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STABLE CAUSALITY OF HIGHLY MOBILE SPACE-TIMES

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Each highly mobile, algebraically special space-time is shown to possess a global time function. It follows from this that these space-times are stably causal.

As is it well known, local causal properties of space-times are the same as those of the flat space-time (Minkowski space), whereas global properties can be quite different and very complicated. There is a number of different types of global causality of general relativistic space-times (see, *e.g.*, [1, 2]). Probably the most important one is the notion of stable causality introduced by Hawking [3, 4]. A space-time V is said to be stably causal if there exist no closed timelike paths in V even if the metric of V is slightly perturbed. Hawking [3] has shown that a time-orientable space-time is stably causal if and only if it admits a cosmic time function, that is a function increasing along every future-directed causal path (*i.e.* a path with a timelike or null tangent vector).

In this paper the stable causality of some algebraically special space-times possessing maximum mobility shall be discussed. Their metrics have been obtained by Petrov [5] by means of local methods. Recently the author [6] has carried out a global investigation of these space-times. The results have been used [7] to investigate the geodesic completeness of these space-times, with an application to the problem of boundary conditions for algebraically special gravitational fields. Obviously, for the same purpose (*i.e.* for correct formulation of the above-mentioned boundary conditions) it is necessary to investigate global causal properties of highly mobile spacetimes, since at large distances algebraically special space-times behave asymptotically as highly mobile ones. We shall prove that all the algebraically special space-times having maximal mobility are stably causal. Stable causality is a rather strong condition: it will follow from this that the same space-times are also causal and strongly causal (for definitions, see [2]). The physical consequences of the theorem will consist in the fact that one can expect no large-scale anomalies in every algebraically special space-time which tends (at infinity) to a highly mobile one.

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Below, we give metrics of highly mobile algebraically special space-times. The metrics are given in their global form and free of coordinate singularities (for details, see [6]).

Petrov type 2, cosmological constant $k = 0$ (space-time denoted by T_2, see [5]):

$$g_x(dx, dx) = 2dx^1dx^4 - \mathrm{ch}^2\, x^4(dx^2)^2 - (dx^3)^2 + A(dx^4)^2, \tag{1}$$

$$A = (x^3)^2 + a, \quad a = \text{const.}$$

Type 2, $k \neq 0$ (space-time T_2^*):

$$g_x(dx, dx) = -e^{2x^4}[2dx^1dx^3 + (dx^2)^2] + be^{x^4}(dx^3)^2 - \frac{3}{k}(dx^4)^2,$$

$$b = \pm 1, \qquad k > 0. \tag{2}$$

Type 3, $k = 0$ (space-time T_3):

$$g_x(dx, dx) = -\tfrac{3}{2}e^{-2x^3}(dx^1)^2 - e^{4x^3+2x^4}(dx^2)^2 - 4e^{2x^4}(dx^3)^2 +$$

$$+2e^{x^4-x^3}(dx^4 - dx^3)dx^1. \tag{3}$$

Type 3, $k \neq 0$ (space-time T_3^*):

$$g_x(dx, dx) = e^{-2x^4}[-2dx^1dx^3 + (dx^2)^2] + 2e^{x^4}dx^2dx^3 - \frac{1}{2}e^{4x^4}(dx^3)^2 - \frac{3}{k}(dx^4)^2,$$

$$k > 0. \tag{4}$$

It should be noted [6] that all these space-times (under the additional assumption of simple connectedness) are homeomorphic to the Euclidean space R^4 and the map (x^i) in every case covers the whole space-time, that is the coordinates used in Eqs (1)–(4) are global. Space-time T_2, moreover, can be homeomorphic also to the topological product R^3S^1 of the Euclidean space and the circle. This space-time is shown [6] to describe a plane gravitational wave with periodic advanced time, to which it is difficult to give a physical interpretation, hence the topological type R^3S^1 will be omitted here.

Consider first of all the space-time T_3. By means of a transformation of coordinates the metric (3) can be given in the form:

$$g_x(dx, dx) = -{}^{(3)}g_x(dx, dx) + (dx^4)^2,$$

$${}^{(3)}g_x(dx, dx) = \tfrac{3}{10}(x^4)^{2/5}e^{-2x^3}(dx^1)^2 + \tfrac{125}{4}(x^4)^{6/5}e^{4x^3}(dx^2)^2 +$$

$$+ \tfrac{75}{8}(x^4)^2(dx^3)^2 + \tfrac{5}{2}(x^4)^{6/5}e^{-x^3}dx^1dx^3. \tag{3a}$$

We see that the space-time T_3 admits a $(3+1)$-decomposition, which shows that T_3 is time-orientable. Since the coordinates x^i in Eq. (3a) are synchronous, the function $t(x) = x^4$ can be taken as a global time in T_3, hence by Hawking's theorem [3], the space-time T_3 is stably causal.

Other metrics can be diagonalized by introducing appropriate differential forms:

$$g_x(dx, dx) = -(\Theta^1)^2-(\Theta^2)^2-(\Theta^3)^2+(\Theta^4)^2, \tag{5}$$

where for the space-time T_2:

$$\Theta^1(dx) = -dx^1+\tfrac{1}{2}(1-A)dx^4,\ \Theta^2(dx) = \mathrm{ch}\, x^4 dx^2,$$
$$\Theta^3(dx) = dx^3,\ \Theta^4(dx) = dx^1+\tfrac{1}{2}(1+A)dx^4; \tag{6}$$

for the space-time $T_2^*(b = 1)$:

$$\Theta^1(dx) = e^{5x^4/2}dx^1, \quad \Theta^2(dx) = e^{x^4}dx^2, \quad \Theta^3(dx) = \sqrt{3/k}\, dx^4,$$
$$\Theta^4(dx) = e^{-x^4/2}dx^3-e^{5x^4/2}dx^1; \tag{7}$$

for the space-time $T_2^*(b = -1)$:

$$\Theta^1(dx) = e^{5x^4/2}dx^1+e^{-x^4/2}dx^3, \quad \Theta^2(dx) = e^{x^4}dx^2,$$
$$\Theta^3(dx) = \sqrt{3/k}\, dx^4, \quad \Theta^4(dx) = e^{5x^4/2}dx^1; \tag{8}$$

for the space-time T_3^*:

$$\Theta^1(dx) = \sqrt{3/k}\,\omega^4, \quad \Theta^2(dx) = \omega^2, \quad \Theta^3(dx) = (\omega^3-2\omega^1-2\omega^2)/\sqrt{2},$$
$$\Theta^4(dx) = \sqrt{2}\,(\omega^1+\omega^2),$$
$$\omega^1 = e^{-4x^4}dx^1, \quad \omega^2 = e^{-x^4}dx^2, \quad \omega^3 = e^{2x^4}dx^3, \quad \omega^4 = dx^4. \tag{9}$$

It is easy to see that all these space-times are orientable, time-orientable and space-orientable. Therefore, the stable causality of these space-times is equivalent to the existence of time functions.

Consider the space-time T_2^* $(b = -1)$. Let $x(s)$ be a future-directed causal path in T_2^*. To normalize the parameter s, let us introduce the auxiliary positively defined metric in the same space-time manifold:

$$\bar{g}_x(dx, dx) = (\Theta^1)^2+(\Theta^2)^2+(\Theta^3)^2+(\Theta^4)^2. \tag{10}$$

Choose s in such a way that

$$\bar{g}_x(\dot{x}, \dot{x}) = 2. \tag{11}$$

Since the path $x(s)$ is causal we have

$$g_x(\dot{x}, \dot{x}) \geqslant 0, \quad \Theta^4(\dot{x}) \geqslant 0. \tag{12}$$

Comparison between Eqs (11) and (12) yields

$$\Theta^4(\dot{x}) \geqslant 1, \tag{13}$$

which in view of Eq. (8) assumes the form:

$$dx^1/ds \geqslant e^{-5x^4/2} > 0.$$

Hence, the function $t(x) = x^1$ increases along every future-directed causal path, hence it can be taken as a global time function in T_2^*. Therefore, the space-time $T_2^*(b = -1)$ is stably causal.

For the remaining space-times, it will be sufficient to prove that they are stably causal in every bounded domain. Indeed, if a space-time V is not stably causal as whole, that is closed causal paths occur in V if the metric of V is slightly perturbed, then every bounded domain $D \subset V$ containing a closed causal path will also be stably acausal. Further, these space-times (namely, T_2; T_2^* at $b = 1$; T_3^*) are homogeneous [6], hence it will be sufficient to prove the stable causality of only bounded domains containing a fixed point, *e. g.* the origin (0, 0, 0, 0). Every such bounded part of V can be enclosed in a M-cube

$$Q_M = \{x \in V : |x^i| \leqslant M(1 \leqslant i \leqslant 4)\}, \quad M > 0. \tag{14}$$

Thus it will be sufficient to show that in every cube (14) a global time function exists.

Consider the space-time T_2. The condition (12) implies

$$2\dot{x}^1\dot{x}^4 + A(\dot{x}^4)^2 \geqslant (\Theta^2)^2 + (\Theta^3)^2 \geqslant 0,$$

that is

$$2\dot{x}^1\dot{x}^4 \geqslant -A(\dot{x}^4)^2. \tag{15}$$

Introduce the time function

$$t_M(x) = x^1 + nx^4, \qquad n = \tfrac{1}{2}(1 + |a| + M^2).$$

In the cube Q_M the function $t_M(x)$ satisfies the condition

$$(\dot{t}_M)^2 \geqslant (\Theta^4(\dot{x}))^2, \tag{16}$$

in view of Eq. (15). Hence, if s is normalized by means of Eq. (11), then Eqs (13), (16) imply $\dot{t}_M \geqslant 1$ for all $x \in Q_M$, and $t_M(x)$ can be taken as a global time in Q_M. Hence, the space-time T_2 is stably causal.

In the case of the space-time $T_2^*(b = 1)$ one can take the function

$$t_M(x) = x^1 - e^{3M}x^3,$$

and then obtains the following evaluation (under the same assumptions):

$$\dot{t}_M \geqslant \Theta^4(\dot{x})e^{-5x^4/2} \geqslant e^{-5M/2},$$

for all x from Q_M. Therefore, Q_M (and hence T_3^*, $b = 1$) is stably causal.

In the case of the space-time T_3^* let us normalize s as follows:

$$\bar{g}_x(\dot{x}, \dot{x}) = 1, \quad \bar{g}_x(dx, dx) = \sum_{i=1}^{4} (\omega^i)^2.$$

Introduce the following time function in Q_M:

$$t_M(x) = x^1 + e^{6M}x^3.$$

It is easy to verify that

$$\dot{t}_M \geqslant e^{-4x^4} \geqslant e^{-4M}$$

for all x from Q_M. Hence, T_3^* also is stably causal.

Therefore, all the highly mobile algebraically special space-times are stably causal.

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