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DUST-FILLED VISCOUS UNIVERSES

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The Friedmann Equation with the bulk viscosity teim (coefficient of bulk viscosity = = const) is integrated for the dust-filled models. It appears that the introduction of dissipation removes the initial singularity provided it is allowed by the Hawking-Penrose Theorem. Many models obtained by this method, although analytically regular, possess regions with negative energy.

1. Introduction

The role of dissipative processes in cosmic evolution has been discussed in several papers [1]–[4]. In a previous paper [5], hereaftei referred to as HKS, we have introduced the bulk viscosity into the frame of Friedmann-Lemaître cosmology. Explicit solutions of Einstein field equations were found for the flat ($k = 0$), dust-filled and radiation-filled universes ($k = 0, \pm 1$), under the highly idealized assumption of constant coefficient of bulk viscosity. It became evident that the introduction of the viscosity term into the cosmological equations does not exclude automatically the appearance of singularities (as is often the case in hydrodynamics). There are, however, some viscous Friedmann-Lemaître models without an initial singulaiity.

In the present paper we enlarge our discussion by considering a wider class of dust-filled Friedmann-Lemaitre universes (with $k = 0, \pm 1$). The bulk viscosity coefficient again is equal to a constant. The solutions which are presented here were found numerically. For the sake of completeness we also review our previous results concerning the case $k = 0$.

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It is worth to mention the work of Treciokas and Ellis [6]. They consider the kinetic world model in which the substratum is regarded as a distribution of particles of proper mass m and 4-momentum p^{α}. This function obeys the relativistic Boltzmann equations. The system is completely described by the Boltzmann-Einstein equations. The kinetic theory indicates that the motion in the Robertson-Walker cosmological models will be, in general, irreversible, corresponding to a non-vanishing bulk viscosity. Treciokas and Ellis have found, moreover, the flat ($k = 0$) solutions of Einstein field equations with the bulk viscosity term. The solutions coincide with those of HKS.

2. *Methods of evaluation*

1. As in HKS, we assume the energy-momentum tensor for an imperfect fluid:

$$T^{\mu\nu} = (c^2\varrho+p)u^{\mu}u^{\nu}-pg^{\mu\nu}+\eta H^{\mu\kappa}H^{\nu\lambda}(u_{\kappa;\lambda}+u_{\lambda;\kappa}-\tfrac{2}{3}g_{\lambda\kappa}u^{\sigma}{}_{;\sigma})+\zeta H^{\mu\nu}u^{\lambda}{}_{;\lambda}, \tag{1}$$

where: $H^{\mu\nu} = g^{\mu\nu}-u^{\mu}u^{\nu}$ and η, ζ are, respectively, the coefficients of shear and bulk viscosity. The Einstein field equations with the Robertson-Walker metric and the above form of energy-momentum tensor give:

$$\kappa\varrho c^2 = -\Lambda+3\frac{kc^2+\dot{R}^2}{c^2R^2}, \tag{2a}$$

$$\kappa p = \Lambda-\frac{2R\ddot{R}+\dot{R}^2+kc^2}{c^2R^2}+2\alpha\frac{\dot{R}}{R}, \tag{2b}$$

where: $\alpha = \frac{3}{2}\zeta\kappa$. The shear viscosity term vanishes on account of isotropy.

The equation (2b) with the equation of state for dust-filled universe: $p = 0$, and the system of units: $c = \kappa = 1$, takes the form:

$$2R\ddot{R}+\dot{R}^2+k-\Lambda R^2-2\alpha\dot{R}R = 0. \tag{3}$$

2. If we define the energy to be equal to ϱR^3, then from (2a) we have:

$$E = 3(\dot{R}^2+k)R-\Lambda R^3. \tag{4}$$

Differentiating (4) and comparing the result with (3) we immediately get:

$$\dot{E} = 6\alpha\dot{R}^2R. \tag{5}$$

Since near the singularity $|\dot{R}| \to \infty$ and $|\ddot{R}| \to \infty$, equation (3) approaches the following form:

$$2R\ddot{R}+\dot{R}^2 = 0. \tag{6}$$

Hence we may notice that the singular behaviour of any model does not depend on any physical constants, and is represented by the Einstein-de Sitter solution:

$$R \sim t^{2/3}. \tag{7}$$

From (4) and (5) we also deduce that:

$$\lim_{R\to 0} \dot{E} = 2\alpha E. \tag{8}$$

3. The equation (3) was integrated numerically with the help of the modified Runge-Kutta method with minimization of truncation error. The serious difficulty in computer calculations is to obtain a "smooth transition" through the singular point of the solution. In order to arrive at proper solutions, the Runge-Kutta method is improved in the following points:

a) We postulate that near the singularity the energy must change continuously. If Δt is the step of integration then, according to (8), relation $\Delta E \approx 2\alpha E \Delta t$ must hold. Applying this to a certain region $R < R_1$ we correct the last value of E and, with the help of (4), the last value of $\dot{R}$, if necessary. The value of R_1 depends on the assumed accuracy.

b) We believe that only positive solutions have a physical meaning. We set the lowest allowed value of R, say R_2. If the result of integration is $R < R_2$, we substitute $R \to R_2$ and $\dot{R} \to |\dot{R}| \operatorname{sign}(\Delta t)$. The value of R_2 is chosen small enough not to decrease the accuracy of the solutions. The solutions obtained by this method are compared with solutions which are known analytically for special cases of $k = 0$ and $\alpha = 0$ and good agreement is found.

3. *Results and discussion*

1. Results of calculations are summarized in Figures 1–3. Three columns in each figure correspond to three different initial values of R (chosen at the moment $E = 1$), the first being the lowest one and the third the highest one. There are some cases in which these initial values cannot be fulfilled, which are labelled with the word "impossible".

If $k = 0$ and $\alpha = 3$ we have the critical value for Λ (see HKS), $\Lambda_\alpha = -\frac{1}{3}\alpha^2$, equal to -3, so the models in the last row (Fig. 1) are identical with CV and CVI from HKS. Note also that in cases (0, 3, 3, b) and (0, 3, 3, c) (the first three numbers denote parameters: k, Λ, α and letters: a, b, c — first, second or third column, respectively) there are regions with negative energy, which had not been marked in HKS (case A II).

2. In considering the problem of singularities in the viscous models we make use of Hawking-Penrose theorem (H–P theorem) [7]. Since in our viscous world models there are no closed timelike curves, we may consider, for our purposes, the H–P theorem to summarize all previous results concerning the appearance of singularities in cosmology (theorems II, III, IV, and V cited in the paper [7], except the theorem I holding also for $\Lambda > 0$; theorem I postulates different, so-called weak, energy condition and applies to the gravitational collapse and to some "open" world models). The H–P theorem is proved for field equations with the cosmological constant $\Lambda \leqslant 0$, however Hawking and Penrose conjecture that this restriction is not essential ("... the larger the curvatures present the smaller is the significance of the value of Λ. Thus, it is hard to imagine that the value of Λ should qualitatively affect the singularity discussion, except in regions where curvatures are still small enough to be comparable with Λ." [7, p. 531]).

The energy condition of the H–P theorem for the energy-momentum tensor (1) and the Robertson-Walker metric are equivalent to:

$$\mathscr{H} \equiv \frac{\ddot{R}}{R} - \frac{2}{3} \leqslant 0 \tag{9}$$

at every point. The thick horizontal lines on some graphs (Fig. 1–3) denote regions in which the energy condition (9) is violated ($\mathscr{H} > 0$). Lack of this line on a given graph does not mean that the corresponding solution satisfies the condition (9).

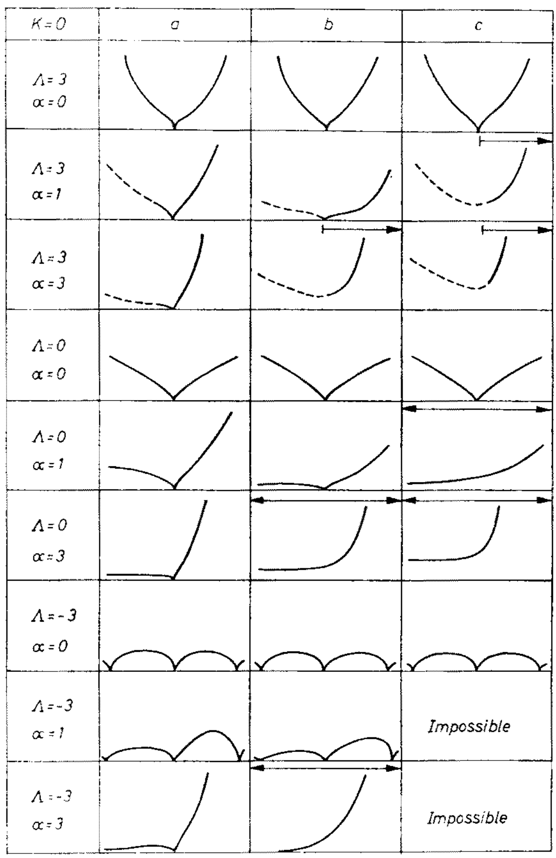

Fig. 1. Models of the universe with $k = 0$: R *vs* t dependence for different values of Λ and α, and for different initial conditions (columns: a, b, c)

3. Most interesting cases are: (0, 0, 1, c), (0, 0, 3, b), (0, 0, 3, c) and $(0, \bar{3}, 3, \mathrm{b})$. In all these cases the energy condition (9) is violated, the singularity disappears and energy $E = \varrho R^3$ is positive everywhere. Let us note that the Friedmann-Lemaître counterparts of these solutions ($\alpha = 0$) possess the initial singularity. Therefore bulk viscosity may be considered here to be an effective mechanism removing the singularity.

There are many cases [(0, 3, 1, c), (0, 3, 3, b), (0, 3, 3, c); ($\bar{1}$, 3, 1, c), ($\bar{1}$, 3, 3, b), ($\bar{1}$, 3, 3, c); ($\bar{1}$, 0, 1, c), ($\bar{1}$, 0, 3, b), ($\bar{1}$, 0, 3, c); ($\bar{1}$, $\bar{3}$, 1, c), ($\bar{1}$, $\bar{3}$, 3, b), ($\bar{1}$, $\bar{3}$, 3, c)] in which $\mathscr{H} > 0$ and the initial singularity is also removed by $\alpha \neq 0$. In these cases the introduction of dissipation makes the energy $E = \varrho R^3$ negative. The achieved solutions are analytically regular. If we decide, however, to treat the regions in which $E < 0$ as unphysical ones and cut them off, we obtain solutions geodesically incomplete, *i. e.* singular in the sense

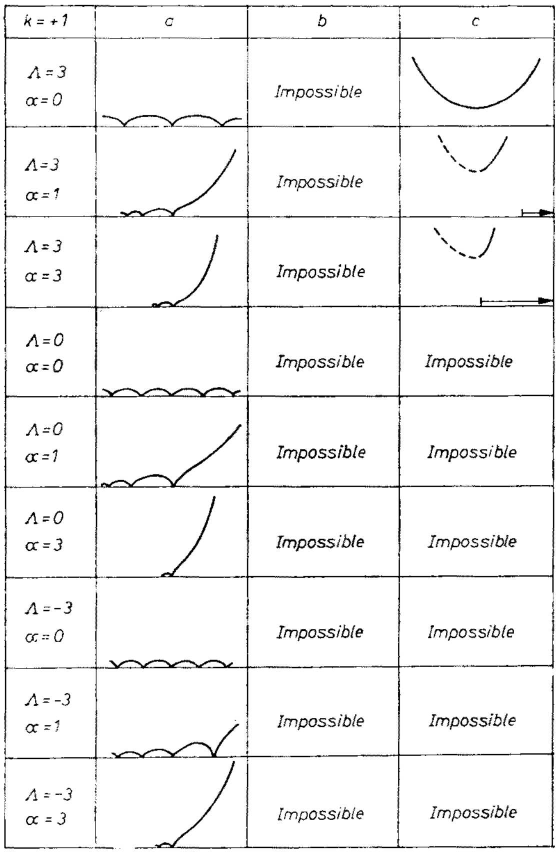

Fig. 2. Models of the universe with $k = +1 : R$ *vs* t dependence

of Geroch [8]. Therefore it will be useful to distinguish between the "point singularity" — the zero-point of the $R(t)$ function, and "geodesical singularity" originating by the removal of a part of the solution. Within the considered class of models such a distinction has a clear meaning.

If we decide to remove regions with negative energy, the models (1, 3, 0, *c*), (1, 3, 1, *c*) and (1, 3, 3, *c*) are especially interesting: the regular model (1, 3, 3, *c*) acquires the geodesical singularity on account of dissipation.

Model (1, 3, 0, *c*) is interesting in itself. It satisfies the energy condition (9) and all other conditions of the $H-P$ theorem and has no singularity. This model supplies an example that the extension of the H–P theorem to models with $\Lambda > 0$ would not be valid. This is precisely the case when "curvatures are still small enough to be comparable with Λ".

4. *Conclusions*

1. The introduction of the bulk viscosity into the frame of Friedmann-Lemaître cosmology removes the initial singularity ("point singularity") provided it is allowed by the H–P theorem.

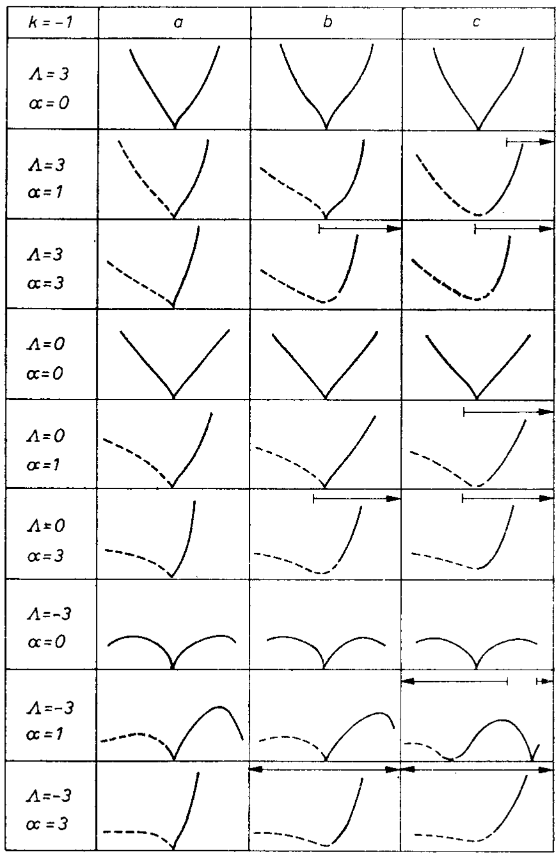

Fig. 3. Models of the universe with $k = -1 : R$ *vs* t dependence

2. Many models obtained with this method, although analytically regular, possess regions with negative energy.

3. The H–P theorem cannot be extended to models with positive value of the cosmological constant.

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