (3.14) results in a form very reminiscent of the uncorrelated jet model: the level density increases as a power of W, and the multiplicity behaves as

$$\bar{N} \sim \ln W. \tag{3.22}$$

For $\nu < -2$, the multiplicity becomes asymptotically constant, since now the momentum space restriction $\prod_1^N p_{i0}^\nu$ leads to a convergent integral even without the energy conservation condition.

In summary we can say:

(1) The level density obtained from conventional phase space (with Boltzmann statistics) increases at most as

$$\tau(W) \sim e^{CW^r}, \quad r < 1. \tag{3.23}$$

(2) The resulting multiplicity increases at most as a power less than unity in W

$$\bar{N}(W) \sim W^r, \quad r < 1.$$

(3) The resulting average energy per secondary increases with total incident energy without bound.

Having seen what asymptotic behaviour we obtain for level densities from conventional phase space volumes, let us now consider in more detail the calculation of level densities in the bootstrap approach.

5. *The statistical bootstrap condition and its solution*

We again begin by considering a somewhat simplified case to illustrate most clearly the essential points. The linear chain model [32] allows each fireball to decay into one stable secondary ("pion") and one further fireball (*cf.* Fig. 2); the resulting bootstrap equation is

$$\tau(P^2) = \Theta(P_0)\delta(P^2-m^2)+\lambda\int\frac{d^3p}{2p_0}\,d^4k\tau(k^2)\delta^{(4)}(p+k-P) \tag{4.1}$$

with λ denoting the fireball-fireball-pion coupling constant. The solution to the problem defined by (4.1) has in fact been shown [16, 32, 33] to yield the dominant decay mode also of fireballs defined by the full bootstrap equation (2.8).

The four-dimensional Laplace transform of (4.1), defined by

$$Z(\beta,\lambda) \equiv \int d^4Pe^{-\beta P}[\tau(P^2)-\Theta(P_0)\delta(P_2^2-m^2)] \tag{4.2}$$

is found to be

$$Z(\beta,\lambda) = \frac{\lambda\varphi^2(\beta)}{1-\lambda\varphi(\beta)}\,. \tag{4.3}$$

Here we require $\beta_0>0$, $\beta^2 = \beta_0^2-\vec{\beta}^2>0$, and the single particle distribution $\varphi(\beta)$ is defined as

$$\varphi(\beta) \equiv \int\frac{d^3p}{2p_0}\,e^{-\beta p} = \frac{2\pi m}{\beta}\,K_1(m\beta), \tag{4.4}$$

$$\beta = \sqrt{\beta_\mu\beta^\mu}$$

which for $m = 0$ has the particularly simple form

$$\varphi_0(\beta) = \frac{2\pi}{\beta^2}\,. \tag{4.5}$$

We note that $Z(\beta,\lambda)$ at fixed λ becomes singular at that value of β for which

$$\lambda\varphi(\beta) = 1. \tag{4.6}$$

It will shortly become clear that (4.6) in fact defines the "maximum temperature" T_0 which governs the exponential increase (2.9) of the bootstrap level density.

To solve equation (4.1) directly, we consider the corresponding equation at fixed N

$$\tilde{\tau}_N(P^2) \equiv \tau_N(P^2) - \Theta(P_0)\delta(P^2 - m^2) =$$

$$= \lambda \int \frac{d^3 p_i}{2p_{i0}} d^4 k_1 \tilde{\tau}_{N-1}(k_1^2)\delta^{(4)}(p_1 + k_1 - P). \tag{4.7}$$

By simply iterating this, we find

$$\tilde{\tau}_N(P^2) = \lambda^{N-1} \int \prod_{i=1}^{N} \frac{d^3 p_i}{2p_{i0}} \delta^{(4)}\left(\sum_1^N p_i - P\right) \tag{4.8}$$

and hence the Yellin form [17] of the bootstrap solution

$$\tilde{\tau}(P^2) = \sum_{N=2}^{\infty} \lambda^{N-1} \int \prod_{i=1}^{N} \frac{d^3 p_i}{2p_{i0}} \delta^{(4)}\left(\sum_1^N p_i - P\right). \tag{4.9}$$

As a check, it is easily verified that (4.9) satisfies (4.3).

For the case $m = 0$, the momentum space integral is, as we know, solvable in closed form (*cf.* (3.11)); substituting this in (4.9) gives

$$\tau_0(P^2) = \frac{1}{2W}\sqrt{2\pi\lambda}\, I_1(\sqrt{2\pi\lambda}\, W), \qquad W = \sqrt{P^2} \tag{4.10}$$

as the exact solution of the linear chain bootstrap condition (4.1). For $W \to \infty$ we regain the familiar linearly exponential increase

$$\tau_0(P^2) \simeq \frac{(2\pi\lambda)^{1/4}}{\sqrt{8\pi W^3}} e^{\sqrt{2\pi\lambda}\, W}. \tag{4.11}$$

From this we see that the Laplace transform (4.2) must in fact diverge unless

$$\beta > \beta_H = \sqrt{2\pi\lambda}\,. \tag{4.12}$$

This value β_H is, however, just that for which $Z(\beta, \lambda)$ becomes singular, as seen by substituting (4.5) in (4.6). In thermodynamics, the temperature is defined as $\beta^{-1} = kT$ with k denoting the Boltzmann constant; hence (4.12) requires with

$$T < T_0 = 1/k\sqrt{2\pi\lambda} \tag{4.13}$$

the existence of a highest possible temperature [15, 34].

The result that the singularity of the Laplace transformed level density, *i.e.* of the partition function $Z(\beta, \lambda)$, determines the maximum temperature and hence the exponent in the level density increase, is in fact very general [33]. We have just shown it for the linear chain case [32], but it holds as well [35] for the full bootstrap equation (2.8) and even for the case [11] of a Pomeranchuk type volume parameter

$$B \to NB_0 \tag{4.14}$$

in the full bootstrap.

Let us briefly indicate here the corresponding results for the full bootstrap. From (2.8) we find by Laplace transform

$$Z(\beta, \lambda) = \varphi(\beta) + \frac{1}{B}\left[e^{BZ(\beta,\lambda)} - 1 - BZ(\beta, \lambda)\right]. \tag{4.15}$$

This relation no longer yields the simple pole structure of (4.3); instead we have a square root branch point singularity [33, 35] at

$$\varphi(\beta_H) = \frac{1}{B}\left[2 \log 2 - 1\right] \equiv \frac{Z_0}{B} \tag{4.16}$$

which now determines β_H. The corresponding Yellin form becomes

$$\tau(P^2) = \sum_{N=2}^{\infty} C_N \left(\frac{B}{Z_0}\right)^{N-1} \int \prod_{1}^{N} \frac{d^3 p_i}{2p_{i0}} \delta^{(4)}\left(\sum_{1}^{N} p_i - P\right) \tag{4.17}$$

with expansion coefficients which satisfy [36]

$$C_{k+1} = \frac{-1}{k+1}\left[kC_k - 2\sum_{l=1}^{k} lC_l C_{k+1-l}\right], \tag{4.18}$$

$$C_1 = 1$$

and which asymptotically in k become [33]

$$C_k \simeq k^{-3/2} \sqrt{\frac{Z_0}{4\pi}}. \tag{4.19}$$

The only difference between full bootstrap and linear chain thus lies in the presence of the factor $k^{-3/2}$ in (4.19) for the former. It leads to the asymptotic result

$$\tau(P^2) \sim W^{\nu} e^{W/T_0} \tag{4.20}$$

with $\nu = -3$ for the full bootstrap, $\nu = -\frac{3}{2}$ for the linear chain. It is thus only on the non-exponential level (for terms small of order $W^{-1} \log W$) that differences occur between the two bootstrap forms (2.8) and (4.1).

In summary, we have as a result of the statistical bootstrap approach a level density increasing linearly exponential in fireball mass, *i.e.*, much stronger than in the case of level densities from conventional phase space. The resulting bootstrap multiplicity increases linearly with N

$$\bar{N} \simeq W/\bar{w}_\infty \tag{4.21}$$

while average energy $\bar{w}$ per secondary approaches the constant value

$$\bar{w}_\infty = -\frac{\partial \log \varphi(\beta_H)}{\partial \beta_H} = \frac{2}{\beta_H} + m\,\frac{K_0(m\beta_H)}{K_1(m\beta_H)} \tag{4.22}$$

as $W \to \infty$. An increase of fireball mass just leads to more secondaries, not faster ones.

Before studying possible dynamical schemes leading to such a behaviour, let us briefly return to the Pomeranchuk model [4]. It had there been suggested that the concept of a free gas should be applicable only once all constituents are separated by distances equal to a greater than the range of hadronic forces, leading to an interaction volume $V = NV_0$, as in (2.5). The resulting level density

$$\tau(P^2) = \sum_{N=2}^{\infty} \frac{(2mNV_0)^N}{N!} \int \prod_1^N \frac{d^3p_i}{2p_{i0}}\, \delta^{(4)}\left(\sum_1^N p_i - P\right) \tag{4.23}$$

can be written

$$\tau(P^2) = \sum_{N=2}^{\infty} C\ (2meV_0)^N \int \prod_1^N \frac{d^3p_i}{2p_{i0}}\, \delta^{(4)}\left(\sum_1^N p_i - P\right) \tag{4.24}$$

with

$$C_N = \frac{1}{N!}\left(\frac{N}{e}\right)^N \xrightarrow[N\to\infty]{} \frac{1}{\sqrt{2\pi N}}\,. \tag{4.25}$$

From (4.17)/(4.19) we see that this in fact is essentially the solution of the full bootstrap equation. Hence we may consider a thermodynamic picture of hadronic matter, expanding until with $V = NV_0$ it has reached a free gas stage, as an alternative road to the fireball description derived above from the bootstrap requirement.

5. Geometrized resonance structure and exponential level degeneracy

In this section we want to investigate further the question of what form of dynamics can lead to a level density increasing linearly exponential in fireball mass [37]. We have already seen in Section 3 that a gas without mutual interaction between the constituents cannot provide such behaviour. To see what has to be changed, consider a world of identical one-dimensional particles capable of taking on discrete energies.

$$q_i = n; \quad n = 0, 1, \ldots; \quad i = 1, 2, \ldots, N. \tag{5.1}$$

The total number of all allowed N particle states at a given over-all energy M then is

$$d(M) = \sum_{N=2}^{\infty} \frac{\lambda}{N!}\left[\prod_{i=1}^{N} \sum_{q_i=0}^{\infty} \delta\left(\sum_{i=1}^{N} q_i, M\right)\right] \tag{5.2}$$

with λ denoting a "volume" parameter. The invariant sub-energies or cluster masses

$$w_i = \sum_{l=1}^{i} q_l; \quad i = 1, 2, \ldots, N \tag{5.3}$$

of course also have discrete values

$$w_i = n; \quad n = 0, 1, \ldots; \quad i = 1, 2, \ldots, N. \tag{5.4}$$

Using (5.3), the level density $d(M)$ becomes

$$d(M) = \sum_{N=2}^{\infty} \frac{\lambda}{N!} \left[\sum_{w_1=0}^{M} \sum_{w_2=w_1}^{M} \cdots \sum_{w_{N-1}=w_{N-2}}^{M} \right] =$$

$$= \sum_{N=2}^{\infty} \frac{M^{N-1}\lambda^N}{N!(N-1)!} = \sqrt{\frac{\lambda}{M}} [I_1(2\sqrt{M\lambda}) - 1] \tag{5.5}$$

which asymptotically gives with

$$d(M) \simeq \text{const } M^{-3/4} e^{2\sqrt{M\lambda}} \tag{5.6}$$

a from increasing exponentially in $M^{1/2}$. Replacing the quantization of subenergies (5.4) by one in squared subenergies in accord with the usual Regge description (for intercept zero and slope κ)

$$\alpha(w_i) = \kappa w_i^2 = n; \quad n = 0, 1, \ldots; \quad i = 1, \ldots, N \tag{5.7}$$

yields in a similar fashion

$$d(M) = \sum_{N=2}^{\infty} \frac{\kappa^N}{N!} \left[\sum_{w_1^2=0}^{M^2} \sum_{w_2^2=w_1^2}^{M^2} \cdots \sum_{w^2_{N-1}=w^2_{N-2}}^{M^2} \right] =$$

$$= \frac{\sqrt{\kappa}}{M} [I_1(2\sqrt{\kappa}\, M) - 1] \tag{5.8}$$

and hence with

$$d(M) \simeq \text{const } M^{-3/2} e^{2\sqrt{\kappa} M} \tag{5.9}$$

a level degeneracy increasing linearly exponential in energy M. It is thus an equidistribution over squared cluster energies, $s_i = w_i^2$, as required by a Regge description of all subsystems, which leads to the desired level density.

Let us now extend these arguments to the real world of three space dimensions. Consider a cascade decay of a hadronic cluster (*cf.* Fig. 2); we introduce cluster variables

$$M_i = \sqrt{P_i^2}\,; \quad P_i = \sum_{l=1}^{i} p_l; \quad i = 1, \ldots, N \tag{5.10}$$

by iterating the relation

$$\frac{d^3p_1}{2p_{10}}\frac{d^3p_2}{2p_{20}} = dM\frac{d^3P}{2P_0}d^2\Omega\frac{K}{2}, \tag{5.11}$$

where M and $\vec{P}$ denote rest mass and three-momentum of the compound system, while Ω and K describe orientation and magnitude of one of the two momenta in the compound CMS. In terms of these variables the covariant momentum space integral ((3.14) with $\nu = 0$) becomes

$$\Omega_N(P^2) = \int \prod_1^N \{dM_i d^3P_i \delta(P_i^2 - M_i^2)\Theta(P_{i0})\}\delta^{(4)}(P_N - P), \tag{5.12}$$

where the cluster masses are restricted by

$$M_k + m \leqslant M_{k+1} \leqslant M - (N-k-1)m,$$
$$k = 1, 2, \ldots, N-1, \quad M_1 = m. \tag{5.13}$$

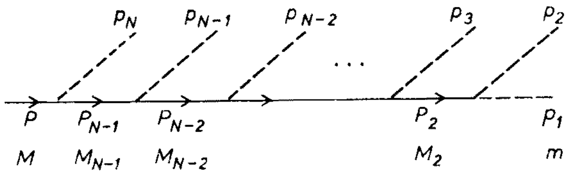


Fig. 2. Linear Chain Decay

From (5.12) we see that the Fermi model in fact assumes an equidistribution also in terms of invariant cluster masses. We would like to replace this assumption by a picture of resonances distributed according to Regge trajectories linear in $s_i = M_i^2$

$$\alpha(s) = \alpha_0 + \alpha' s. \tag{5.14}$$

It is clear that in comparison to an equidistribution in M_i this will greatly increase the number of states per unit mass interval. To obtain the resulting Regge space level density, we write

$$\tau(M^2) = \sum_{N=2}^{\infty} \frac{(\alpha')^N}{N!} \int \prod_1^N \{ds_i d^3P_i \delta(P_i^2 - s_i)\Theta(P_{i0})\}\delta^{(4)}(P_N - P) \tag{5.15}$$

with α' as in (5.14). The evaluation of (5.15) for large P^2 yields [37] (assuming for simplicity zero mass secondaries)

$$\tau(M^2) \simeq \text{const}\, M^{-5/2} e^{2\sqrt{2\pi\alpha'}\, M} \tag{5.16}$$

giving us the expected linear exponential increase in M. It should be emphasized that the agreement between fireball behaviour from bootstrap arguments (or a Pomeranchuk picture) and the level density of Regge space only holds if the trajectories are linear in s; any other power would not lead to (5.16). We can therefore interpret the linearly exponential increase of the level density as a statistical statement of hadron production through resonance cascade with underlying Regge dynamics an in (5.14).

6. *DRM features and statistical bootstrap results*

The essential outcome of the statistical bootstrap approach is, as we have seen, the linearly exponential increase of the level degeneracy and the resulting asymptotically bounded energy per emitted secondary. In the last section we showed the form of the level degeneracy to be intimately connected with hadron production through intermediate resonances. It is therefore natural to see if fireball behaviour can be found also in dynamical models incorporating resonance production.

The most complete formulation of a dynamical description in this vein is the dual resonance model [38, 39] (DRM), which combines low energy resonance production with high energy Regge exchange in a crossing-symmetric fashion. The resulting N-particle dual resonance amplitude [40] B_N provides a description rather similar to the cascade decay discussed above: one can indeed expand the decay amplitude of a heavy resonance as a sum over intermediate resonances, as in Fig. 2. One of the first properties to be established for B_N in this context was the high degree of degeneracy $\tau(M)$ exhibited by heavy resonances, and it was shown [41] that this increases with increasing resonance mass

$$\tau(M) \sim \exp\left(\sqrt{\frac{8\pi^2}{3}}\,\alpha' M\right) \tag{6.1}$$

in precisely the same way as had been found previously in bootstrap considerations. We have seen in the last section that this result is not so surprising: the number of states (6.1) is necessary simply to accommodate all kinematically possible configurations, if we require the additional (dynamical) condition of resonance distribution governed by linear Regge trajectories.

The level density itself has, however, no direct predictive power in a dynamical theory. We have defined statistical descriptions as those where the level density defines the decay probability. In a dynamical description such as the DRM, one has an amplitude different from zero for precisely all states counted in $\tau(M)$; the amplitude, however, will generally not be constant over all these states, but attribute different weights to their importance in the decay both by modulus and phase.

To make this role more transparent, consider the Regge space $\mathscr{R} = \{s_i, \vec{p}_i\}$ in (5.15) as the relevant momentum space for hadronic cluster decay. The statistical bootstrap model then is a Fermi approach in this space: decay probabilities are compared simply by comparing $\mathscr{R}$-space volumes. The DRM amplitude B_N is defined in precisely the same space, but it varies both in magnitude and phase for different regions of $\mathscr{R}$-space. Thus the agreement in level density in the two approaches in no way implies that *e.g.* the decay spectra should also be the same.

There thus remains the question if it is possible to find fireball behaviour in a DRM description. It can be shown [42] that to some extent this is indeed the case, although the presence of selection rules and other inherent non-statistical effects can considerably modify the resulting production properties. This problem will be treated in detail in a forthcoming paper [43].

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