ON SOME PROPERTIES OF FORM FACTORS WITH NON-NEGATIVE IMAGINARY PARTS*

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Considered are properties of the form factor $F(t)$ which follow from the assumptions that $F(t)$ satisfies an unsubtracted dispersion relation and its imaginary part is non-negative for $t \geqslant t_0$.

In this note, we shall consider some properties of the form factor under the following assumptions: (*i*) $F(t)$ satisfies an unsubtracted dispersion relation, *i. e.*,

$$F(t) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\mathrm{Im}\, F(W+i\varepsilon)dW}{W-t}\,; \tag{1}$$

(*ii*) the imaginary part of $F(t)$ is non-negative [1] for $t \geqslant t_0$, *i. e.*, $\mathrm{Im}\, F(t+i\varepsilon) \geqslant 0$. Note that $F(t)$ is a real analytic function of t, $F^*(t^*) = F(t)$.

Consider the charge form factor of the pion where $t_0 = 4M_\pi^2$. Let $h(W) = \mathrm{Im}\, F(W+i\varepsilon)/\pi$ and use the variable Z where $Z = t/4M_\pi^2$. Thus, Eq. (1) becomes

$$F(Z) = \int_{1}^{\infty} \frac{h(W)dW}{W-Z}\,. \tag{2}$$

We may also expand $F(Z)$ in a Taylor series for $|Z| < 1$,

$$F(Z) = 1+a_1Z+a_2Z^2+\ldots, \tag{3}$$

where we have used the fact that $F(0) = 1$.

Let $z = x+iy$; thus Eq. (2) becomes

$$F(Z) = \int_{1}^{\infty} \frac{(W-x)h(W)dW}{(W-x)^2+y^2} + iy\int_{1}^{\infty} \frac{h(W)dW}{(W-x)^2+y^2}\,. \tag{4}$$

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The following results are an immediate consequence of Eq. (4):

$$\operatorname{Re} F(Z) > 0,\ |Z| < 1; \tag{5a}$$

$$\operatorname{Im} F(Z)/\operatorname{Im} Z > 0,\ |Z| < 1; \tag{5b}$$

$$F(Z) \text{ has no zeroes in } |Z| < 1. \tag{5c}$$

The class of functions with the property given by Eq. (5a) is known as "functions with a positive real part". In addition to the properties (5b) and (5c) they also have the following properties [2]

$$|a_N| \leqslant 2, \tag{6}$$

where a_N is defined by Eq. (3); the upper and lower bounds,

$$\frac{1-|Z|}{1+|Z|} \leqslant |F(Z)| \leqslant \frac{1+|Z|}{1-|Z|}, \quad |Z| < 1. \tag{7}$$

Note that

$$\frac{1}{N!} F^{(N)}(Z) = \int_1^\infty \frac{h(W)dW}{(W-Z)^{N+1}}, \tag{8a}$$

and

$$a_N = \frac{1}{N!} F^{(N)}(0); \tag{8b}$$

thus since $h(W)$ is real and non-negative, Eq. (6) becomes

$$a_N \leqslant 2. \tag{6a}$$

From Eq. (8a), it easily follows that,

$$F^{(N)}(Z) \geqslant 0, \quad -\infty < Z < 1. \tag{9}$$

The derivative of $F(Z)$ has the following upper bound [3, 4],

$$\left|\frac{dF(Z)}{dZ}\right| \leqslant \frac{4[1+|Z|]}{[1-|Z|]\,[1-|Z|^2]}, \quad |Z| < 1. \tag{10}$$

We note the following very important fact. The class of functions, $F(Z) = 1+a_1Z+$ $+a_2Z^2+\ldots$, that are regular in the disk $|Z| < 1$ and satisfy the inequality $\operatorname{Re} F(Z) > 0$ for $|Z| < 1$, can be represented by a "structural formula" given by the following Stieltjes integral [5],

$$F(Z) = \int_{-\pi}^{\pi} \frac{e^{it}+Z}{e^{it}-Z}\, d\mu(t), \tag{11}$$

where $\mu(t)$ is a nondecreasing function in $[-\pi, \pi]$ such that $\mu(\pi)-\mu(-\pi) = 1$.

Note that since Eq. (1) and (4) hold for any value Z outside the cut, $1 < Z < \infty$, we have in general that for $\text{Im}\, Z > 0$, $\text{Im}\, F(Z) > 0$. A function with this property is called a Herglotz function and it admits the following integral representation [6, 7],

$$F(Z) = A + BZ + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im}\, F(W)(1+ZW)dW}{(1+W^2)(W-Z)}, \tag{12}$$

with $B \geqslant 0$. Thus, $F(Z)$ as given in Eq. (2) with assumptions (*i*) and (*ii*) is a Herglotz function. Thus, we immediately obtain the following lower bounds on $F(Z)$, [7]

$$|F(Z)| > \frac{C_1}{|Z|}, \quad \varepsilon < \text{Arg}\, Z < \pi - \varepsilon; \tag{13a}$$

$$|F(Z)| > \frac{C_2}{|Z|\, |N|Z|}, \quad \text{Arg}\, Z = 0, \pi. \tag{13b}$$

Lastly, we turn to the particular case of the pion charge form factor. Let R be the root-mean-square charge radius. Its value in terms of $F(t)$ is,

$$\frac{1}{6} R^2 \equiv \left.\frac{dF(t)}{dt}\right|_{t=0} = \left.\frac{dF(Z)}{dZ}\frac{dZ}{dt}\right|_{t=0} = a_i \left(\frac{1}{4M_\pi^2}\right). \tag{14}$$

Using the upper bound given by Eq. (6a), we obtain an upper bound on R^2,

$$R^2 \leqslant \frac{3}{M_\pi^2} = \frac{1}{2}\left(\frac{M_e}{M_\pi}\right)^2 R_{\text{VD}}^2, \tag{15}$$

where M_ϱ is the mass of the rho-meson and $R_{\text{VD}}^2 = 6/M_\varrho^2$ is the value for R obtained from vector-meson-dominance theory [8].

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