PARAMETRIZATION OF REGGE Δ EXCHANGE FOR BACKWARD πN QUASI-TWO-BODY SCATTERING*

By F. Schneider** and D. W. Schlitt

Department of Physics, University of Nebraska, Lincoln***

(Received September 17, 1973)

A parametrization of Regge residue for Δ exchange which incorporates MacDowell symmetry, sense-nonsense zeros, constraints and cancellation of kinematic singularities is proposed. Absence of the parity partner of the (3,3) resonance is included through a zero in the residue. Explicit results are given for the cases of $\pi^-p \to \pi^-p$, ϱ^-p, A_1^-p, and A_2^-p using the Gell-Mann choosing nonsense mechanism. Other sense-nonsense behaviours are discussed.

1. Introduction

The parametrization of the Regge residue function is one of the areas of choice in making Regge fits to scattering data. We present here a parametrization for fermion exchange which incorporates the general features which are expected on theoretical grounds. This parametrization incorporates MacDowell symmetry sense-nonsense zeros, constraints and cancellation of kinematic singularities. For concreteness, as well as for use in a specific application [1], we will deal with Δ exchange in the processes

$$\pi^-p \to \begin{cases} \pi^-p \\ \varrho^-p \\ A_1^-p \\ A_2^-p. \end{cases} \tag{1}$$

The general approach is applicable to other processes.

There are four Δ trajectories denoted Δ_α, Δ_β, Δ_δ, and Δ_γ. The Δ_δ trajectory has positive parity and includes the well established (3, 3) resonance with mass 1236 MeV. The Δ_α and Δ_β trajectories form a pair of trajectories with opposite parities and lie much lower

* Supported in part by NSF grant GP-13959. A more detailed discussion of many of the results reported here are contained in the thesis submitted by one of us (F. S.) in partial fulfillment of the requirements for the Ph. D. degree at the University of Nebraska.

** Present address: Southeastern Illinois College, Harrisburg, Illinois 62946, USA.

*** Address: Department of Physics, University of Nebraska, Lincoln, Nebraska 68508, USA

than the Δ_δ trajectory. The Δ_γ trajectory is the parity partner of the Δ_δ required by MacDowell symmetry. The experimental evidence is that these negative parity resonances are absent. This experimental fact is included in the parametrization by requiring that the appropriate residue have a zero at the mass of the parity partner.

The parametrization given here differs from those previously used in Δ exchange by including all of the features listed above. Previous parametrizations have omitted one or another of the above. The specific parametrizations given here, *e.g.* Eq. (27), involve choices which can be termed arbitrary. We have tried to point out each of these choices as it is made and indicate the alternatives.

2. *Parametrization of the Regge residue*

We will not include a detailed derivation of the reggeization of the amplitudes, the removal of kinematic singularities and the conditions imposed by the constraints since the methods have been adequately discussed elsewhere [2]. We will begin, however, by setting down sufficient definitions and intermediate results to establish notation, choices of phases, *etc.*

Kinematics

For the process

$$a+b \to c+d \tag{2}$$

we use the Mandelstam variables

$$\begin{aligned} s &= (p_a+p_b)^2 = (p_c+p_d)^2, \\ t &= (p_a+p_c)^2 = (p_b+p_d)^2, \\ u &= (p_a+p_d)^2 = (p_b+p_c)^2, \end{aligned} \tag{3}$$

where the 4 momenta p_i are all taken as incoming. In addition to these Lorentz invariant variables we will make use of Z_s, the cosine of the c.m. scattering angle when s is the energy variable and similarly for Z_t and Z_u. The variables are related by

$$s+t+u = \Sigma = m_a^2+m_b^2+m_c^2+m_d^2, \tag{4}$$

$$Z_s = \frac{2st+s^2-s\,\Sigma\,+(m_a^2-m_b^2)\,(m_c^2-m_d^2)}{\{[s-(m_a-m_b)^2]\,[s-(m_a+m_b)^2]\,[s-(m_c-m_d)^2]\,[s-(m_c+m_d)^2]\}^{\frac{1}{2}}}, \tag{5}$$

and

$$Z_u = \frac{-2su-u^2+u\,\Sigma\,+(m_c^2-m_b^2)\,(m_a^2-m_d^2)}{\{[u-(m_c-m_b)^2]\,[u-(m_c+m_b)^2]\,[u-(m_a-m_d)^2]\,[u-(m_a+m_d)^2]\}^{\frac{1}{2}}}. \tag{6}$$

The helicities of the particles will be denoted λ_a *etc.*, their spins by S_a *etc.*, and their intrinsic parities by η_a *etc.*

Reggeization

Reggeization is accomplished using the parity conserving helicity amplitudes by the method of Gell-Mann *et al.* [3]. The phases of the amplitudes are those of Jacob and Wick [4].

The reggeized s-channel parity conserving helicity amplitude is

$$F_h^{\pm}(s, Z_s) = (-1)^{\lambda-\mu}\left[\sum_{\tau,\,\mathrm{Re}\,\alpha>-M} (2\alpha^{\pm}+1)\beta_h^{\pm\tau}\frac{E^{+\tau}_{-\lambda\mu}(-Z_s, -\alpha^{\pm}-1)}{\cos\pi(\alpha^{\pm}-\lambda)} + \right.$$

$$\left. + \sum_{\tau,\,\mathrm{Re}\,\alpha>-M} (2\alpha^{\mp}+1)\beta_h^{\mp\tau}\frac{E^{-\tau}_{-\lambda\mu}(-Z_s, -\alpha^{\mp}-1)}{\cos\pi(\alpha^{\mp}-\lambda)}\right], \tag{7}$$

where $E^{\pm\tau}$ is given by:

$$E^{\pm\tau}_{-\lambda\mu}(-Z, -\alpha-1) = \frac{1}{4}\left[\left(\frac{1+Z}{2}\right)^{-\frac{1}{2}|\lambda+\mu|}\left(\frac{1-Z}{2}\right)^{-\frac{1}{2}|\lambda-\mu|} e_{-\lambda\mu}(-Z, -\alpha-1) \pm \right.$$

$$\left. \pm(-1)^{m-\lambda}\left(\frac{1-Z}{2}\right)^{-\frac{1}{2}|\lambda+\mu|}\left(\frac{1+Z}{2}\right)^{-\frac{1}{2}|\lambda-\mu|} e_{-\lambda-\mu}(-Z, -\alpha-1)\right] \times$$

$$\times[1+\tau\exp[-i\pi(\alpha-\mu)]] \tag{8}$$

and where

$$h = \{\lambda_c\lambda_d, \lambda_a\lambda_b\}, \qquad \lambda = \lambda_a-\lambda_b,$$

$$m = \max(|\lambda|, |\mu|), \qquad \mu = \lambda_c-\lambda_d,$$

$$\tau = \text{signature}.$$

The functions $e_{\lambda\mu}(-Z, -\alpha-1)$ are the rotation functions of the second kind defined by Andrews and Gunson [6] and for $\lambda > |\mu|$ they are given by

$$e_{\lambda\mu}(-Z, -\alpha-1) = \frac{1}{2\Gamma(-2\alpha)}\{\Gamma(-\alpha+\lambda)\Gamma(-\alpha+\mu)\Gamma(-\alpha-\lambda)\Gamma(-\alpha-\mu)\}^{\frac{1}{2}}\times$$

$$\times\left(\frac{1-Z}{2}\right)^{-\frac{1}{2}(\lambda+\mu)}\left(\frac{1+Z}{2}\right)^{-\frac{1}{2}(\lambda-\mu)}\left(\frac{-Z-1}{2}\right)^{\alpha-\lambda} F\left(-\alpha-\lambda, \alpha-\mu; -2\alpha; \frac{2}{1+Z}\right). \tag{9}$$

The parity conserving helicity amplitude is related to the ordinary helicity amplitude f_h by

$$F_h^{\pm}(s, Z_s) = \left(\frac{1+Z_s}{2}\right)^{-\frac{1}{2}|\lambda+\mu|}\left(\frac{1-Z_s}{2}\right)^{-\frac{1}{2}|\lambda-\mu|} f_h(s, Z_s) \pm$$

$$\pm\eta_c\eta_d(-1)^{S_a+S_b-v}(-1)^{\lambda+m}\left(\frac{1+Z_s}{2}\right)^{-\frac{1}{2}|\lambda-\mu|}\left(\frac{1-Z_s}{2}\right)^{-\frac{1}{2}|\lambda+\mu|} f_{h-}, \tag{10}$$

where

$$h_- = \{-\lambda_c - \lambda_d;\ \lambda_a \lambda_b\} \quad \text{and} \quad v = \begin{cases} 0 \text{ for fermion-fermion or boson-boson} \\ \tfrac{1}{2} \text{ for fermion-boson.} \end{cases}$$

In terms of the helicity amplitudes the cross-section is

$$\frac{d\sigma}{du} = \frac{1}{\pi[s-(m_a-m_b)^2]\,[s-(m_a+m_b)^2]}\,\frac{1}{(2S_a+1)\,(2S_b+1)} \sum_h |f_h(s, Z_s)|^2. \qquad (11)$$

Δ exchange in $\pi^- p$ scattering

The s-channel will be taken as the $\pi^- p$ channel. For backward scattering the Regge poles of interest will be in the u-channel. For this reason the cross-section should be expressed in terms of the u-channel helicity amplitudes. Because the crossing matrix is a real orthogonal matrix we have the relation

$$\sum_h |f_h(s, Z_s)|^2 = \sum_{\bar{h}} |f_{\bar{h}}(u, Z_u)|^2. \qquad (12)$$

It is convenient to define

$$\bar{h} = \{\lambda_{\bar{a}}\lambda_d;\ \lambda_{\bar{c}}\lambda_b\}, \qquad \bar{\lambda} = \lambda_{\bar{c}} - \lambda_b,$$

$$\bar{h}_- = \{-\lambda_{\bar{a}} - \lambda_d;\ \lambda_{\bar{c}}\lambda_b\}, \qquad \bar{\mu} = \lambda_{\bar{a}} - \lambda_d.$$

For the processes of Eq. (1) we have, as previously noted, only one contributing trajectory for each parity and both have $\tau = -1$.

Additional simplification comes from the fact that for large Z

$$E^{\pm}_{-\lambda\mu}(-Z, -\alpha-1) = \frac{1}{2}\,\frac{\sin\pi\alpha}{\cos\pi\alpha}\,\frac{\Gamma(\frac{1}{2})\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\Bigg[(\alpha+1-\lambda)\,(\alpha+1-\mu)\times$$

$$\times \prod_{n=0}^{|\lambda-\frac{1}{2}|}\left(\frac{\alpha+n+\frac{1}{2}-|\lambda-\frac{1}{2}|}{\alpha-n+\frac{3}{2}-|\lambda-\frac{1}{2}|}\right)\prod_{n'=0}^{|\mu-\frac{1}{2}|}\left(\frac{\alpha+n'+\frac{1}{2}-|\mu-\frac{1}{2}|}{\alpha-n'+\frac{3}{2}-|\mu-\frac{1}{2}|}\right)\Bigg]^{\frac{1}{2}}(-2Z)^{\alpha-\lambda}\times$$

$$\times(1\pm 1+0(Z^{-2})). \qquad (13)$$

Thus, to leading order in s, we need only to keep the terms in E^+ in Eq. (7). Finally for the fermion trajectories $W = \sqrt{u}$ is a more convenient variable. With these simplifications the amplitudes will have the general form

$$F^{\pm}_{\bar{h}} = Q(\alpha^{\pm}(W))\xi(\alpha^{\pm}(W))\left(\frac{S}{S_0}\right)^{\alpha^{\pm}-\frac{1}{2}} S^{\frac{1}{2}-\bar{\lambda}} K^{\pm}_{\bar{\lambda}\bar{\mu}}(W)\gamma^{\pm}_{\bar{h}}(W), \qquad (14)$$

where $Q(\alpha)$ is a polynomial in α which contains the ghost killing factors; $\xi(\alpha)$ is the signature factor

$$\xi(\alpha) = \{1-\exp[i\pi(\alpha-\tfrac{1}{2})]\}/\cos\pi\alpha. \qquad (15)$$

$K^{\pm}_{\bar{\lambda}\bar{\mu}}(W)$ is the factor which gives the kinematic singularities in W at the thresholds and pseudo-thresholds (which we will call by the collective name "thresholds".) These functions have been determined by Henyey [7]. The function $\gamma^{\pm}_{\bar{h}}(W)$ is the reduced residue which we wish to parametrize subject to further constraints.

The kinematic factors $K^{\pm}_{\bar{\lambda}\bar{\mu}}$ involve the factors $W^2-(m_{\bar{a}}\pm m_c)^2$ and $W^2-(m_{\bar{c}}\pm m_b)^2$ so it is convenient to introduce the following notation which makes use of the fact that particles d and b are both protons for the problem of interest;

$$N^{\pm}(\bar{a}) = W\pm(m_d+m_{\bar{a}}), \quad P^{\pm}(\bar{a}) = W\pm(m_d-m_{\bar{a}}), \quad M^{\pm}(\bar{a}) = m_{\bar{a}}\pm m_d. \tag{16}$$

From $F^{\pm}_{\bar{h}}$ the u-channel helicity amplitudes are obtained by inverting the relation in Eq. (10) and use of conservation of parity. The required relations are

$$f_{\bar{h}} = \frac{1}{2}\left(\frac{1+Z_u}{2}\right)^{\frac{1}{2}|\bar{\lambda}+\bar{\mu}|}\left(\frac{1-Z_u}{2}\right)^{\frac{1}{2}|\bar{\lambda}-\bar{\mu}|}(F^{+}_{\bar{h}}+F^{-}_{\bar{h}}), \tag{17}$$

$$f_{\bar{h}_-} = \frac{1}{2}\left(\frac{1+Z_u}{2}\right)^{\frac{1}{2}|\bar{\lambda}-\bar{\mu}|}\left(\frac{1-Z_u}{2}\right)^{\frac{1}{2}|\bar{\lambda}+\bar{\mu}|}(F^{+}_{\bar{h}}-F^{-}_{\bar{h}}), \tag{18}$$

and

$$f_{-\bar{h}} = \frac{\eta_{\bar{a}}\eta_d}{\eta_{\bar{c}}\eta_b}(-1)^{S_a+S_b-S_c-S_d}(-1)^{\bar{\lambda}-\bar{\mu}}f_{\bar{h}}. \tag{19}$$

In order to extract the kinematic factors $K^{\pm}_{\bar{\lambda}\bar{\mu}}(W)$ it was necessary to define the parity conserving amplitudes. These amplitudes are related by a generalization of MacDowell symmetry:

$$F^{+}_{\bar{h}}(W, Z_u) = (-1)^{|\bar{\lambda}|+|\bar{\mu}|+|\bar{\lambda}-\varepsilon_{\bar{c}d}\varepsilon_{\bar{a}d}\bar{\mu}|}F^{-}_{\bar{h}}(-W, Z_u), \tag{20}$$

where

$$\varepsilon_{ij} = \frac{(m_i-m_j)}{|m_i-m_j|}.$$

The kinematic factors $K^{\pm}_{\bar{\lambda}\bar{\mu}}$ contain the singularities at the thresholds, however, in addition, the helicity amplitudes are not independent there. The method for establishing the relationship is discussed by Jackson and Hite [8]. These constraint equations are the major condition which determine the parametrization of the reduced residues, γ.

There is one more condition on the γ's that arises from the threshold singularities in the u-channel amplitudes. The amplitudes $f_h(s, Z_s)$ do not have singularities at the u-channel thresholds. This means that $\sum_h |f_h(s, Z_s)|^2$ does not have singularities. Equation (13) therefore implies that $\sum_{\bar{h}}|f_{\bar{h}}(u, Z_u)|^2$ has no singularities at the thresholds. Since the individual terms have singularities the reduced residues must be chosen in such a way that the singularities cancel between the various terms. We impose this condition for $u<0$ where $W = i\sqrt{|u|}$, and can then continue into the region $u>0$ with no singularities appearing.

Finally we must consider the coupling of nonsense helicity states, *i.e.*, those with helicity with magnitude incompatable with the total angular momentum. The sense--nonsense structure is discussed by Gell-Mann [3]. The behaviour of the residue at a sense--nonsense point and its image point $-(\lambda+1)$ is given by:

$$\beta_{\rm sn} \propto (J-\lambda)^{\frac{1}{2}}(J+\lambda+1)^{\frac{1}{2}}. \tag{21}$$

Factorization requires

$$(\beta_{\rm sn})^2 = \beta_{ss}\beta_{nn}. \tag{22}$$

This implies that the square root zeros in $\beta_{\rm sn}$ must occur as full zeros in the product $\beta_{ss}\beta_{nn}$.

There are four generally accepted solutions which avoid the introduction of additional singularities and satisfy factorization. In summary, for α near λ they are [2]

	β_{ss}	β_{nn}	β_{ns}
Sense-choosing	1	$(\alpha-\lambda)$	$(\alpha-\lambda)^{1/2}$
Gell-Mann	$(\alpha-\lambda)$	1	$(\alpha-\lambda)^{1/2}$
Chew	$(\alpha-\lambda)$	$(\alpha-\lambda)^2$	$(\alpha-\lambda)^{3/2}$
No compensation	$(\alpha-\lambda)^2$	$(\alpha-\lambda)$	$(\alpha-\lambda)^{3/2}$. (23)

The ghost-killing factor in what follow will be determined according to the Gell-Mann mechanism.

There is some additional arbitrariness in the choice of the ghost-killing factors, and this is how many different values of λ to include. The choice we have made is to include the correct factors of $(\alpha+\frac{1}{2})$ and for each sense-nonsense point, λ, the correct factors of $(\alpha-\lambda)(\alpha+\lambda+1)$.

We assume that the trajectory functions are linear in u. This requirement combined with MacDowell symmetry means that there is only one trajectory function

$$\alpha(W) = \alpha^{\pm}(W) = \alpha_0+\alpha' W^2. \tag{24}$$

π^-p elastic scattering

In the case of π^-p elastic scattering it is sufficient to compute only $F^{\pm}_{0-\frac{1}{2}0-\frac{1}{2}}$; all of the helicity amplitudes can be computed from them using Eqs (17), (18), and (19).

The kinematic factors in this case are

$$K^{\pm}_{\frac{1}{2}\frac{1}{2}} = W^{-1}N^{\mp}(\pi)P^{\mp}(\pi) \tag{25}$$

and the ghost-killing factor is

$$Q(\alpha) = (\alpha+\tfrac{1}{2}). \tag{26}$$

For this process there is no coupling between sense and nonsense states so that the only constraint that is imposed on γ^{+}_{h} is that it vanish at parity partner of the (3, 3) resonance. The simplest parametrization satisfying this requirement is

$$\gamma^{+}_{0-\frac{1}{2}0-\frac{1}{2}} = A\left(1-\frac{W}{m_{33}}\right). \tag{27}$$

With the choice made above about the inclusion of ghost-killing factors, this parametrization applies to all four mechanisms. It does, however, omit the wrong signature nonsense zero at $\alpha = 1/2$ which Igi *et al.* [9] find useful in slightly improving their fit to the data.

$\pi^- p \to \varrho^- p$

For ϱ production we must consider six parity conserving amplitudes which involve four different kinematic factors.

The kinematic factors in this case are

$$K^{\pm}_{\frac{1}{2}\frac{1}{2}} = W^{-1}\left\{\frac{N^{\mp}(\pi)P^{\mp}(\pi)}{N^{\mp}(\varrho)P^{\mp}(\varrho)}\right\}^{\frac{1}{2}} \tag{28}$$

and

$$K^{\pm}_{\frac{1}{2}\frac{1}{2}} = W^{-2}P^{\mp}(\pi)N^{\mp}(\pi)\,\{P^{\pm}(\pi)N^{\pm}(\pi)P^{\pm}(\varrho)N^{\pm}(\varrho)\}^{\frac{1}{2}}. \tag{29}$$

The ghost-killing factor which also includes the factors from the Gell-Mann choosing nonsense mechanism is

$$Q(\alpha) = (\alpha-\tfrac{1}{2})(\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2}). \tag{30}$$

The remaining problem is the parametrization of the six reduced residue functions $\gamma^{\pm}_{0-\frac{1}{2}1-\frac{1}{2}}$, $\gamma^{\pm}_{0-\frac{1}{2}1\frac{1}{2}}$ and $\gamma^{\pm}_{0-\frac{1}{2}0-\frac{1}{2}}$. It is sufficient to find the 3 with positive parity and determine the others from MacDowell symmetry.

The constraints of the thresholds are determined by the method of Jackson and Hite [8]. The resulting equations are

$$-2W\gamma^{+}_{0-\frac{1}{2}1-\frac{1}{2}} = \gamma^{+}_{0-\frac{1}{2}1\frac{1}{2}} = \frac{1}{\sqrt{2}}\gamma^{+}_{0-\frac{1}{2}0-\frac{1}{2}} \quad \text{at} \quad N^{-}(\varrho) = 0,$$

$$-2W\gamma^{+}_{0-\frac{1}{2}1-\frac{1}{2}} = \gamma^{+}_{0-\frac{1}{2}1\frac{1}{2}} = \frac{-1}{\sqrt{2}}\gamma^{+}_{0-\frac{1}{2}0-\frac{1}{2}} \quad \text{at} \quad P^{-}(\varrho) = 0,$$

$$2W\gamma^{+}_{0-\frac{1}{2}1-\frac{1}{2}} = \gamma^{+}_{0-\frac{1}{2}1\frac{1}{2}} \pm \sqrt{2}\,\gamma^{+}_{0-\frac{1}{2}0-\frac{1}{2}} \quad \text{at} \quad \begin{cases} N^{+}(\varrho) = 0, \\ P^{+}(\varrho) = 0. \end{cases} \tag{31}$$

Cancellation of the singularities in the cross-section requires that

$$4W^2|\gamma^{+}_{0-\frac{1}{2}1-\frac{1}{2}}|^2 - |\gamma^{+}_{0-\frac{1}{2}1\frac{1}{2}}|^2 - |\gamma^{+}_{0-\frac{1}{2}0-\frac{1}{2}}|^2 \sim P^{-}(\varrho)P^{+}(\varrho)N^{-}(\varrho)N^{+}(\varrho). \tag{32}$$

Since the γ's are smooth analytic functions we represent them by polynomials in W. In addition γ^{+} must vanish at $W = m_{33}$. This leads to the parametrization

$$\gamma^{+}_{0-\frac{1}{2}1-\frac{1}{2}} = \left[A\left(1-\frac{W^2}{M^{+}(\varrho)^2}\right) + B\left(1-\frac{W^2}{M^{-}(\varrho)^2}\right)\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^{+}_{0-\frac{1}{2}1\frac{1}{2}} = -2W\left[A\left(1-\frac{W^2}{M^{+}(\varrho)^2}\right) + B\left(1-\frac{W^2}{M^{-}(\varrho)^2}\right)\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} = -2\sqrt{2}\left[AM^{-}(\varrho)\left(1-\frac{W^2}{M^{+}(\varrho)^2}\right)+BM^{+}(\varrho)\left(1-\frac{W^2}{M^{-}(\varrho)^2}\right)\right]\times$$
$$\times\left(1-\frac{W}{m_{33}}\right) \tag{33}$$

for the Gell-Mann mechanism.

For the other mechanisms the constraint equations are modified by the presence of $(\alpha-\lambda)(\alpha+\lambda+1)$ factors for the sense-nonsense point at $\lambda=\frac{1}{2}$. The resulting parametrization for the sense mechanism is:

$$\gamma^{+}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \left\{A\left(1-\frac{W^2}{M^{+}(\varrho)^2}\right)+B\left(1-\frac{W^2}{M^{-}(\varrho)^2}\right)\right\}\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^{+}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} = \frac{-2W}{\left(\alpha(W)-\frac{1}{2}\right)\left(\alpha(W)+\frac{3}{2}\right)}\left\{A\left[\alpha(M^{-}(\varrho))-\frac{1}{2}\right]\left[\alpha(M^{-}(\varrho))+\frac{3}{2}\right]\times\right.$$
$$\left.\times\left(1-\frac{W^2}{M^{+}(\varrho)^2}\right)+B\left[\alpha(M^{+}(\varrho))-\frac{1}{2}\right]\left[\alpha(M^{+}(\varrho)+\frac{3}{2}\right]\left(1-\frac{W^2}{M^{-}(\varrho)^2}\right)\right\}\times$$
$$\times\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} = \frac{-2\sqrt{2}}{\left(\alpha(W)-\frac{1}{2}\right)\left(\alpha(W)+\frac{3}{2}\right)}\left\{AM^{-}(\varrho)\left[\alpha(M^{-}(\varrho))-\frac{1}{2}\right]\left[\alpha(M^{-}(\varrho))+\frac{3}{2}\right]\times\right.$$
$$\left.\times\left(1-\frac{W^2}{M^{+}(\varrho)^2}\right)+BM^{+}(\varrho)\left[\alpha(M^{+}(\varrho))-\frac{1}{2}\right]\left[\alpha(M^{+}(\varrho))+\frac{3}{2}\right]\left[1-\frac{W^2}{M^{-}(\varrho)^2}\right]\right\}\times$$
$$\times\left(1-\frac{W}{m_{33}}\right). \tag{34}$$

The parametrizations for the remaining two mechanisms differ from that for the sense mechanism only in the factors containing $\alpha(W)$ which multiply the curly brackets. For the Chew mechanism they are chosen so that

$$F^{\pm}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} \propto (\alpha+\tfrac{1}{2})(\alpha-\tfrac{1}{2})^2(\alpha+\tfrac{3}{2})^2,$$
$$F^{\pm}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} \propto (\alpha+\tfrac{1}{2})(\alpha-\tfrac{1}{2})(\alpha+\tfrac{3}{2}),$$
$$F^{\pm}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \propto (\alpha+\tfrac{1}{2})(\alpha-\tfrac{1}{2})(\alpha+\tfrac{3}{2}) \tag{35}$$

and for the no compensation mechanism they are chosen so that

$$F^{\pm}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} \propto (\alpha+\tfrac{1}{2})(\alpha-\tfrac{1}{2})^2(\alpha+\tfrac{3}{2})^2,$$

$$F^{\pm}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} \propto (\alpha+\tfrac{1}{2})\,(\alpha-\tfrac{1}{2})^2(\alpha+\tfrac{3}{2})^2,$$
$$F^{\pm}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \propto (\alpha+\tfrac{1}{2})\,(\alpha-\tfrac{1}{2})^2(\alpha+\tfrac{3}{2})^2. \tag{36}$$

$\pi^- p \to A_1^- p$

This process differs from ϱ production only in the intrinsic parity of the A_1. There are again 6 parity conserving amplitudes to be determined.

The kinematic factors are

$$K^{\pm}_{\frac{1}{2}\frac{1}{2}} = W^{-1}\left\{\frac{N^{\mp}(\pi)P^{\mp}(\pi)}{N^{\pm}(A_1)P^{\pm}(A_1)}\right\}^{\frac{1}{2}} \tag{37}$$

and

$$K^{\pm}_{\frac{1}{2}\frac{1}{2}} = W^{-2}N^{\mp}(\pi)P^{\mp}(\pi)\,\{N^{\mp}(A_1)P^{\mp}(A_1)N^{\pm}(\pi)P^{\pm}(\pi)\}^{\frac{1}{2}}. \tag{38}$$

The Gell-Mann mechanism ghost-killing factor is

$$Q(\alpha) = (\alpha+\tfrac{1}{2})\,(\alpha-\tfrac{1}{2})\,(\alpha+\tfrac{3}{2}). \tag{39}$$

The constraints at the thresholds are

$$2W\gamma^{+}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \gamma^{+}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} = -\frac{1}{\sqrt{2}}\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \quad \text{at} \quad P^{+}(A_1)=0,$$

$$-2W\gamma^{+}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \gamma^{+}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} \pm\sqrt{2}\,\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \quad \text{at} \quad \begin{cases} N^{-}(A_1)=0 \\ P^{-}(A_1)=0, \end{cases}$$

$$2W\gamma^{+}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \gamma^{+}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} = \frac{1}{\sqrt{2}}\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \quad \text{at} \quad N^{+}(A_1)=0 \tag{40}$$

and cancellation of the singularities in the cross-section requires

$$4W^2|\gamma^{+}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}}|^2 - |\gamma^{+}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}}|^2 - |\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}}|^2 \propto N^{+}(A_1)N^{-}(A_1)P^{+}(A_1)P^{-}(A_1). \tag{41}$$

A parametrization satisfying these constraints is

$$\gamma^{+}_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \left[A\left(1-\frac{W^2}{M^{+}(A_1)^2}\right)+B\left(1-\frac{W^2}{M^{-}(A_1)^2}\right)\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^{+}_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} = 2W\left[A\left(1-\frac{W^2}{M^{+}(A_1)^2}\right)+B\left(1-\frac{W^2}{M^{-}(A_1)^2}\right)\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^{+}_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} = -2\sqrt{2}\left[AM^{-}(A_1)\left(1-\frac{W^2}{M^{+}(A_1)^2}\right)+BM^{+}(A_1)\left(1-\frac{W^2}{M^{-}(A_1)^2}\right)\right]\times$$
$$\times\left(1-\frac{W}{m_{33}}\right). \tag{42}$$

For the other sense-nonsense couplings the parametrization can be modified as was done for ϱ production.

$\pi^- p \to A_2^- p$

In treating this reaction we will ignore the possibility of a split A_2 and treat it as a single particle with $J^P = 2^+$. In this case there are ten parity conserving amplitudes to be determined.

The relevant kinematic factors are

$$K^{\pm}_{\frac{1}{2}\frac{1}{2}} = W^{-1}(N^{\mp}(A_2)P^{\mp}(A_2))^{-1}\left\{\frac{N^{\mp}(\pi)P^{\mp}(\pi)}{N^{\pm}(A_2)P^{\pm}(A_2)}\right\}^{\frac{1}{2}}, \tag{43}$$

$$K^{\pm}_{\frac{3}{2}\frac{1}{2}} = W^{-2}N^{\mp}(\pi)P^{\mp}(\pi)\left\{\frac{N^{\pm}(\pi)P^{\pm}(\pi)}{N^{\mp}(A_2)P^{\mp}(A_2)}\right\}^{\frac{1}{2}} \tag{44}$$

and

$$K^{\pm}_{\frac{5}{2}\frac{1}{2}} = W^{-3}N^{\pm}(\pi)P^{\pm}(\pi)\,\{N^{\pm}(A_2)P^{\pm}(A_2)N^{\mp}(\pi)^3P^{\mp}(\pi)^3\}^{\frac{1}{2}}. \tag{45}$$

The Gell-Mann mechanism ghost-killing factor is

$$Q(\alpha) = (\alpha+\tfrac{1}{2})(\alpha+\tfrac{3}{2})(\alpha+\tfrac{5}{2})(\alpha-\tfrac{1}{2})(\alpha-\tfrac{3}{2}). \tag{46}$$

There are five positive parity reduced residue functions to be determined in such a way that they satisfy the constraints at the thresholds. For the Gell-Mann mechanism the required constraint equations are

$$8W^2\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} = -4W\gamma^+_{0\,-\frac{1}{2}\,2\,\frac{1}{2}} = -2W\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} =$$
$$= \frac{2}{\sqrt{6}}\gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \quad \text{at} \quad N^-(A_2) = 0,$$

$$\left.\begin{aligned} 6W\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} &= \gamma^+_{0\,-\frac{1}{2}\,2\,\frac{1}{2}} - 2\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} \\ -4W^2\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} &= \gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} - \frac{3}{\sqrt{6}}\gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \\ -16W^2\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} &= 6W\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} + \gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \end{aligned}\right\} \text{ at } P^+(A_2) = 0,$$

$$-8W^2\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} = 4W\gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} = -2W\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = \gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} =$$
$$= -\frac{2}{\sqrt{6}}\gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \quad \text{at} \quad P^-(A_2) = 0,$$

$$\left.\begin{aligned} 6W\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} &= \gamma^+_{0\,-\frac{1}{2}\,2\,\frac{1}{2}} + 2\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} \\ 4W^2\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} &= \gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} + \frac{3}{\sqrt{6}}\gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} \\ 16W^2\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} &= 6W\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} + \gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} \end{aligned}\right\} \text{ at } N^+(A_2) = 0. \tag{47}$$

The condition for cancellation of the singularities in the cross-section is

$$16W^4|\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}}|^2-4W^2|\gamma^+_{0\,-\frac{1}{2}\,2\,\frac{1}{2}}|^2-4W^2|\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}}|^2+$$
$$+|\gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}}|^2+|\gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}}|^2 \propto [N^+(A_2)N^-(A_2)P^+(A_2)P^-(A_2)]^2. \tag{48}$$

A parametrization (which we have not established as unique) satisfying these conditions is

$$\gamma^+_{0\,-\frac{1}{2}\,2\,-\frac{1}{2}} = \left[A\left(1-\frac{W^2}{M^+(A_2)^2}\right)^2+B\left(1-\frac{W^2}{M^-(A_2)^2}\right)^2\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^+_{0\,-\frac{1}{2}\,2\,\frac{1}{2}} = -2W\left[A\left(1-\frac{W^2}{M^+(A_2)^2}\right)^2+B\left(1-\frac{W^2}{M^-(A_2)^2}\right)^2\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^+_{0\,-\frac{1}{2}\,1\,-\frac{1}{2}} = -4\left[AM^-(A_2)\left(1-\frac{W^2}{M^+(A_2)^2}\right)^2+BM^+(A_2)\left(1-\frac{W^2}{M^-(A_2)^2}\right)^2\right]\times$$
$$\times\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^+_{0\,-\frac{1}{2}\,1\,\frac{1}{2}} = 8W\left[AM^-(A_2)\left(1-\frac{W^2}{M^+(A_2)^2}\right)^2+\right.$$
$$\left.+BM^+(A_2)\left(1-\frac{W^2}{M^-(A_2)^2}\right)^2\right]\left(1-\frac{W}{m_{33}}\right),$$

$$\gamma^+_{0\,-\frac{1}{2}\,0\,-\frac{1}{2}} = \frac{4\sqrt{2}}{3}\left[A(W^2+2M^-(A_2)^2)\left(1-\frac{W^2}{M^+(A_2)^2}\right)^2+\right.$$
$$\left.+B(W^2+2M^+(A_2)^2)\left(1-\frac{W^2}{M^-(A_2)^2}\right)^2\right]\left(1-\frac{W}{m_{33}}\right). \tag{49}$$

Similar parametrizations can be found for the other sense-nonsense mechanism.

3. *Conclusions*

We have presented a parametrization for Regge residues which incorporates those general features which are expected on theoretical grounds. We have not been able to establish any sort of uniqueness for these parametrizations. There are some general features of the way in which the constraints at the thresholds are satisfied which are worth pointing out.

The constraints on the residues at $W = +(m_i \pm m_j)$ are identical to those at $W = -(m_i \pm m_j)$ except for the signs of certain of the reduced residues. This allows the constraints to be satisfied by terms of the form $(1-W^2/(m_i \pm m_j))^k$ where k is the magnitude of the difference between the initial spin and the final spin. In order to satisfy the condition that the singularities cancel in the s-channel differential cross-section k must be greater than one. The differences in sign were taken into account by multiplying by $W/(m_i \pm m_j)$ where needed. Such a factor is needed in those terms where the proton helicity changes sign.

We gratefully acknowledge a number of useful discussions with Dr P. Finkler.

REFERENCES

[1] F. Schneider, *Ph. D. Thesis*, University of Nebraska.
[2] G. E. Hite, *Rev. Mod. Phys.*, **41**, 669 (1969) and references cited there.
[3] M. Gell-Mann, M. L. Goldberger, F. Low, E. Marx, F. Zachariasen, *Phys. Rev.*, **133**, B145. (1964).
[4] M. Jacob, G. C. Wick, *Ann. Phys.*, (N.Y.), **7**, 404 (1959).
[5] M. Jacobs, M. Vaughn, *Phys. Rev.*, **185**, 2045 (1969).
[6] M. Andrews, J. Gunson, *J. Math. Phys.*, **5**, 1391 (1964).
[7] F. S. Henyey, *Phys. Rev.*, **171**, 1509 (1968).
[8] J. D. Jackson, G. E. Hite, *Phys. Rev.*, **169**, 1248 (1968).
[9] K. Igi, S. Matsuda, Y. Oyemgi, H. Sato, *Phys. Rev. Lett.*, **21**, 580 (1968).